

BANACH'S ISOMETRIC SUBSPACE PROBLEM IN DIMENSION FOUR

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ABSTRACT. We prove that if all intersections of a convex body $B \subset \mathbb{R}^4$ with 3-dimensional linear subspaces are linearly equivalent then B is a centered ellipsoid. This gives an affirmative answer to the case $n = 3$ of the following question by Banach from 1932: Is a normed vector space V whose n -dimensional linear subspaces are all isometric, for a fixed $2 \leq n < \dim V$, necessarily Euclidean?

The dimensions $n = 3$ and $\dim V = 4$ is the first case where the question was unresolved. Since the 3-sphere is parallelizable, known global topological methods do not help in this case. Our proof employs a differential geometric approach.

1. INTRODUCTION

In this paper we give an affirmative answer to the case $n = 3$ of the following problem:

Problem 1.1. *Let $(V, \|\cdot\|)$ be a normed vector space (over $\mathbb{R}$) such that for some fixed n , $2 \leq n < \dim V$, all n -dimensional linear subspaces of V are isometric. Is $\|\cdot\|$ necessarily a Euclidean norm (i.e., an inner product one)?*

The problem goes back to Banach [2, Remarks on Chapter XII] and is also known as “Banach’s conjecture”: it is conjectured that the answer is always affirmative. This has been proven for many but not all dimensions. Auerbach, Mazur and Ulam [1] proved the conjecture in the case $n = 2$. Dvoretzky [10] proved it for an infinite-dimensional V and any $n \geq 2$ as a corollary of his famous theorem on approximately Euclidean subspaces. Gromov [11] proved that the answer is affirmative if n is even or $\dim V \geq n + 2$. Recently Bor, Hernández-Lamonedá, Jiménez-Desantiago and Montejano [4] extended Gromov’s approach and proved the conjecture for n congruent to 1 modulo 4 except $n = 133$. The problem also makes sense for complex normed spaces, see [5, 11, 17] for some results in this direction.

Recall that (due to the parallelogram law) a normed vector space V is Euclidean if and only if all its 2-dimensional linear subspaces are. This fact reduces Problem 1.1 to the case $\dim V = n + 1$, and below we focus only on this case.

We say that two sets $A, B \subseteq V$ are *linearly equivalent* if there is a linear bijection $L: \text{LinSpan}(A) \rightarrow \text{LinSpan}(B)$ such that $L(A) = B$. Considering the unit ball of the norm, one can reformulate Problem 1.1 in geometric terms as follows.

Problem 1.2. *Let $n \geq 2$ and let $B \subset \mathbb{R}^{n+1}$ be a convex body symmetric with respect to the origin. Assume that all intersections of B with n -dimensional linear subspaces are linearly equivalent. Is B necessarily an ellipsoid?*

The symmetry assumption for B can be removed due to the following result of Montejano [18, Theorem 1]: If $B \subset \mathbb{R}^{n+1}$ is a convex body and all intersections of B with n -dimensional linear subspaces are affine equivalent, then either B is symmetric with respect to 0, or B is a (not necessarily centered) ellipsoid. In particular, linear equivalence of the intersections implies that B is symmetric with respect to 0. Keeping this fact in mind, we do not assume the symmetry in the sequel as some of our intermediate results may be of interest in the non-symmetric case.

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Our main result is the following theorem, which implies affirmative answers to the above problems for $n = 3$.

Theorem 1.3. *Let $B \subset \mathbb{R}^4$ be a convex body such that all intersections of B with 3-dimensional linear subspaces of $\mathbb{R}^4$ are linearly equivalent. Then B is an ellipsoid centered at 0.*

In contrast with the aforementioned works [1, 4, 11], the proof of Theorem 1.3 is essentially local: for the most part it deals with an arbitrarily small neighborhood of one cross-section. A similar approach was developed in [14] where a “local” variant of Banach’s problem was solved for $n = 2$ and smooth strictly convex norms. An analogous generalization is possible for Theorem 1.3 but it is outside the scope of this paper. The present paper borrows many ideas from [14], in particular Proposition 2.4 is a generalization of constructions in [14, §3]. We believe that this approach to Banach’s problem is viable in all dimensions, but so far we were able to make it work only for $n = 2$ and $n = 3$.

“Integrable” and “non-integrable” problems. The proofs in [1, 4, 11] are based on topological methods. In fact, the arguments in these proofs also solve the following “non-integrable” version of the problem where one considers a family of n -dimensional convex bodies in linear subspaces without requiring that they are cross-sections of a single body in V :

Let V be a vector space, $2 \leq n < \dim V$, and let $K \subset \mathbb{R}^n$ be a convex body. Assume there exists a continuous family $\{K_X\}$ where X ranges over all n -dimensional linear subspaces of V and each K_X is a convex body in X linearly equivalent to K . For what types of K is this possible?

This problem is topological by its nature and it can be reduced to the study of structure groups of certain bundles, see [4, 11]. Gromov [11] showed that if $n = 2k$ or $\dim V \geq n + 2$ then K in this problem must be an ellipsoid. This obviously implies the respective cases of Banach’s conjecture. Although the “non-integrable” generalization fails for $n = 2k + 1 = \dim V - 1$, in some cases one can obtain strong restrictions on the geometry of the body in question. Namely, in [4] it is shown that for $n = 4k + 1$, except $n = 133$, K must be an affine body of revolution (i.e., has an $SO(n - 1)$ symmetry), and then it follows that in the original “integrable” setting K is an ellipsoid.

We note that considering the non-integrable problem does not help if $n = 3$ and $\dim V = 4$ because in this case any centrally symmetric convex body can be chosen for K . This follows from the fact that the 3-sphere is parallelizable, see [6] for a detailed explanation.

Structure of the proof and organization of the paper. The plan of the proof of Theorem 1.3 is detailed in Section 2, here is an overview.

The proof has two main steps. In the first step, stated as Proposition 2.4, we obtain differential geometric implications of “local integrability” near a generic cross-section $K = X \cap B$ where X is an n -dimensional linear subspace. Loosely speaking, Proposition 2.4 provides us with an algebraic $n(n - 1)$ -parameter family of vectors tangent to the boundary of B at points of K . Furthermore it includes an $(n - 1)^2$ -parameter sub-family of vectors tangent to the cross-section K itself, see (2.2).

In the second step we study restrictions on the geometry of the cross-section K imposed by the existence of such a family of tangent vectors. The resulting statement is Proposition 2.5. It asserts that, at least for $n = 3$, the linear vector fields provided by Proposition 2.4 are tangent to ∂K everywhere rather than only at certain planes as in (2.2). The proof of Proposition 2.5 is the most substantial part of the paper, see §2.2 for an outline.

Theorem 1.3 follows from Propositions 2.4 and 2.5 with an application of Blaschke-Kakutani ellipsoid characterization, see §2.3. The specifics of the dimension $n = 3$ are used only in the proof of Proposition 2.5, while the proof of Proposition 2.4 and the deduction of Theorem 1.3 from the propositions work for all $n \geq 2$.

The rest of the paper is organized as follows. In Section 2 we introduce definitions and notation, formulate Propositions 2.4 and 2.5, and deduce Theorem 1.3 from them. Section 3 is a collection of

technical lemmas (about polynomial vector fields tangent to convex hypersurfaces) used throughout the paper. The remaining sections are devoted to the proofs of Propositions 2.4 and 2.5. The proof of Proposition 2.4 is given in Section 4. In Section 5 we consider degenerate cases of Proposition 2.5, and in Section 6 we handle the non-degenerate case and finish the proof of Proposition 2.5.

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2. PRELIMINARIES AND THE PLAN OF THE PROOF

In this section we set up terminology and notation, formulate the two main propositions and deduce Theorem 1.3 from them.

2.1. Notation and preliminaries. In this paper, a “vector space” always means a finite-dimensional real vector space. For vector spaces X and Y , $\text{Hom}(X, Y)$ denotes the space of linear maps from X to Y . We denote $\text{Hom}(X, \mathbb{R})$ by X^* .

For a vector space X , we denote by $\text{Gr}_k(X)$ the Grassmannian manifold of (unoriented) k -dimensional linear subspaces of X and by $GL(X)$ the group of linear bijections from X to itself.

For a convex set $K \subset X$, we denote by ∂K the relative boundary of K , that is the boundary in the topology of its affine span, and by $\mathcal{I}(K)$ the *group of linear self-equivalences of K* :

$$(2.1) \quad \mathcal{I}(K) = \{L \in GL(X) : L(K) = K\}.$$

A convex set $K \subset X$ is called a *convex body* if it is compact and its interior is non-empty. If K is a convex body then $\mathcal{I}(K)$ is a compact subgroup of $GL(X)$.

By an *ellipsoid* we mean the unit ball of an inner product norm in a vector space. In other words, all ellipsoids are assumed to be centered at 0 (unless explicitly stated otherwise). The same terminology adjustment applies to ellipses in dimension 2.

By a *Minkowski norm* on a vector space V we mean a function $\Psi: V \rightarrow \mathbb{R}_+$ which is positively 1-homogeneous, subadditive (and hence convex), and positive on $V \setminus \{0\}$. The difference from usual norms is that a Minkowski norm is not assumed symmetric. There is a standard 1-to-1 correspondence between Minkowski norms on V and convex bodies in V with 0 in the interior. Namely, to each Minkowski norm Ψ one associates its unit ball $B_\Psi = \{x \in V : \Psi(x) \leq 1\}$, and for every convex body $B \subset V$ with 0 in the interior there is a corresponding Minkowski norm $\Psi^B(x) = \inf\{\lambda > 0 \mid x/\lambda \in B\}$.

The convexity of Ψ implies (see e.g. [9, 2.1.1 and 2.2.7]) that the *one-sided directional derivative*

$$\partial_x^+ \Psi(v) = \lim_{s \downarrow 0} \frac{\Psi(x + sv) - \Psi(x)}{s}$$

is well-defined for all $x, v \in V$. Moreover, $\partial_x^+ \Psi: V \rightarrow \mathbb{R}$ is positively 1-homogeneous and subadditive for all $x \in V$.

There are many equivalent definitions of one-sided tangent directions of a convex hypersurface. In the present paper we use the one by Bouligand:

Definition 2.1. Let V be a vector space, $B \subset V$ a convex body, $x \in \partial B$, and $v \in V$. We say that v is *forward tangent to ∂B at x* if there exist sequences $\{x_i\}_{i=1}^\infty \subset \partial B$ and $\{c_i\}_{i=1}^\infty \subset \mathbb{R}_+$ such that $x_i \rightarrow x$ and $c_i(x_i - x) \rightarrow v$ as $i \rightarrow \infty$.

We say that v is *tangent to B at x* if both v and $-v$ are forward tangent to B at x .

The following two lemmas are standard. We include their proofs for the reader’s convenience and to avoid a discussion on various definitions of tangency.

Lemma 2.2. *Let V be a vector space and $B \subset V$ a convex body with 0 in the interior. Let $x \in \partial B$ and $v \in V$. Then the following conditions are equivalent:*

- (1) *v is forward tangent to ∂B at x .*
- (2) *$\partial_x^+ \Psi(v) = 0$ where Ψ is the Minkowski norm associated to B .*

Proof. Both conditions are obviously satisfied for $v = 0$, so we assume that $v \neq 0$. Let v be forward tangent to ∂B at x and let $\{x_i\}_{i=1}^\infty \subset \partial B$, $\{c_i\}_{i=1}^\infty \subset \mathbb{R}_+$ be the corresponding sequences from Definition 2.1. Since $v \neq 0$, we have $c_i \rightarrow +\infty$, hence may assume that $\{c_i\}_{i=1}^\infty$ is an increasing sequence. We have

$$\partial_x^+ \Psi(v) = \lim_{i \rightarrow \infty} \frac{\Psi(x + v/c_i) - \Psi(x)}{1/c_i} = \lim_{i \rightarrow \infty} \Psi(c_i x + v) - c_i \Psi(x) = \lim_{i \rightarrow \infty} \Psi(c_i x + v) - \Psi(c_i x_i).$$

By the sub-additivity of Ψ ,

$$-\Psi(c_i(x_i - x) - v) \leq \Psi(c_i x + v) - \Psi(c_i x_i) \leq \Psi(c_i(x - x_i) + v).$$

Since $c_i(x_i - x) \rightarrow v$ as $i \rightarrow \infty$, we conclude that $\partial_x^+ \Psi(v) = 0$.

On the other hand, if $\partial_x^+ \Psi(v) = 0$ then we can choose any sequence $\{s_i\}_{i=1}^\infty \subset \mathbb{R}_+$ decreasing to 0 and set x_i and c_i required in Definition 2.1 to be $x_i = \frac{x + s_i v}{\Psi(x + s_i v)}$ and $c_i = \frac{\Psi(x + s_i v)}{s_i}$. $\square$

Lemma 2.3. *Let V be a vector space, $B \subset V$ a convex body with 0 in the interior, and $x \in \partial B$. Then the set of all vectors tangent to ∂B at x is a linear subspace of V .*

Proof. By Lemma 2.2, $v \in V$ is tangent to ∂B at x if and only if $\partial_x^+ \Psi(v) = \partial_x^+ \Psi(-v) = 0$. Since $\partial_x^+ \Psi$ is positively 1-homogeneous, this implies that $\partial_x^+ \Psi(tv) = 0$ for all $t \in \mathbb{R}$.

Let v, w be tangent to ∂B at x . Then, by the sub-additivity of $\partial_x^+ \Psi$,

$$\partial_x^+ \Psi(v + w) \leq \partial_x^+ \Psi(v) + \partial_x^+ \Psi(w) = 0$$

and

$$\partial_x^+ \Psi(v + w) \geq \partial_x^+ \Psi(v) - \partial_x^+ \Psi(-w) = 0.$$

Hence $\partial_x^+ \Psi(v + w) = 0$ and therefore $v + w$ is forward tangent to ∂B at x . Similarly, $-(v + w)$ is forward tangent to ∂B at x , thus $v + w$ is tangent to ∂B at x and the lemma follows. $\square$

2.2. Steps of the proof of the main theorem. We are now in a position to formulate two main intermediate results in the proof of Theorem 1.3. The first one is the following proposition, which works in all dimensions.

Proposition 2.4. *Let V be a vector space, $\dim V = n + 1 \geq 3$, and let $B \subset V$ be a convex body such that all intersections of B with n -dimensional linear subspaces of V are linearly equivalent. Then for almost every $X \in \text{Gr}_n(V)$ there exist a vector $\nu \in V \setminus X$ and a linear map*

$$R: X^* \rightarrow \text{Hom}(X, X)$$

such that for every $\lambda \in X^*$ the linear operator $R_\lambda = R(\lambda): X \rightarrow X$ satisfies:

- (1) *Trace $R_\lambda = 0$.*
- (2) *For every $x \in \partial B \cap X$, the vector $R_\lambda(x) + \lambda(x)\nu$ is tangent to ∂B at x .*

Note that the assumptions of Proposition 2.4 imply that B contains 0 in its interior.

The proof of Proposition 2.4 is given in Section 4. In the case when B is smooth, the argument is a simple differential geometric calculation, see §4.1. In the general case the proof simulates the same calculation replacing derivatives by suitable partial limits. We make use of the fact that Lipschitz functions (in particular, convex ones) are differentiable almost everywhere, this is the reason why “almost every X ” appears in the statement.

Let V, B, X and R be as in Proposition 2.4 and denote $K = B \cap X$. Proposition 2.4(2) implies the following property, which plays a central role in subsequent arguments:

(2.2) for every $\lambda \in X^*$ and $x \in \partial K \cap \ker \lambda$, the vector $R_\lambda(x)$ is tangent to ∂K at x .

Indeed, if $\lambda(x) = 0$ then the vector $R_\lambda(x) + \lambda(x)\nu$ from Proposition 2.4(2) equals $R_\lambda(x)$ and hence belongs to X , therefore this vector is also tangent to $\partial K = \partial B \cap X$. Note that (2.2) does not involve the $(n+1)$ -dimensional space V and the body B , i.e. it is a statement about K and R that live in the n -dimensional space X .

The second step in the proof of Theorem 1.3 is the following proposition.

Proposition 2.5. *Let X be a vector space, $\dim X = 3$, and let $K \subset X$ be a convex body with 0 in its interior. Let $R: X^* \rightarrow \text{Hom}(X, X)$ be a linear map such that for every $\lambda \in X^*$ the map $R_\lambda = R(\lambda)$ satisfies $\text{Trace } R_\lambda = 0$ and (2.2) holds. Then $R_\lambda(x)$ is tangent to ∂K for all $x \in \partial K$.*

The following view on Proposition 2.5 is helpful. For each $\lambda \in X^*$, we regard the operator $R_\lambda \in \text{Hom}(X, X)$ as a linear vector field on X . Proposition 2.5 asserts that this vector field is tangent to ∂K . Therefore the 1-parameter subgroup $\{e^{tR_\lambda}\}_{t \in \mathbb{R}}$ generated by R_λ is contained in the group of linear self-equivalences of K . Since K is 3-dimensional, this leaves us with three possibilities: either K is an ellipsoid and all operators R_λ are skew-symmetric with respect to the inner product corresponding to K , or K is an affine body of revolution and all operators R_λ are proportional to the infinitesimal generator of the respective group of affine rotations, or $R = 0$.

In the proof of Proposition 2.5 we handle the three cases separately but the unified formulation allows us to deduce the main theorem without case chasing. In some of the cases we show directly that K is an ellipsoid and finish the proof with the following lemma.

Lemma 2.6. *Proposition 2.5 holds true if K is an ellipsoid.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the inner product associated to K , and $|\cdot|$ the respective Euclidean norm. We have to show that, under the assumptions of Proposition 2.5, $\langle R_\lambda(v), v \rangle = 0$ for all $\lambda \in X^*$ and $v \in X$ with $|v| = 1$. Fix $v \in X$ such that $|v| = 1$ and observe that the expression $\langle R_\lambda(v), v \rangle$ is linear in λ , therefore it suffices to verify the identity for all λ from some basis of X^* . Pick an orthonormal basis $e_1, \dots, e_n$ in X such that $e_n = v$, and let $f_1, \dots, f_n \in X^*$ be the dual basis. Then $\langle R_{f_i}(e_n), e_n \rangle = 0$ for all $i \leq n-1$ by the assumption (2) of Proposition 2.5. Similarly, $\langle R_{f_n}(e_i), e_i \rangle = 0$ for all $i \leq n-1$. Since $\text{Trace } R_{f_n} = 0$, it follows that $\langle R_{f_n}(e_n), e_n \rangle = 0$. Thus $\langle R_{f_i}(v), v \rangle = \langle R_{f_i}(e_n), e_n \rangle = 0$ for all $i = 1, \dots, n$ and the lemma follows. $\square$

The proof of Proposition 2.5 occupies Sections 5 and 6. We divide the problem into degenerate and non-degenerate cases (Definition 5.1) and handle them using different structures arising from the tensor R . For the degenerate case (Section 5) we construct a collection of quadratic vector fields tangent to ∂K (Lemma 5.4) and analyze them by algebraic methods. The degeneration implies certain cancellations that lead to the desired conclusion (Proposition 5.10).

The non-degenerate case (Section 6) is the most interesting one. In this case R determines K uniquely up to a homothety and we show that K is an ellipsoid. The key construction is given in Lemma 6.2, which shows that R induces a flow on the space of 2-dimensional cross-sections of K with trajectories consisting of linearly equivalent cross-sections. Furthermore, the non-degeneracy assumption implies that cross-sections corresponding to fixed points of the flow are ellipses. Then, with some help from two-dimensional topological dynamics (Lemma 3.7), we deduce that K has a one-parameter family of ellipses among its cross-sections. The way how R determines K implies that ∂K is a real analytic surface, and then it is not hard to show that K is an ellipsoid and finish the proof, see the end of Section 6.

2.3. Proof of Theorem 1.3 assuming Propositions 2.4 and 2.5. The final step of the proof of Theorem 1.3 is to deduce it from the two above propositions.

Let $B \subset V = \mathbb{R}^4$ be a convex body satisfying the assumptions of Theorem 1.3. As explained after Problem 1.2, B is symmetric with respect to 0. Let $\|\cdot\|$ be the norm with unit ball B .

Let X , ν and R be as in Proposition 2.4, and let $K = B \cap X$ be the corresponding cross-section. As shown in §2.2, K and R satisfy (2.2) and therefore Proposition 2.5 applies. Fix $x \in \partial K$ and pick $\lambda \in X^*$ such that $\lambda(x) \neq 0$. Let $w_1 = R_\lambda(x)$ where $R_\lambda = R(\lambda)$ and $w_2 = w_1 + \lambda(x)\nu$. Propositions 2.4 and 2.5 imply that both w_1 and w_2 are tangent to ∂B at x . Since ν is a linear combination of w_1 and w_2 , namely $\nu = \frac{1}{\lambda(x)}(w_1 - w_2)$, Lemma 2.3 implies that ν is also tangent to ∂B at x .

Since x is an arbitrary point of $\partial K = \partial B \cap X$, it follows that B is contained in the cylinder $K + \mathbb{R}\nu$. Let $P_X: V \rightarrow X$ be the projection along ν , that is, P_X is the unique linear map from V to X such that $P_X|_X = \text{id}_X$ and $P_X(\nu) = 0$. The fact that $B \subset K + \mathbb{R}\nu$ implies that $P_X(B) \subset B$. Equivalently, the projector P_X does not increase the norm, that is, $\|P_X(v)\| \leq \|v\|$ for all $v \in V$.

Recall that X in Proposition 2.4 can be chosen arbitrarily from a set of full measure in $\text{Gr}_3(V)$ and observe that the existence of a norm non-increasing projector is a closed condition. Thus for every hyperplane $X \in \text{Gr}_3(V)$ there exist a projector $P_X: V \rightarrow X$ onto X not increasing the norm.

To finish the proof we apply the Blaschke-Kakutani characterization of ellipsoids [15]. We use the following formulation that can be found in e.g. [12]:

Theorem 2.7 ([15], [12, Theorem 12.5]). *Let $(V, \|\cdot\|)$ be a finite-dimensional Banach space and $2 \leq k < \dim V$. Suppose that for every k -dimensional linear subspace $X \in \text{Gr}_k(V)$ there exists a linear projector P_X from V onto X not increasing the norm. Then the norm $\|\cdot\|$ is Euclidean.*

Applying Theorem 2.7 to $V = \mathbb{R}^4$, the norm $\|\cdot\|$ associated to B and $k = 3$, we conclude that B is an ellipsoid and complete the proof of Theorem 1.3. $\square$

3. VECTOR FIELDS TANGENT TO CONVEX SURFACES

We often interpret maps from a vector space X to itself as vector fields on X . A map $W: X \rightarrow X$ is called a *homogeneous polynomial vector field of degree k* if the coordinates of $W(x)$, $x \in X$, are homogeneous polynomials of degree k in coordinates of x . Clearly this property does not depend on the choice of coordinates. Homogeneous polynomial vector fields of degree 1, 2 and 3 are called *linear*, *quadratic* and *cubic*, respectively.

A *trajectory* of a vector field $W: X \rightarrow X$ is a curve $\gamma: (a, b) \subseteq \mathbb{R} \rightarrow X$ such that $\dot{\gamma}(t) = W(\gamma(t))$ for all $t \in (a, b)$. We say that a vector field W is (forward) tangent to ∂K , where $K \subset X$ is a convex body, if $W(x)$ is (forward) tangent to ∂K at x for every $x \in \partial K$.

In this section we prove a number of technical facts about polynomial vector fields tangent to convex hypersurfaces. The most essential ones are Lemma 3.4 and Lemma 3.7.

Lemma 3.1. *Let X be a vector space, $K \subset X$ a convex body with 0 in the interior, and Ψ the Minkowski norm associated to K . Let $W: X \rightarrow X$ be a homogeneous polynomial vector field forward tangent to ∂K . Then Ψ is constant along every trajectory of W .*

Proof. By Lemma 2.2 we have $\partial_x^+ \Psi(W(x)) = 0$ for all $x \in \partial K$. By homogeneity, this identity extends to all $x \in X$. Let $\gamma: (a, b) \rightarrow X$ be a trajectory of W . Then $\partial_{\gamma(t)}^+ \Psi(\dot{\gamma}(t)) = 0$ for all t , hence the function $\Psi \circ \gamma$ has zero right derivative everywhere. Since this function is locally Lipschitz, it follows that it is constant. $\square$

The next lemma is a generalization of Lemma 3.1 used in Section 5.

Lemma 3.2. *Let X be a vector space, $K \subset X$ a convex body with 0 in the interior, and Ψ the Minkowski norm associated to K . Let $W: X \rightarrow X$ be a homogeneous polynomial vector field and $f: X \rightarrow \mathbb{R}$ a nonzero homogeneous polynomial function. Assume that the vector field fW (that is,*

$x \mapsto f(x)W(x)$ is forward tangent to ∂K . Then Ψ is constant along every trajectory of W and therefore W is tangent to ∂K .

Proof. By Lemma 3.1, Ψ is constant along the trajectories of fW , but trajectories of W can be different because of zeroes of f . Let $\gamma: (a, b) \rightarrow X$ be a trajectory of W and consider the set

$$T := \{t \in (a, b) : f(\gamma(t)) = 0\}.$$

Since $f \circ \gamma$ is a real analytic function, we have either $T = (a, b)$ or T is a discrete subset of (a, b) .

First consider the case when $f \circ \gamma$ is not everywhere zero on (a, b) . Then T is discrete: for any $t_0 \in (a, b)$ there exists $\varepsilon > 0$ such that $f \circ \gamma$ has no zeroes on $(t_0 - \varepsilon, t_0) \cup (t_0, t_0 + \varepsilon)$. This implies that $\gamma|_{(t_0 - \varepsilon, t_0)}$ and $\gamma|_{(t_0, t_0 + \varepsilon)}$ coincide with trajectories of fW up to a change of parametrization. By Lemma 3.1 applied to fW it follows that $\Psi \circ \gamma$ is constant on each of the intervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$. By continuity both constants equal $\Psi(\gamma(t_0))$, therefore $\Psi \circ \gamma$ is constant on $(t_0 - \varepsilon, t_0 + \varepsilon)$. Thus we have shown that $\Psi \circ \gamma$ is locally constant and hence constant on (a, b) .

Now consider the case when $f \circ \gamma \equiv 0$. Fix $t_0 \in (a, b)$ and let $p = \gamma(t_0)$. Since f is a nonzero polynomial, the set of its zeroes is nowhere dense in X . Hence there is a sequence $\{p_i\} \subset X$ such that $p_i \rightarrow p$ and $f(p_i) \neq 0$ for all i . Let $\gamma_i: (a_i, b_i) \rightarrow X$ be the maximal trajectory of W with initial data $\gamma_i(t_0) = p_i$. Then γ_i pointwise converge to γ on (a, b) . Since $f(p_i) \neq 0$, each γ_i falls under the above case where $f \circ \gamma_i$ is not everywhere zero, thus $\Psi \circ \gamma_i$ is constant for every i . Passing to the limit as $i \rightarrow \infty$ we conclude that $\Psi \circ \gamma$ is constant as well.

To prove that W is tangent to ∂K , consider $x \in \partial K$ and let $\gamma: (-\varepsilon, \varepsilon) \rightarrow X$ be a trajectory of W with initial data $\gamma(0) = x$. Since $\Psi \circ \gamma$ is constant, we have $\gamma(t) \in \partial K$ for all t . Applying Definition 2.1 to sequences $x_i = \gamma(\frac{1}{i})$ and $c_i = i$, $i = 1, 2, \dots$, yields that $W(x) = \dot{\gamma}(0)$ is forward tangent to ∂K at x . Considering $x_i = \gamma(-\frac{1}{i})$ similarly shows that $-W(x)$ is forward tangent to ∂K at x . Thus W is tangent to ∂K . $\square$

Lemma 3.3. *Let X be a vector space, $K \subset X$ a convex body with 0 in the interior, and $L: X \rightarrow X$ a linear vector field forward tangent to ∂K . Then $\text{Trace } L = 0$ and $e^{tL} \in \mathcal{I}(K)$ for all $t \in \mathbb{R}$, where e^{tL} is the matrix exponential of tL and $\mathcal{I}(K)$ is the group of linear self-equivalences of K , see (2.1).*

Proof. Recall that trajectories of L have the form $t \mapsto e^{tL}(p)$ where $p \in X$ is the initial value. By Lemma 3.1, every trajectory is either contained in K or disjoint with K . Therefore $e^{tL}(K) = K$ for all $t \in \mathbb{R}$, thus $e^{tL} \subset \mathcal{I}(K)$. Since $e^{tL}(K) = K$, the operator e^{tL} is volume-preserving, therefore $\det e^{tL} = 1$ and $\text{Trace } L = 0$. $\square$

Lemma 3.4. *Let $K \subset \mathbb{R}^2$ be a convex body with 0 in the interior and W a nonzero homogeneous polynomial vector field on $\mathbb{R}^2$ of degree $\deg W \leq 3$. Suppose that W is tangent to ∂K . If $\deg W = 3$, assume in addition that W vanishes at some point other than the origin. Then K is an ellipse.*

Proof. Clearly $\deg W \neq 0$. Consider the three cases according to the possible values of $\deg W$.

Case 1: $\deg W = 1$. By Lemma 3.3, the self-equivalence group $\mathcal{I}(K)$ contains a one-parameter subgroup $G = \{e^{tW}\}_{t \in \mathbb{R}}$. As $\mathcal{I}(K)$ is a compact subgroup of $\text{GL}(2)$, this is possible only if G is conjugate to $SO(2)$ and thus K is an ellipse.

Case 2: $\deg W = 2$. Consider a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x) = x \wedge W(x)$ where the $\wedge$ -product is just the determinant of the 2×2 matrix composed of vectors x and $W(x)$. Observe that f is a homogeneous polynomial of degree 3, in particular, $W(-x) = -W(x)$ for all $x \in \mathbb{R}^2$. Hence f attains values of opposite signs on ∂K , therefore $f(p) = 0$ for some $p \in \partial K$.

The vector $W(p)$ is collinear to p and tangent to ∂K at the same time, hence $W(p) = 0$. By homogeneity, W vanishes on the entire line $\ell = \{tp \mid t \in \mathbb{R}\}$. Therefore W can be decomposed as a product $W = \lambda L$ where $\lambda: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a linear function with $\ker \lambda = \ell$ and $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear vector field. By Lemma 3.2, L is tangent to ∂K and then the result of Case 1 implies that K is an ellipse.

Case 3: $\deg W = 3$. By assumption there exists $p \in \mathbb{R}^2 \setminus \{0\}$ such that $W(p) = 0$. As in Case 2, W vanishes at the line $\ell = \{tp \mid t \in \mathbb{R}\}$ and hence $W = \lambda Q$ where λ is a linear function and Q is a quadratic vector field. By Lemma 3.2, Q is tangent to ∂K and then the result of Case 2 implies that K is an ellipse. $\square$

In Section 6 we will need some facts about quadratic vector fields on smooth convex surfaces. The following dynamical systems terminology will be handy.

Definition 3.5. Let S be a smooth surface (i.e., a two-dimensional smooth manifold) and W a complete smooth vector field on S . An *orbit* of W is a subset of S of the form $\{\gamma(t) : t \in \mathbb{R}\}$ where γ is a trajectory of W . Sets of the form $\{\gamma(t) : t \geq 0\}$ and $\{\gamma(t) : t \leq 0\}$ are called *half-orbits* (forward and backward ones, respectively). A *closed orbit* is an orbit corresponding to a non-constant periodic trajectory. By abuse of notation, a *fixed point* of W is the same as a zero of W , i.e., a point $p \in S$ such that $W(p) = 0$.

The following routine lemma is a preparation to Lemma 3.7.

Lemma 3.6. *Let W be a smooth vector field on a closed smooth surface S and let $p \in S$ be an isolated zero of W . Assume that p does not belong to the closure of any non-constant orbit of W .*

Then every neighborhood U of p contains a closed orbit of W separating p from $S \setminus U$.

Furthermore $\text{ind}_p(W) = 1$ where $\text{ind}_p(W)$ is the index of W at p (as in the Poincaré-Hopf Index Theorem, see e.g. [13, Chapter 3, §5]).

Proof. We may assume that U is a topological disc and that it does not contain zeroes of W other than p . Let $U_1 \ni p$ be a smaller neighborhood whose closure $\overline{U}_1$ is contained in U .

First we show that there exist a non-constant half-orbit of W contained in $\overline{U}_1$. Pick a sequence $\{p_i\}_{i=1}^\infty \subset U_1 \setminus \{p\}$ such that $p_i \rightarrow p$. For each i , let γ_i be the trajectory of W with $\gamma_i(0) = p_i$ and let $(-a_i, b_i)$ be the maximal interval of $\mathbb{R}$ containing 0 and contained in the set $\{t \in \mathbb{R} : \gamma_i(t) \in U_1\}$. Since $p_i \rightarrow p$ and p is a fixed point of W , we have $a_i \rightarrow +\infty$. If $a_i = +\infty$ for some i , then $\{\gamma_i(t) : t \leq 0\}$ is a desired half-orbit. Otherwise we have a sequence of points $\{\gamma_i(-a_i)\}$ in the boundary of U_1 . Let q be a partial limit of this sequence and let γ be the trajectory of W with $\gamma(0) = q$. Then γ is the limit of a subsequence of trajectories $t \mapsto \gamma_i(t - a_i)$. Since $a_i \rightarrow +\infty$, it follows that $\gamma(t) \in \overline{U}_1$ for all $t \geq 0$, so $\{\gamma(t) : t \geq 0\}$ is a desired half-orbit.

Thus we have proved the existence of a non-constant half-orbit of W in $\overline{U}_1$. The Poincaré-Bendixson Theorem (see e.g. [16, Theorem 14.1.1]) implies that the closure of this half-orbit contains a fixed point or a closed orbit. The case of a fixed point is ruled out by the assumption of the lemma since p is the only fixed point in U . Hence U contains a closed orbit of W .

Let $D \subset U$ be the disc bounded by this closed orbit. Then D is invariant under the flow generated by W . Hence, by Brouwer's fixed-point theorem, D contains a fixed point of the flow. Since p is the unique fixed point in U , it follows that $p \in D$, hence the closed orbit separates p from $S \setminus U$.

To prove the second claim of the lemma, calculate the index $\text{ind}_p(W)$ in a local coordinate chart which maps p to $0 \in \mathbb{R}^2$ and D to the standard disc in $\mathbb{R}^2$. In this chart the relation $\text{ind}_p(W) = 1$ is immediate from the definition of the index, see [13, p. 134]. $\square$

Lemma 3.7. *Let W be a nonzero quadratic homogeneous vector field on a vector space $Y \simeq \mathbb{R}^3$, and let $S \subset Y$ be a smooth closed strictly convex surface symmetric with respect to 0. (The strict convexity means that S is a boundary of a convex body and S contains no straight line segments).*

Assume that W is tangent to S . Then there exist a non-constant orbit of W in S whose closure contains a zero of W .

Proof. Suppose to the contrary that no zero of W belongs to the closure of a non-constant orbit. We restrict W to S and regard it as a vector field on S . Note that S is diffeomorphic to the 2-sphere.

First consider the case when W has finitely many zeroes. By the Poincaré-Hopf Theorem (see e.g. [13, Chapter 3, §5]), the sum of indices $\text{ind}_p(W)$ where p ranges over all zeroes of W , equals 2. By Lemma 3.6 each index equals 1, hence W has exactly two zeroes on S . Denote one of them by p , then the other one is $-p$ due to the symmetry of S and W .

Choose a basis e_1, e_2, e_3 in Y such that $e_3 = p$ and e_1, e_2 are parallel to the tangent plane to S at p . The choice of e_1 and e_2 will be refined later. Identify Y with $\mathbb{R}^3$ by means of this basis and denote the respective coordinates by x, y, z . We call the xy -plane *horizontal*.

We write W as a function of coordinates x, y, z :

$$W(x, y, z) = (W_1(x, y, z), W_2(x, y, z), W_3(x, y, z))$$

where $W_i(x, y, z)$ are quadratic forms, and decompose W_i as

$$W_i(x, y, z) = Q_i(x, y) + z \cdot L_i(x, y)$$

where Q_i and L_i are a quadratic form and a linear function on the xy -plane, $i = 1, 2, 3$. The lack of z^2 in the formula follows from the fact that $W(0, 0, 1) = W(p) = 0$.

Since the tangent plane to S at $p = (0, 0, 1)$ is horizontal, S is smooth and W is homogeneous, we have

$$\frac{W_3(x, y, 1)}{\|W(x, y, 1)\|} \rightarrow 0 \quad \text{as } x, y \rightarrow 0.$$

This implies that $L_3 = 0$, so $W_3(x, y, z) = Q_3(x, y)$ for all x, y, z .

Observe that Q_3 cannot be positive or negative definite, as otherwise the z -coordinate would be strictly monotone along a periodic trajectory provided by Lemma 3.6. Therefore Q_3 vanishes at some nonzero horizontal vector, and we now require that the basis is chosen so that e_2 is such a vector. Then $W_3(0, y, z) = Q_3(0, y) = 0$ for all y, z .

Consider the restriction of the quadratic form W_1 to the yz -plane. We are going to show that this restriction is semi-definite, vanishing only on the line $\{x = y = 0\}$ containing p . Suppose to the contrary that W_1 vanishes on some other line in the yz -plane. Then there is a point $q = (0, y_0, z_0) \in S$ such that $q \notin \{p, -p\}$ and $W_1(q) = 0$. Observe that $|z_0| < 1$ since all points of S except p and $-p$ lie in the open strip $\{-1 < z < 1\}$ due to the strict convexity of S .

Since $W_1(q) = 0$ and W_3 vanishes on the yz -plane, $W(q)$ is proportional to $(0, 1, 0)$. Note that $W(q) \neq 0$ since p and $-p$ are the only zeroes of W . Recall that $W(q)$ is tangent to S at q , hence the straight line

$$\{q + tW(q) : t \in \mathbb{R}\} = \{(0, y_0 + t, z_0) : t \in \mathbb{R}\}$$

does not intersect the interior of the body bounded by S . However this line contains the point $(0, 0, z_0)$ which belongs to the interior of the body (since $|z_0| < 1$). This contradiction shows that the restriction of W_1 to the yz -plane is semi-definite. Hence it has a constant sign, either positive or negative, on the set $\{x = 0, y \neq 0\}$. Replacing, if necessary, W with $-W$, we can assume that the sign is positive.

Consider the disc (“half-sphere”) $D = S \cap \{x \geq 0\}$. The positivity of W_1 on $\{x = 0, y \neq 0\}$ implies that W points inwards D everywhere on the boundary of D except the points p and $-p$ where W vanishes. This implies that any forward half-orbit of W starting at a point of D never leaves D . However by Lemma 3.6 there exists a periodic trajectory of W separating p from $-p$. This trajectory enters and leaves D infinitely many times, a contradiction. This proves Lemma 3.7 in the case when W has finitely many zeroes.

Now consider the case when W has infinitely many zeroes on S . Identify Y with $\mathbb{R}^3$ and denote the set of zeroes of W on S by Γ . Then Γ is the intersection of S with zero sets of the three coordinate functions of W . These functions are quadratic forms on $\mathbb{R}^3$, so each of the three zero sets is either all of $\mathbb{R}^3$ or a conical surface representing a quadric in the projective plane. The intersection of plane quadrics is infinite only if it is a straight line or all the quadrics coincide.

Hence Γ is the intersection of S with either a plane, or a union of two planes, or an elliptic cone. In any case $S \setminus \Gamma$ has a component U homeomorphic to the open disc. Pick a non-constant orbit of W contained in U and apply the Poincaré-Bendixson Theorem as in the proof of Lemma 3.6. Since by our assumption the closure of the orbit cannot contain a fixed point, it contains a closed orbit. This closed orbit bounds a disc $D \subset U$ which is invariant under the flow generated by W and hence contains a fixed point of the flow by Brouwer's fixed-point theorem. This fixed point is a zero of W that does not belong to Γ , a contradiction. This finishes the proof of Lemma 3.7. $\square$

4. PROOF OF PROPOSITION 2.4

In this section we prove Proposition 2.4. To facilitate understanding, we begin in §4.1 with a short argument proving the proposition in the smooth case, where it holds for all rather than for almost all $X \in \text{Gr}_n(V)$. Then in §4.2 we explain the choice of X and ν , and finally in §4.3 we construct the desired tensor R . We fix V and B satisfying the assumptions of Proposition 2.4 for the rest of the section. Recall that B contains 0 in its interior.

4.1. Smooth case. First we assume that ∂B is C^∞ -smooth and prove Proposition 2.4 under this assumption.

Fix $X \in \text{Gr}_n(V)$ and $\nu \in V \setminus X$. We decompose V as $X \oplus \mathbb{R}\nu \simeq X \times \mathbb{R}$ and parametrize a neighborhood of X in $\text{Gr}_n(V)$ by elements of X^* as follows: to each $\lambda \in X^*$ we associate its graph $H_\lambda \subset V$. That is,

$$(4.1) \quad H_\lambda = \{x + \lambda(x)\nu \mid x \in X\}.$$

Note that $H_\lambda \in \text{Gr}_n(V)$ for all λ and $H_0 = X$. Denote $X \cap B$ by K .

Since all cross-sections of B are linearly equivalent, for every $\lambda \in X^*$ there exists a linear map $F_\lambda: X \rightarrow V$ such that $F_\lambda(X) = H_\lambda$ and $F_\lambda(K) = H_\lambda \cap B$. The smoothness of Ψ_B implies (see [3, §2.1]) that one can choose the family $\{F_\lambda\}$ to be smooth in a neighborhood of $\lambda = 0$ and such that $F_0 = \text{id}_X$. Since $F_\lambda(X) = H_\lambda$, F_λ can be written in the form

$$(4.2) \quad F_\lambda(x) = G_\lambda(x) + \lambda(G_\lambda(x)) \cdot \nu$$

for $G_\lambda \in \text{Hom}(X, X)$ defined by $G_\lambda = \text{pr}_\nu \circ F_\lambda$ where $\text{pr}_\nu: V \rightarrow X$ is the projection along ν .

Now define R as the differential of the map $\lambda \mapsto G_\lambda$ at $\lambda = 0$. By construction, R is a linear map from X^* to $\text{Hom}(X, X)$. We denote $R(\lambda)$ by R_λ for all $\lambda \in X^*$.

Since $F_\lambda(K)$ is a cross-section of B , we have $F_\lambda(x) \in \partial B$ for all $x \in \partial K$. Therefore the vector $\frac{d}{dt}\big|_{t=0} F_{t\lambda}(x)$ is tangent to B at $F_0(x) = x$. From (4.2) we have

$$\frac{d}{dt}\bigg|_{t=0} F_{t\lambda}(x) = \frac{d}{dt}\bigg|_{t=0} G_{t\lambda}(x) + \lambda(G_0(x))\nu = R_\lambda(x) + \lambda(x)\nu.$$

Thus for every $x \in \partial K$ the vector $R_\lambda(x) + \lambda(x)\nu$ is tangent to B at x . This is the second property of R_λ claimed in Proposition 2.4.

To obtain the remaining property $\text{Trace } R_\lambda = 0$, we have to adjust the choice of ν . In the next subsection we describe a construction for ν based on the derivative of the cross-section area and prove the identity $\text{Trace } R_\lambda = 0$ in Lemma 4.2. Alternatively, in the smooth case one can use the following short argument.

Having constructed R as above, let us replace ν by $\nu' = \nu + w$, where $w \in X$, and adjust R accordingly. Namely for $\lambda \in X^*$ define $R'_\lambda \in \text{Hom}(X, X)$ by

$$R'_\lambda(x) = R_\lambda(x) - \lambda(x)w.$$

Then the linear map $R': X^* \rightarrow \text{Hom}(X, X)$ is defined by $R'(\lambda) = R'_\lambda$ for all $\lambda \in X^*$. By construction,

$$R'_\lambda(x) + \lambda(x)\nu' = R_\lambda(x) + \lambda(x)\nu \quad \text{for all } x \in X,$$

hence (2) in Proposition 2.4 is satisfied for R' and ν' . For the traces we have

$$\text{Trace } R'_\lambda = \text{Trace } R_\lambda - \lambda(w).$$

Since the map $\lambda \mapsto \text{Trace } R_\lambda$ is linear, there exists $w \in X$ such that $\lambda(w) = \text{Trace } R_\lambda$ for all $\lambda \in X^*$. Choosing this w for the above construction and substituting ν' and R' for ν and R finishes the proof of Proposition 2.4 in the smooth case.

4.2. Choice of X and ν . Now we return to the general case where no smoothness is assumed. Pick an auxiliary Euclidean metric on V and consider the cross-section area function

$$(4.3) \quad X \mapsto \text{vol}_n(B \cap X), \quad X \in \text{Gr}_n(V),$$

where vol_n is the n -dimensional Euclidean volume. This function is Lipschitz and therefore, by Rademacher's Theorem, it is differentiable almost everywhere. Our plan is to prove the claims of Proposition 2.4 for any $X \in \text{Gr}_n(V)$ where this function is differentiable. Though vol_n depends on the choice of the auxiliary metric, the differentiability property does not.

In what follows we use an affine-invariant version of the cross-section area. It is a function on the exterior product $\Lambda^n V$ defined in (4.4) below.

Since $\dim V = n + 1$, every n -vector $\sigma \in \Lambda^n V$ can be written in the form $\sigma = v_1 \wedge \cdots \wedge v_n$ for some $v_1, \dots, v_n \in V$. If $\sigma \neq 0$ then $v_1, \dots, v_n$ are linearly independent. In this case we denote by Π_σ the linear span of $v_1, \dots, v_n$, and by vol_σ the Haar measure on the hyperplane Π_σ normalized so that the measure of the parallelotope with edges $v_1, \dots, v_n$ equals 1. One easily sees that Π_σ and vol_σ do not depend on the choice of vectors $v_1, \dots, v_n$ representing σ . Now define

$$(4.4) \quad A(\sigma) = \text{vol}_\sigma(B \cap \Pi_\sigma)^{-1}.$$

The resulting function $A: \Lambda^n V \setminus \{0\} \rightarrow \mathbb{R}_+$ is positively 1-homogeneous and it can be extended to 0 by setting $A(0) = 0$. If V is equipped with a Euclidean metric then $A(\sigma) = \|\sigma\| \text{vol}_n(B \cap \Pi_\sigma)^{-1}$, where $\|\sigma\|$ is the Euclidean norm of σ . Hence A is differentiable at σ if and only if the cross-section area (4.3) is differentiable at Π_σ . One can see that A is essentially the Busemann-Hausdorff area ([7], see also [19, Chapter 7]) of the Minkowski norm associated to B , except for a normalization constant, which we omit for convenience.

Let σ be a differentiability point of A and $X = \Pi_\sigma$. We now construct a vector ν required in Proposition 2.4. Consider the differential $d_\sigma A$ of A at σ . It is a linear function on $\Lambda^n V$, and we regard it as an exterior form on V , that is, $d_\sigma A \in \Lambda^n V^*$. Since $\dim V = n + 1$, there exists a vector $\nu \in V \setminus \{0\}$ such that

$$(4.5) \quad d_\sigma A(\nu \wedge \beta) = 0 \quad \text{for all } \beta \in \Lambda^{n-1} V$$

Indeed, the exterior n -form $d_\sigma A$ can be written as $d_\sigma A = \ell_1 \wedge \cdots \wedge \ell_n$ for some $\ell_1, \dots, \ell_n \in V^*$. In order to satisfy (4.5), it suffices to pick ν from $\bigcap_{i=1}^n \ker \ell_i \setminus \{0\}$. Such a vector exists since $\bigcap_{i=1}^n \ker \ell_i$ is at least one-dimensional. Note that

$$(4.6) \quad d_\sigma A(\sigma) = A(\sigma) \neq 0$$

due to the homogeneity of A . This and (4.5) imply that $\nu \notin X$.

We fix $X \in \text{Gr}_n(V)$ and $\nu \in V \setminus X$ constructed above for the rest of this section, and denote $B \cap X$ by K .

Remark 4.1. The vector ν can be characterized in Euclidean terms as follows: If a Euclidean structure on V is defined in such a way that $\nu \perp X$, then the cross-section area (4.3) has zero derivative at X . We do not prove this fact as we do not use it in this paper.

The choice of ν is essential for (1) in Proposition 2.4. Namely, we have the following lemma.

Lemma 4.2. *Let X , K and ν be as above. Let $\lambda \in X^*$ and $L \in \text{Hom}(X, X)$ be such that for every $x \in \partial K$ the vector $L(x) + \lambda(x)\nu$ is forward tangent to B at x . Then $\text{Trace } L = 0$.*

Proof. Pick a basis $e_1, \dots, e_n$ of X and let $\sigma = e_1 \wedge \dots \wedge e_n$. We equip X with a Euclidean metric such that (e_i) is an orthonormal basis. For $t \in \mathbb{R}$, define $F_t \in \text{Hom}(X, V)$ by

$$F_t(x) = x + t(L(x) + \lambda(x)\nu)$$

and $\sigma_t \in \Lambda^n V$ by

$$(4.7) \quad \sigma_t = (F_t)_*(\sigma) = \bigwedge_{i=1}^n (e_i + tL(e_i) + t\lambda(e_i)\nu).$$

Define $a(t) = A(\sigma_t)^{-1}$ where A is the function from (4.4). The definitions imply that $a(t) = |F_t^{-1}(B)|$ where $|\cdot|$ denotes the Euclidean volume in $X \simeq \mathbb{R}^n$. We are going to show that the right derivative of $a(t)$ at $t = 0$ equals 0.

Let Ψ be the norm on V associated to B . Then $F_t^{-1}(B)$ is the unit ball of the norm $\Psi_t := \Psi \circ F_t$ on X . Observe that for every $x \in \partial K$,

$$(4.8) \quad \frac{d}{dt} \Big|_{t=0+} \Psi_t(x) = \partial_x^+ \Psi \left(\frac{d}{dt} \Big|_{t=0} F_t(x) \right) = \partial_x^+ \Psi(L(x) + \lambda(x)\nu) = 0,$$

where the last equality follows from Lemma 2.2 and the forward tangency assumption in the statement of the lemma. Furthermore, (4.8) holds for all $x \in X$ due to homogeneity. We now express the volume of the unit ball of Ψ_t in terms of its radial function:

$$a(t) = \frac{1}{n} \int_{S(X)} \Psi_t(x)^{-n} d\text{vol}_{n-1}(x)$$

where $S(X)$ is the unit sphere in X . Since $\Psi_t(x)$ is a convex function of (x, t) , the formula has enough regularity to allow differentiation under the integral. This and (4.8) imply that $\frac{d}{dt} \Big|_{t=0+} a(t) = 0$, as claimed.

On the other hand, since $a(t) = A(\sigma_t)^{-1}$ and A is differentiable at $\sigma_0 = \sigma$, we have

$$0 = \frac{d}{dt} \Big|_{t=0+} a(t) = d_\sigma(A^{-1}) \left(\frac{d}{dt} \Big|_{t=0} \sigma_t \right) = -A(\sigma)^{-2} \cdot d_\sigma A \left(\frac{d}{dt} \Big|_{t=0} \sigma_t \right).$$

Thus $d_\sigma A \left(\frac{d}{dt} \Big|_{t=0} \sigma_t \right) = 0$. Define $\tilde{\sigma}_t = \Lambda_{i=1}^n (e_i + tL(e_i))$ and observe that

$$d_\sigma A \left(\frac{d}{dt} \Big|_{t=0} \tilde{\sigma}_t \right) = d_\sigma A \left(\frac{d}{dt} \Big|_{t=0} \sigma_t \right) = 0$$

because the terms involving ν in the definition (4.7) of σ_t vanish under $d_\sigma A$ due to (4.5). Expanding the derivative of a wedge-product yields

$$\frac{d}{dt} \Big|_{t=0} \tilde{\sigma}_t = \text{Trace } L \cdot \sigma$$

Since $d_\sigma A(\sigma) \neq 0$ by (4.6), the last two equations imply that $\text{Trace } L = 0$. $\square$

4.3. Construction of R . Let X and ν be as above and $K = B \cap X$. Recall that $\mathcal{I}(K)$ denotes the group of linear self-equivalences of $K \subset X$ (see (2.1)). Since $\mathcal{I}(K)$ is a compact subgroup of $GL(X)$, there is a Euclidean inner product $\langle \cdot, \cdot \rangle$ that is invariant under $\mathcal{I}(K)$ (see [4, Lemma 2.2] and [11, Lemma 1] for some explicit constructions). We extend this inner product to V in such a way that $\nu \perp X$ and $\langle \nu, \nu \rangle = 1$. Throughout the rest of this section V is regarded as a Euclidean space with this inner product.

The Euclidean structures on X and V induce a Euclidean inner product on $\text{Hom}(X, V)$ in a standard way: for $F, G \in \text{Hom}(X, V)$,

$$(4.9) \quad \langle F, G \rangle = \text{Trace}(F \circ G^*) = \text{Trace}(G \circ F^*)$$

where $F^*, G^* \in \text{Hom}(V, X)$ are the adjoint operators to F and G . We use the Euclidean norm on $\text{Hom}(X, V)$ defined by $\|F\|^2 = \langle F, F \rangle$.

There is a natural right action (by composition) of $\mathcal{I}(K)$ on $\text{Hom}(X, V)$. Since all elements of $\mathcal{I}(K)$ are orthogonal operators on X , this action is isometric: $\langle F \circ L, G \circ L \rangle = \langle F, G \rangle$ for all $F, G \in \text{Hom}(X, V)$ and $L \in \mathcal{I}(K)$.

As in §4.1, we denote by pr_ν the projection from V to X along ν , and by H_λ the hyperplane corresponding to $\lambda \in X^*$ as in (4.1). Since ν is now a unit normal vector to X , we can rewrite the definition of H_λ as follows:

$$(4.10) \quad H_\lambda = \{x \in V : \langle x, \nu \rangle = \lambda(\text{pr}_\nu(x))\}.$$

Note that $H_0 = X$ and the map $\lambda \mapsto H_\lambda$ is a homeomorphism between X^* and the set of hyperplanes from $\text{Gr}_n(V)$ that do not contain ν .

Recall that every cross-section $B \cap H_\lambda$ is linearly equivalent to K . We denote by $\mathcal{I}_\lambda$ the set of all linear equivalences between them, more precisely,

$$\mathcal{I}_\lambda = \{F \in \text{Hom}(X, V) : F(K) = B \cap H_\lambda\} \text{ for all } \lambda \in X^*.$$

Clearly $\mathcal{I}_\lambda$ is an orbit of the aforementioned action of $\mathcal{I}(K)$ and the stabilizer of each $F \in \mathcal{I}_\lambda$ is trivial. In particular, $\mathcal{I}_\lambda$ is a compact smooth submanifold of $\text{Hom}(X, V)$. The set $\mathcal{I}_0$ is the image of $\mathcal{I}(K)$ under the natural embedding $\text{Hom}(X, X) \hookrightarrow \text{Hom}(X, V)$. Denote by i_0 the inclusion $X \hookrightarrow V$, thus i_0 is the element of $\mathcal{I}_0$ corresponding to the identity of $\mathcal{I}(K)$.

Let us fix $\lambda \in H_0^* \setminus \{0\}$. We are going to construct an operator $R_\lambda \in \text{Hom}(X, X)$ satisfying the properties from Proposition 2.4.

Let $\{t_k\}_{k=1}^\infty$ be a sequence of positive reals decreasing to 0. For each k , let $F_k \in \mathcal{I}_{t_k\lambda}$ be a nearest point to i_0 in $\mathcal{I}_{t_k\lambda}$:

$$\|F_k - i_0\| = \inf_{F \in \mathcal{I}_{t_k\lambda}} \|F - i_0\|$$

where $\|\cdot\|$ is the Euclidean norm on $\text{Hom}(X, V)$ defined above. Note that $F_k \neq i_0$ since $\mathcal{I}_{t_k\lambda}$ and $\mathcal{I}_0$ are disjoint.

Lemma 4.3. *The sequence $\{F_k\} \subset \text{Hom}(X, V)$ constructed above satisfies:*

- (1) *For every k one has $F_k - i_0 \in (T_{i_0}\mathcal{I}_0)^\perp$ where $T_{i_0}\mathcal{I}_0$ is the tangent space to $\mathcal{I}_0$ at i_0 regarded as a linear subspace of $\text{Hom}(X, V)$, and the orthogonal complement is taken in $\text{Hom}(X, V)$ with respect to the inner product (4.9).*
- (2) *$F_k \rightarrow i_0$ as $k \rightarrow \infty$.*

Proof. Since the group $\mathcal{I}(K)$ acts on $\text{Hom}(X, V)$ by isometries and both $\mathcal{I}_{t_k\lambda}$ and $\mathcal{I}_0$ are orbits of this action, $\|F_k - i_0\|$ is in fact the minimum distance between these orbits:

$$(4.11) \quad \|F_k - i_0\| = \text{dist}(\mathcal{I}_{t_k\lambda}, \mathcal{I}_0) := \inf\{\|F - G\| : F \in \mathcal{I}_{t_k\lambda}, G \in \mathcal{I}_0\}.$$

In particular, i_0 is nearest to F_k among points of $\mathcal{I}_0$. Hence the derivative at i_0 of the function $G \mapsto \|F_k - G\|^2$ is zero along any vector from $T_{i_0}\mathcal{I}_0$. This derivative along a vector $W \in \text{Hom}(X, V)$ equals $2\langle F_k - i_0, W \rangle$. Thus $F_k - i_0 \perp W$ for all $W \in T_{i_0}\mathcal{I}_0$ and the first claim of the lemma follows.

As $\{F_k\}$ is a bounded sequence, it either converges to i_0 or has a partial limit different from i_0 . Let $F_\infty \in \text{Hom}(X, V)$ be a partial limit of $\{F_k\}$. Passing to a subsequence we may assume that $F_k \rightarrow F_\infty$ as $k \rightarrow \infty$. Since $F_k(K) = B \cap H_{t_k\lambda}$ for all k and the hyperplanes $H_{t_k\lambda}$ converge to $H_0 = X$, we have $F_\infty(K) = K$, therefore $F_\infty \in \mathcal{I}_0$. By (4.11),

$$\|F_k - i_0\| = \text{dist}(\mathcal{I}_{t_k\lambda}, \mathcal{I}_0) \leq \|F_k - F_\infty\| \xrightarrow{m \rightarrow \infty} 0,$$

therefore $F_k \rightarrow i_0$ and hence $F_\infty = i_0$. The second claim of the lemma follows. $\square$

Passing to a subsequence we may assume that there exists a limit

$$(4.12) \quad W_\lambda = \lim_{k \rightarrow \infty} \frac{F_k - i_0}{\|F_k - i_0\|} \in \text{Hom}(X, V).$$

Lemma 4.4. *The operator W_λ defined by (4.12) satisfies:*

- (1) $W_\lambda \neq 0$.
- (2) $W_\lambda \in (T_{i_0}\mathcal{I}_0)^\perp$.
- (3) *For every $x \in \partial K$, the vector $W_\lambda(x)$ is forward tangent to ∂B .*
- (4) *There exists $C > 0$ such that $\langle W_\lambda(x), \nu \rangle = C\lambda(x)$ for all $x \in X$.*

Proof. The first claim follows from the fact that $\|W_\lambda\| = 1$. The second one follows from Lemma 4.3(1). The third one follows from the definition of forward tangency (see Definition 2.1) applied to the sequence $\{F_k(x)\}_{k=1}^\infty \subset \partial B$; this sequence converges to x due to Lemma 4.3(2).

To prove the fourth claim, recall that $F_k(X) = H_{t_k\lambda}$. Hence, by (4.10),

$$\langle F_k(x), \nu \rangle = t_k \lambda(\text{pr}_\nu(F_k(x)))$$

for all $x \in X$. Since $\langle x, \nu \rangle = 0$ and $\text{pr}_\nu(x) = x$, this identity can be rewritten as follows:

$$\langle F_k(x) - x, \nu \rangle = t_k \lambda(\text{pr}_\nu(F_k(x) - x)) + t_k \lambda(x).$$

As $\frac{F_k(x) - x}{\|F_k - i_0\|} \rightarrow W_\lambda(x)$ and $t_k \rightarrow 0$, dividing the last equality by $\|F_k - i_0\|$ and passing to the limit yields that

$$\langle W_\lambda(x), \nu \rangle = \lambda(x) \lim_{k \rightarrow \infty} \frac{t_k}{\|F_k - i_0\|},$$

in particular the limit in the right-hand side exists. Denote this limit by C , and we obtain the desired identity $\langle W_\lambda(x), \nu \rangle = C\lambda(x)$. Clearly $C \geq 0$.

It remains to prove that $C \neq 0$. Suppose to the contrary that $\langle W_\lambda(x), \nu \rangle = 0$ for all $x \in X$. Hence $W_\lambda(X) \subset X$, so we may now regard W_λ as an element of $\text{Hom}(X, X)$ and as a linear vector field on X . By Lemma 3.3, the 1-parameter subgroup $\{e^{tW_\lambda}\}_{t \in \mathbb{R}}$ of $GL(X)$ is contained in $\mathcal{I}(K)$. Therefore W_λ belongs to the tangent space $T_e\mathcal{I}(K)$ of $\mathcal{I}(K)$ regarded as a linear subspace of $\text{Hom}(X, X)$. However by (1) and (2), W_λ is nonzero and orthogonal to this tangent space, a contradiction. Thus $C \neq 0$ and the last claim of Lemma 4.4 follows. $\square$

Now define $R_\lambda \in \text{Hom}(X, X)$ by

$$(4.13) \quad R_\lambda = C^{-1} \text{pr}_\nu \circ W_\lambda$$

where W_λ is defined by (4.12) and C is the constant from Lemma 4.4(4). By Lemma 4.4(4) we have

$$R_\lambda(x) + \lambda(x)\nu = C^{-1}W_\lambda(x)$$

for all $x \in X$, and by Lemma 4.4(3) this vector is forward tangent to ∂B whenever $x \in \partial K$. This tangency and Lemma 4.2 imply that $\text{Trace } R_\lambda = 0$. In addition, Lemma 4.4(2) implies that $R_\lambda \perp T_e\mathcal{I}(K)$ in $\text{Hom}(X, X)$.

To finish the proof of Proposition 2.4, it remains to show that R_λ depends linearly on λ . This is verified in the following proposition.

Proposition 4.5. *For every $\lambda \in X^*$ there exists a unique operator $R_\lambda \in \text{Hom}(X, X)$ such that*

- (1) $\text{Trace } R_\lambda = 0$.
- (2) $R_\lambda(x) + \lambda(x)\nu$ *is forward tangent to ∂B for all $x \in \partial K$.*
- (3) $R_\lambda \perp T_e\mathcal{I}(K)$ *in $\text{Hom}(X, X)$.*

Moreover the map $\lambda \mapsto R_\lambda$ is linear.

Proof. The existence follows from (4.13) and subsequent arguments (for $\lambda = 0$, define $R_\lambda = 0$). Before showing uniqueness, we first prove that $R_\lambda + R_{-\lambda} = 0$ for any pair of operators R_λ and $R_{-\lambda}$ satisfying (1)–(3) with parameters λ and $-\lambda$, respectively.

Consider $L = R_\lambda + R_{-\lambda}$ and let Ψ be the Minkowski norm on V associated to B . As $\partial_x^+ \Psi$ is a subadditive function on X , we have

$$\partial_x^+ \Psi(L(x)) = \partial_x^+ \Psi(R_\lambda(x) + R_{-\lambda}(x)) \leq \partial_x^+ \Psi(R_\lambda(x) + \lambda(x)\nu) + \partial_x^+ \Psi(R_{-\lambda}(x) - \lambda(x)\nu) = 0$$

where the last identity follows from (2) and Lemma 2.2. With this inequality and the identity $\text{Trace } L = 0$ at hand, we argue as in Lemma 3.3. Namely, we regard L as a linear vector field on X and observe that, due to the above relation $\partial_x^+ \Psi(L(x)) \leq 0$, Ψ is non-increasing along trajectories of L . Therefore $e^{tL}(K) \subset K$ for all $t \geq 0$. On the other hand, since $\text{Trace } L = 0$, the operators e^{tL} preserve the volume on X . Since $e^{tL}(K)$ is contained in K and has the same volume, we conclude that in fact $e^{tL}(K) = K$ for all $t \geq 0$. Hence the 1-parameter group $\{e^{tL}\}$ is contained in $\mathcal{I}(K)$, therefore $L \in T_e \mathcal{I}(K)$, and then the orthogonality (3) implies that $L = 0$.

Thus we have shown that $R_\lambda + R_{-\lambda} = 0$ for any choice of R_λ and $R_{-\lambda}$. This implies that the choice of R_λ is, in fact, unique.

The same identity $R_\lambda + R_{-\lambda} = 0$ shows that one can replace “forward tangent” in (2) by “tangent” (see Definition 2.1). Upon this replacement all conditions (1)–(3) become linear in λ , see Lemma 2.3. This and the uniqueness of R_λ imply that R_λ depends on λ linearly. $\square$

Proof of Proposition 2.4. Define $R: X^* \rightarrow \text{Hom}(X, X)$ by $R(\lambda) = R_\lambda$ where R_λ is provided by Proposition 4.5. The requirements of Proposition 2.4 follow from the respective properties of R_λ in Proposition 4.5. Recall that the above constructions work for all $X \in \text{Gr}_n(V)$ where the cross-section area is differentiable, and this condition is satisfied almost everywhere on $\text{Gr}_n(X)$. This finishes the proof of Proposition 2.4. $\square$

Remark 4.6. The arguments in this section do not require *all* cross-sections of B to be linearly equivalent. They work just as well if we only assume this for hyperplanes from an arbitrarily small neighborhood of X in $\text{Gr}_n(V)$. Thus we also have the following local version of Proposition 2.4 (compare with [14]):

Let V be a vector space, $\dim V = n + 1 \geq 3$. Let $B \subset V$ be a convex body with 0 in the interior and $\mathcal{U} \subset \text{Gr}_n(V)$ a nonempty open set such that all cross-sections of B by hyperplanes from $\mathcal{U}$ are linearly equivalent. Then for almost every $X \in \mathcal{U}$ there exist a vector $\nu \in V \setminus X$ and a linear map $R: X^* \rightarrow \text{Hom}(X, X)$ with the same properties as in Proposition 2.4.

5. DEGENERATE CASES OF PROPOSITION 2.5

Let X be a vector space, $\dim X = n \geq 2$, and let $R: X^* \rightarrow \text{Hom}(X, X)$ be a linear map such that $\text{Trace } R(\lambda) = 0$ for all $\lambda \in X^*$ (cf. Proposition 2.5). As in the previous sections, we denote $R(\lambda)$ by R_λ . To avoid repetitions of the formula $\text{Trace } R_\lambda = 0$, we regard R as a map from X^* to the space

$$\text{sl}(X) = \{L \in \text{Hom}(X, X) : \text{Trace } L = 0\}.$$

For $p \in X$ we denote by $p^\perp$ the set $\{\lambda \in X^* : \lambda(p) = 0\}$ of covectors vanishing at p . Consider the family

$$S^R = \{S_p^R\}_{p \in X}$$

of linear maps $S_p^R: p^\perp \rightarrow X$ defined by

$$(5.1) \quad S_p^R(\lambda) = R_\lambda(p), \quad \lambda \in p^\perp.$$

We denote the image of S_p^R by T_p^R , i.e.,

$$(5.2) \quad T_p^R = S^R(p^\perp) = \{R_\lambda(p) \mid \lambda \in p^\perp\}.$$

The assumption (2.2) of Proposition 2.5 can be restated as follows: for every $p \in \partial K$, T_p^R consists of vectors tangent to ∂K at p .

Clearly T_p^R is a linear subspace of X and $T_0^R = \{0\}$. If $p \neq 0$, then $\dim T_p^R \leq \dim p^\perp = n - 1$.

Definition 5.1. For a point $p \in X \setminus \{0\}$, we say that R is *degenerate at p* if $\dim T_p^R < n - 1$. We say R is *degenerate* if there exists $p \in X \setminus \{0\}$ such that R is degenerate at p .

The goal of this section is to prove Proposition 2.5 assuming that R is degenerate, see Proposition 5.10. We begin with a couple of algebraic lemmas that do not depend on the dimension and do not involve K .

Lemma 5.2. *A linear map $R: X^* \rightarrow \mathfrak{sl}(X)$ is uniquely determined by S^R (see (5.1)). In other words, if R and R' are linear maps from X^* to $\mathfrak{sl}(X)$ such that $S^R = S^{R'}$, then $R = R'$.*

Proof. Since S^R is linear in R , it suffices to show that $R = 0$ whenever $S^R = 0$. The relation $S^R = 0$ boils down to the following: $R_\lambda(p) = 0$ for all $\lambda \in X^*$ and $p \in X$ such that $\lambda(p) = 0$. Equivalently, $\ker \lambda \subset \ker R_\lambda$ for all $\lambda \in X^*$. This inclusion implies that for every $\lambda \in X^* \setminus \{0\}$ there is a vector $v_\lambda \in X$ such that $R_\lambda(x) = \lambda(x)v_\lambda$ for all $x \in X$. We are going to show that v_λ is independent of λ .

Let $\lambda, \mu \in X^* \setminus \{0\}$. Since R_λ is linear in λ , we have

$$0 = R_{\lambda+\mu}(x) - R_\lambda(x) - R_\mu(x) = \lambda(x)(v_{\lambda+\mu} - v_\lambda) + \mu(x)(v_{\lambda+\mu} - v_\mu)$$

for all $x \in X$. If λ and μ are linearly independent then there exists $x_0 \in X$ such that $\mu(x_0) = 0$ and $\lambda(x_0) \neq 0$. Substituting $x = x_0$ in the above identity we obtain that $v_{\lambda+\mu} = v_\lambda$. Similarly we have $v_{\lambda+\mu} = v_\mu$, hence $v_\lambda = v_\mu$. We have shown that $v_\lambda = v_\mu$ for any linearly independent $\lambda, \mu \in X^*$, this implies that v_λ is independent of $\lambda \in X^* \setminus \{0\}$.

Thus we have a fixed vector $v \in X$ such that $R_\lambda(x) = \lambda(x)v$ for all $\lambda \in X^* \setminus \{0\}$ and $x \in X$. The trace of the operator $x \mapsto \lambda(x)v$ equals $\lambda(v)$, hence $0 = \text{Trace } R_\lambda = \lambda(v)$ for all $\lambda \in X^* \setminus \{0\}$. This implies that $v = 0$ and thus $R = 0$. $\square$

For $\lambda, \mu \in X^*$ and a linear map $R: X^* \rightarrow \mathfrak{sl}(X)$, define a quadratic homogeneous map $V_{\lambda,\mu}^R: X \rightarrow X$ by

$$(5.3) \quad V_{\lambda,\mu}^R(x) = \mu(x)R_\lambda(x) - \lambda(x)R_\mu(x), \quad x \in X.$$

Note that $V_{\lambda,\mu}^R$ is bi-linear and skew-symmetric in λ and μ . We regard $V_{\lambda,\mu}^R$ as a vector field on X .

Lemma 5.3. *Let $f_1, \dots, f_n$ be a basis of X^* and $V_{ij} = V_{f_i, f_j}^R$ for $i, j \in \{1, \dots, n\}$, see (5.3). Then*

(1) *For every $p \in X$,*

$$T_p^R = \{V_{\lambda,\mu}^R(p) \mid \lambda, \mu \in X^*\} = \text{LinSpan}\{V_{ij}^R(p) \mid 1 \leq i < j \leq n\}$$

where T_p^R is defined in (5.2).

(2) *R is uniquely determined by the collection $\{V_{ij}^R\}_{1 \leq i < j \leq n}$. That is, if R and R' are linear maps from X^* to $\mathfrak{sl}(X)$ such that $V_{ij}^R = V_{ij}^{R'}$ for all $1 \leq i < j \leq n$, then $R = R'$. In particular, if $V_{ij}^R = 0$ for all $1 \leq i < j \leq n$, then $R = 0$.*

Proof. The first claim is trivial for $p = 0$. Let $p \in X \setminus \{0\}$ and $\lambda, \mu \in X^*$. The linearity of R and (5.3) imply that $V_{\lambda,\mu}^R(p) = R_\alpha(p)$ where $\alpha = \mu(p)\lambda - \lambda(p)\mu$. Clearly $\alpha(p) = 0$, thus $\alpha \in p^\perp$ and therefore $R_\alpha(p) = S_p^R(\alpha) \in T_p^R$. This proves the inclusion $\{V_{\lambda,\mu}^R(p)\} \subseteq T_p^R$. To prove the converse, choose $\mu \in X^*$ such that $\mu(p) = 1$ and observe that $R_\lambda(p) = V_{\lambda,\mu}^R(p)$ for all $\lambda \in p^\perp$. This and (5.2) imply that $T_p^R \subseteq \{V_{\lambda,\mu}^R(p)\}$ and the first equality in (1) follows.

The second equality in (1) follows from the fact that $V_{\lambda,\mu}^R$ is bi-linear and skew-symmetric in λ and μ . Indeed, this fact implies that each $V_{\lambda,\mu}^R$ is a linear combination of V_{ij}^R , $1 \leq i < j \leq n$.

To prove (2), observe again that the collection $\{V_{ij}^R\}_{1 \leq i < j \leq n}$ determines the family $\{V_{\lambda,\mu}^R\}_{\lambda,\mu \in X^*}$ by means of linear combinations. For $p \in X \setminus \{0\}$ and $\lambda \in p^\perp$ pick $\mu \in X^*$ such that $\mu(p) = 1$

and conclude again that $S_p^R(\lambda) = R_\lambda(p) = V_{\lambda,\mu}^R(p)$. Thus S^R is determined by the collection $\{V_{ij}^R\}_{1 \leq i < j \leq n}$. By Lemma 5.2, S^R uniquely determines R and the claim follows. $\square$

Now let $K \subset X$ be a convex body with 0 in the interior satisfying the assumptions of Proposition 2.5 with respect to R . We emphasize the following implication of Lemma 5.3:

Lemma 5.4. *Let X , K and R be as in Proposition 2.5 and let $V_{\lambda,\mu}^R$ be defined by (5.3). Then for all $\lambda, \mu \in X^*$, the vector field $V_{\lambda,\mu}^R$ is tangent to ∂K .*

Proof. Let $p \in \partial K$. By Lemma 5.3(1) we have $V_{\lambda,\mu}^R(p) \in T_p^R$. By (2.2) and the definition of T_p^R (see (5.2)), all vectors from T_p^R are tangent to ∂K at p , hence the result. $\square$

From now on, we restrict ourselves to dimension $n = 3$.

5.1. Degeneration everywhere. In this subsection we handle the case when R is degenerate at all points. Our goal is to prove the following.

Proposition 5.5. *Let X , K and R be as in Proposition 2.5 and assume that R is degenerate at all points $p \in X \setminus \{0\}$ in the sense of Definition 5.1. Then the conclusion of Proposition 2.5 holds, namely for every $\lambda \in X^*$ the map R_λ regarded as vector field on X is tangent to ∂K .*

Proof. Since $n = 3$, the degeneracy means that $\dim T_p^R \leq 1$ for all $p \in X$. Fix a basis e_1, e_2, e_3 of X and the corresponding dual basis f_1, f_2, f_3 of X^* . Define quadratic vector fields V_1, V_2, V_3 on X by

$$V_1 = V_{23}^R, \quad V_2 = V_{31}^R, \quad V_3 = V_{12}^R$$

where $V_{ij}^R = V_{f_i, f_j}^R$ as in Lemma 5.3. For $x = \sum x_i e_i$ the definitions expand to

$$\begin{aligned} V_1(x) &= x_3 R_{f_2}(x) - x_2 R_{f_3}(x), \\ V_2(x) &= x_1 R_{f_3}(x) - x_3 R_{f_1}(x), \\ V_3(x) &= x_2 R_{f_1}(x) - x_1 R_{f_2}(x). \end{aligned} \tag{5.4}$$

Observe that

$$x_1 V_1(x) + x_2 V_2(x) + x_3 V_3(x) = 0 \tag{5.5}$$

for all $x = \sum x_i e_i \in X$; this follows immediately from (5.4).

By Lemma 5.4, the vector fields V_1, V_2, V_3 are tangent to ∂K . By Lemma 5.3(1) and the assumption that $\dim T_p^R \leq 1$, the vectors $V_i(p)$, $i = 1, 2, 3$, are collinear for every $p \in X$. The following algebraic lemma characterizes such triples of vector fields.

Lemma 5.6. *Let V_1, V_2, V_3 be quadratic homogeneous vector fields on $X \simeq \mathbb{R}^3$ such that*

$$\dim \text{LinSpan}\{V_1(x), V_2(x), V_3(x)\} \leq 1 \tag{5.6}$$

for all $x \in X$. Then at least one of the following holds:

- (1) *There exist a quadratic vector field Q on X and constants $C_1, C_2, C_3 \in \mathbb{R}$ such that $V_i = C_i \cdot Q$ for $i = 1, 2, 3$.*
- (2) *There exist a linear vector field L on X and linear functions $\ell_1, \ell_2, \ell_3 \in X^*$ such that $V_i = \ell_i \cdot L$ for $i = 1, 2, 3$.*
- (3) *There exist a vector $v \in X$ and quadratic forms $Q_1, Q_2, Q_3: X \rightarrow \mathbb{R}$ such that $V_i = Q_i \cdot v$ for $i = 1, 2, 3$.*

Proof. We identify X with $\mathbb{R}^3$ and regard V_1, V_2, V_3 as $\mathbb{R}^3$ -valued polynomials in variables x_1, x_2, x_3 which represent coordinates in $\mathbb{R}^3$. Consider a 3×3 matrix $\mathbf{V}$ whose columns are composed of coordinate components of V_1, V_2, V_3 . We assume that $\mathbf{V} \neq 0$, otherwise the lemma is trivial.

The entries of $\mathbf{V}$ are quadratic forms in x_1, x_2, x_3 . By (5.6) all the 2×2 minors of $\mathbf{V}$ vanish pointwise, hence they are zero as polynomials. We now consider $\mathbf{V}$ as a matrix over the field $\mathbb{R}(x_1, x_2, x_3)$ of rational functions and conclude that its rank is no greater than 1. This implies that $\mathbf{V}$ can be represented as a matrix product

$$(5.7) \quad \mathbf{V} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \cdot (G_1 \quad G_2 \quad G_3)$$

where $F_1, F_2, F_3, G_1, G_2, G_3$ are some rational functions. Since the ring $\mathbb{R}[x_1, x_2, x_3]$ is a unique factorization domain and $\mathbb{R}(x_1, x_2, x_3)$ is its field of fractions, all F_i and G_j can in fact be chosen to be polynomials rather than general rational functions.

Indeed, represent all F_i and G_j as irreducible fractions of polynomials and assume that some of them, say F_1 , has a nontrivial denominator. Let q be a prime factor of this denominator. Since $F_1 \cdot G^j$ is a polynomial for every j , the polynomial q should divide numerators of all G^j . Now replace each F_i by $F_i q$ and each G_j by G_j/q . This operation preserves the identity (5.7) and reduces the total number of prime factors in the denominators. It can be applied to any of F_i and G_j in place of F_1 and repeated until all nontrivial denominators disappear. Thus we may assume that F_i and G_j in (5.7) are polynomials.

Since $\mathbf{V} \neq 0$, at least one of the products $F_i G_j$ is nonzero. Assume without loss of generality that $F_1 G_1 \neq 0$. Since $F_1 G_1$ is a homogeneous polynomial of degree 2, the polynomials F_1 and G_1 are themselves homogeneous and $\deg F_1 + \deg G_1 = 2$. Let $d = \deg F_1$, then $\deg G_1 = 2 - d$. For every nonzero F_i , the product $F_i G_1$ is a nonzero homogeneous polynomial of degree 2, hence F_i is homogeneous of degree $2 - (2 - d) = d$. If any of F_i is zero, then it is a homogeneous polynomial of any degree, in particular of degree d . Similarly, every G_j is a homogeneous polynomial of degree $2 - d$. Depending on the value of $d \in \{0, 1, 2\}$, the identity (5.7) now translates to one of the possibilities (1), (2) and (3) listed in the lemma. $\square$

Now we continue the proof of Proposition 5.5. If $V_1 = V_2 = V_3 = 0$ then $R = 0$ by Lemma 5.3(2) and the proposition follows trivially. Assume that at least one of V_1, V_2, V_3 is nonzero and consider the three cases from Lemma 5.6.

Case 1: $V_i = C_i Q$ where $C_i \in \mathbb{R}$ and Q is a quadratic vector field on X . In this case the relation (5.5) takes the form

$$(C_1 x_1 + C_2 x_2 + C_3 x_3) \cdot Q(x) = 0$$

for all $x = \sum x_i e_i \in X$. This identity implies that $Q = 0$ or $C_1 = C_2 = C_3 = 0$. In both cases it follows that $V_1 = V_2 = V_3 = 0$, contrary to our assumption.

Case 2: $V_i = \ell_i L$ where $\ell_i \in X^*$ and L is a linear vector field. Assume without loss of generality that $V_1 \neq 0$. Then $\ell_1 \neq 0$ and $L \neq 0$. Recall that V_1 is tangent to ∂K by Lemma 5.4. This and Lemma 3.2 imply that L is tangent to ∂K . Then $\text{Trace } L = 0$ by Lemma 3.3.

Write ℓ_i as $\ell_i(x) = \sum_j C_{ij} x_j$ and substitute into (5.5):

$$(5.8) \quad \left(\sum_{1 \leq i, j \leq 3} C_{ij} x_i x_j \right) \cdot L(x) = 0$$

for all $x = \sum x_i e_i \in X$. Since $L \neq 0$, the first factor in (5.8) vanishes for all x . Therefore $C_{ii} = 0$ and $C_{ij} + C_{ji} = 0$ for $i \neq j$.

We now claim that, for every $\lambda = \sum \lambda_i f_i \in X^*$,

$$(5.9) \quad R_\lambda = C_\lambda L \quad \text{where} \quad C_\lambda = C_{32} \lambda_1 + C_{13} \lambda_2 + C_{21} \lambda_3.$$

To prove this, define $R'(\lambda) = C_\lambda L$ and observe that $\text{Trace } R'(\lambda) = C_\lambda \text{Trace } L = 0$, so R' is a linear map from X^* to $\mathfrak{sl}(X)$. Further, for $V_{ij}^{R'}$ defined in Lemma 5.3 and $x = \sum x_i e_i$ we have

$$V_{12}^{R'}(x) = x_2 R'(f_1)(x) - x_1 R'(f_2)(x) = (C_{32}x_2 - C_{13}x_1)L(x) = \ell_3(x)L(x) = V_3(x) = V_{12}^R(x).$$

Here we used the identities $C_{13} = -C_{31}$ and $C_{33} = 0$. Similarly (via cyclic permutation of indices), $V_{23}^{R'} = V_{23}^R$ and $V_{31}^{R'} = V_{31}^R$. By Lemma 5.3 these identities imply that $R' = R$ and (5.9) follows.

As L is tangent to ∂K , (5.9) implies that R_λ is tangent to ∂K as well. Thus Proposition 5.5 holds in this case.

Case 3: $V_i = Q_i v$ where Q_i is a quadratic form and v is a fixed vector. Assume without loss of generality that $V_1 \neq 0$. Then $Q_1 \neq 0$ and $v \neq 0$. Since V_1 is tangent to ∂K , Lemma 3.2 applied to the constant vector field v and a polynomial factor Q_1 implies that the Minkowski norm associated to K is constant along the ray $t \mapsto tv$ ($t > 0$), a contradiction. This completes the proof of Proposition 5.5. $\square$

5.2. Degeneration at one point. In this subsection we consider the case when R is degenerate at some point but not at all points. Our goal is to prove the following.

Proposition 5.7. *Let X , K and R be as in Proposition 2.5. Assume that there exist $p_0, p_1 \in X \setminus \{0\}$ such that R is degenerate at p_0 and non-degenerate at p_1 (see Definition 5.1). Then K is an ellipsoid.*

Proof. Recall that (since $n = 3$) the degeneracy of R at p means that $\dim T_p^R \leq 1$. Consider the degenerate set

$$\Sigma = \{p \in X : \dim T_p^R \leq 1\}.$$

Clearly Σ is a cone: if $p \in \Sigma$ and $t \in \mathbb{R}$ then $tp \in \Sigma$ as well.

Lemma 5.8. *$X \setminus \Sigma$ is an open dense set in X .*

Proof. Similarly to Lemma 5.6, introduce coordinates in X and consider a 3×3 matrix $\mathbf{V}$ whose entries are polynomials representing vector fields V_{ij}^R from Lemma 5.3. A point $x \in X$ belongs to Σ if and only if all 2×2 minors of $\mathbf{V}$ vanish at x . Thus Σ is a null set of a system of polynomial equations. Since we are assuming that $\Sigma \neq X$, it follows that $X \setminus \Sigma$ is a dense open set. $\square$

Lemma 5.9. *Let $Z \in \text{Gr}_2(X)$ be such that $Z \setminus \{0\}$ intersects both Σ and $X \setminus \Sigma$. Then $K \cap Z$ is an ellipse.*

Proof. Choose a basis $e_1, e_2, e_3 \in X$ such that $e_1, e_2 \in Z$, $e_1 \in \Sigma$ and $e_2 \notin \Sigma$. Let $f_1, f_2, f_3 \in X^*$ be the corresponding dual basis. Define a linear map

$$F: Z \rightarrow X, \quad F(x) = R_{f_3}(x) \text{ for } x \in Z,$$

and a quadratic homogeneous map

$$G: Z \rightarrow X, \quad G(x) = R_{x_2 f_1 - x_1 f_2}(x) = x_2 R_{f_1}(x) - x_1 R_{f_2}(x) \text{ for } x = x_1 e_1 + x_2 e_2 \in Z.$$

Then construct a linear combination $W(x)$ of $F(x)$ and $G(x)$ killing the third coordinate:

$$W(x) = f_3(G(x)) \cdot F(x) - f_3(F(x)) \cdot G(x).$$

Since $W(x) \in Z$ for all $x \in Z$, we may regard W as a vector field on Z . Note that W is homogeneous polynomial of degree 3.

Observe that for every $x = x_1 e_1 + x_2 e_2 \in Z$ the covectors f_3 and $x_2 f_1 - x_1 f_2$ from the definitions of $F(x)$ and $G(x)$ form a basis of $x^\perp \subset X^*$. Therefore $\text{LinSpan}\{F(x), G(x)\} = T_x^R$ (see (5.2)). In particular both $F(x)$ and $G(x)$ are tangent to ∂K if $x \in \partial K \cap Z$. This and Lemma 2.3 imply that $W(x)$ is tangent to ∂K and hence to $\partial(K \cap Z)$.

Since $e_1 \in \Sigma$, we have $\dim T_{e_1}^R \leq 1$, therefore $F(e_1)$ and $G(e_1)$ are collinear, hence $W(e_1) = 0$. Since $e_2 \notin \Sigma$, we have $\dim T_{e_2}^R = 2$, so $F(e_2)$ and $G(e_2)$ are linearly independent. This implies that $W(e_2) \neq 0$. Indeed, if $W(e_2) = 0$ then $F(e_2), G(e_2) \in \ker f_3 = Z$, hence $F(e_2)$ and $G(e_2)$ are tangent to $\partial(K \cap Z)$ and therefore linearly dependent, a contradiction. Thus W is a nonzero cubic vector field on $Z \simeq \mathbb{R}^2$, W is tangent to $\partial(K \cap Z)$ and it vanishes at e_1 . By Lemma 3.4, these properties imply that $K \cap Z$ is an ellipse. $\square$

Fix $p_0 \in \partial K \cap \Sigma$, such a point exists since Σ is a cone and $\Sigma \neq \{0\}$. Consider the set (homeomorphic to the circle) of all planes from $\text{Gr}_2(X)$ that contain p_0 . Lemma 5.8 imply that this set contains a dense subset of planes intersecting $X \setminus \Sigma$. By Lemma 5.9, the intersection of every such a plane with K is an ellipse. By continuity of intersection it follows that *all* cross-sections of K by planes containing 0 and p_0 are ellipses.

Consider the tangent lines to these elliptic cross-sections at p_0 . These lines are tangent to ∂K at p_0 , therefore (cf. Lemma 2.3) they form a tangent plane to ∂K at p_0 . Let H be the plane through 0 parallel to this tangent plane. Choose a basis e_1, e_2, e_3 in X such that $e_3 = p_0$ and $e_1, e_2 \in H$, and let $f_1, f_2, f_3 \in X^*$ be the dual basis. We equip X with the Euclidean structure such that e_1, e_2, e_3 is an orthonormal basis. We call vectors from H *horizontal* and vectors collinear to p_0 *vertical*. A plane $Z \in \text{Gr}_2(X)$ is called *vertical* if it contains p_0 .

For every vertical plane Z the cross-section $\partial K \cap Z$ is an ellipse containing p_0 and having a horizontal tangent line at p_0 . Such an ellipse has axes on the vertical and horizontal lines in Z and it is uniquely determined by its horizontal axis $Z \cap H \cap K$. Thus K is uniquely determined by the point p_0 and the horizontal cross-section $K \cap H$. We are going to show that $K \cap H$ is an ellipse, this will imply that K is an ellipsoid.

Observe that for every vertical plane Z , the ellipse $\partial K \cap Z$ has a vertical tangent line at the point of intersection with H . Thus the vertical vector $e_3 = p_0$ is tangent to ∂K at every point from $\partial K \cap H$. Consider a linear vector field L on H defined by

$$L(x) = \text{pr}_H(R_{f_3}(x)), \quad x \in H$$

where pr_H is the coordinate projection to H . For every $x \in \partial K \cap H$, both $R_{f_3}(x)$ and e_3 are tangent to ∂K at x , hence $L(x)$ is also tangent to ∂K at x by Lemma 2.3. Thus L is a linear vector field on H tangent to $\partial(K \cap H)$. If $L \neq 0$ then Lemma 3.4 applied to $K \cap H$ implies that $K \cap H$ is an ellipse.

If $L = 0$, consider the third coordinate $f_3(R_{f_3}(x))$ as a function of $x \in H$. Since this function is linear, it vanishes at some point $q \in \partial K \cap H$. For this point we have $L(q) = \text{pr}_H(R_{f_3}(q)) = 0$ and $f_3(R_{f_3}(q)) = 0$, hence $R_{f_3}(q) = 0$. Since $f_3 \in q^\perp$, this implies that R is degenerate at q . Now we can repeat the above arguments for q in place of p_0 and conclude that all cross-sections of K by planes containing 0 and q are ellipses. In particular this applies to the plane H .

Thus we have shown that $K \cap H$ is an ellipse. Since K is uniquely determined by $K \cap H$ and p_0 as explained above, K coincides with the ellipsoid having a vertical axis and a given elliptic horizontal cross-section. This finishes the proof of Proposition 5.7. $\square$

Now we combine Proposition 5.5 and Proposition 5.7 to obtain the main result of this section:

Proposition 5.10. *Proposition 2.5 holds true if R is degenerate in the sense of Definition 5.1.*

Proof. If R is degenerate at all points then the result follows from Proposition 5.5, otherwise it follows from Proposition 5.7 and Lemma 2.6. $\square$

6. NON-DEGENERATE CASE

In this section we prove Proposition 2.5 in the case when R is non-degenerate in the sense of Definition 5.1, that is,

$$(6.1) \quad \dim T_p^R = n - 1 = 2 \quad \text{for all } p \in X \setminus \{0\},$$

where T_p^R is defined by (5.2). We fix X , K and R from Proposition 2.5 and assume (6.1) for the rest of this section.

Let Ψ be the Minkowski norm on X associated to K . By Lemma 2.2 and the homogeneity of Ψ and R , the assumption (2.2) of Proposition 2.5 implies that

$$(6.2) \quad \partial_x^+ \Psi(\pm R_\lambda(x)) = 0 \quad \text{for all } \lambda \in X^* \text{ and } x \in \ker \lambda.$$

Lemma 6.1. *K is symmetric with respect to 0 and ∂K is a real analytic surface.*

Proof. Recall that for every $p \in \partial K$ all vectors from T_p^R are tangent to ∂K at p (cf. (2.2) and (5.2)). Since $\dim T_p^R = 2$, this means that T_p^R is the tangent plane of ∂K at p . Thus ∂K is an integral surface of the two-dimensional distribution $\{T_p^R\}_{p \in X \setminus \{0\}}$ on $X \setminus \{0\}$. Since this distribution is generated by algebraic vector fields (see Lemma 5.3), it is real analytic, hence so is ∂K .

To prove the symmetry of K , consider the intersection of ∂K with an arbitrary straight line passing through 0. It is a pair of points $p_1, p_2 \in \partial K$ such that $p_2 = cp_1$ where $c < 0$. Define a function $F: X \rightarrow \mathbb{R}$ by $F(p) = \Psi(cp)$ for all $p \in X$. For every $p \in \partial K$, the differential $d_p F = c \cdot d_{cp} \Psi$ vanishes on T_p^R . Indeed, consider $p' = \Psi(cp)^{-1} cp \in \partial K$ and observe that $T_{p'}^R = T_p^R$ since the distribution $\{T_p^R\}$ is generated by homogeneous vector fields. Since $T_{p'}^R$ is tangent to ∂K at p' , it follows that $d_{p'} \Psi$ vanishes on $T_{p'}^R = T_p^R$. Hence, by the homogeneity of Ψ , $d_{cp} \Psi$ and hence $d_p F$ vanish on T_p^R .

Since ∂K is connected and $d_p F$ vanishes on its tangent plane T_p^R for every $p \in \partial K$, the restriction $F|_{\partial K}$ is constant. Substituting p_1 one sees that this constant is $F(p_1) = \Psi(p_2) = 1$. On the other hand, it equals $F(p_2) = \Psi(c^2 p_1) = c^2$. Therefore $c^2 = 1$, hence $c = -1$ and $p_2 = -p_1$. Thus for every $p_1 \in \partial K$ we have $-p_1 \in \partial K$, therefore K is symmetric. $\square$

We reuse some notation from §4.2 but here the constructions are applied to X in place of V . For linearly independent vectors $u, v \in X$ define a plane $\Pi_{u \wedge v} \in \text{Gr}_2(X)$ by

$$\Pi_{u \wedge v} = \text{LinSpan}(u, v).$$

As in (4.4), define a positively 1-homogeneous area function $A: \Lambda^2 X \rightarrow \mathbb{R}_+$ for K by $A(0) = 0$ and

$$(6.3) \quad A(u \wedge v) = \text{vol}_{u \wedge v}(K \cap \Pi_{u \wedge v})^{-1} \text{ for linearly independent } u, v \in X$$

where $\text{vol}_{u \wedge v}$ is the Haar measure on $\Pi_{u \wedge v}$ normalized in such a way that the parallelogram with edges u, v has measure 1.

Let $IK \subset \Lambda^2 X$ be the intersection body of K , that is,

$$(6.4) \quad IK = \{\sigma \in \Lambda^2 X \mid A(\sigma) \leq 1\}.$$

We place the intersection body in $\Lambda^2 X$ rather than in X or X^* (cf. [8, 19]) to ensure its independence of additional structures (such as an inner product) on X . By a classical result of Busemann [8], the symmetry of K (see Lemma 6.1) implies that IK is convex. Note that A from (6.3) is the Minkowski norm associated to IK .

The second claim of Lemma 6.1 implies that ∂IK is a real analytic surface in $\Lambda^2 X$. Note that the analyticity implies that IK is strictly convex in the sense that its boundary does not contain straight line segments. Indeed, if ∂IK contains a segment of a straight line then by analyticity it contains the entire line, contrary to the fact that it is the boundary of a compact set. See also [8, IIa] for general comments about strict convexity of intersection bodies.

The next lemma is the key construction of the proof.

Lemma 6.2. *There exist a quadratic homogeneous vector field W on $\Lambda^2 X$ such that*

- (1) *W is tangent to ∂IK .*
- (2) *For every trajectory γ of W in ∂IK , all cross-sections $K \cap \Pi_{\gamma(t)}$, $t \in \mathbb{R}$, are linearly equivalent.*
- (3) *If $\sigma \in \partial IK$ is such that $W(\sigma) = 0$ then $K \cap \Pi_\sigma$ is an ellipse.*

Proof. Fix a linear isomorphism

$$\varphi: \Lambda^2 X \rightarrow X^*$$

such that $u, v \in \ker \varphi(u \wedge v)$ for all $u, v \in X$. Such φ can be constructed, for instance, by choosing a volume form ω on X and setting $\varphi(u \wedge v) = \omega(u, v, \cdot)$.

We define the desired vector field $W: \Lambda^2 X \rightarrow \Lambda^2 X$ by

$$(6.5) \quad W(\sigma) = R_{\varphi(\sigma)}(u) \wedge v + u \wedge R_{\varphi(\sigma)}(v) \quad \text{if } \sigma = u \wedge v \text{ where } u, v \in X.$$

The definition does not depend on the choice of u and v representing σ . Indeed, if $\sigma \neq 0$, any other pair of vectors representing σ can be obtained from the original one by a series of elementary transformations of the form $(u, v) \mapsto (v, -u)$, $(u, v) \mapsto (cu, c^{-1}v)$ and $(u, v) \mapsto (u, v + cu)$ where $c \in \mathbb{R} \setminus \{0\}$. Applying any of these transformations to u and v in (6.5) and using the linearity of $R_{\varphi(\sigma)}$ and the bi-linearity and skew-symmetry of $\wedge$, one sees that the result does not change.

First we check that W is indeed a quadratic vector field. Choose a basis e_1, e_2, e_3 in X and introduce coordinates on $\Lambda^2 X$ by means of the basis $e_2 \wedge e_3$, $e_3 \wedge e_1$, $e_1 \wedge e_2$. Consider $\sigma \in \Lambda^2 X$ and let x, y, z be the coordinates of σ :

$$\sigma = xe_2 \wedge e_3 + ye_3 \wedge e_1 + ze_1 \wedge e_2.$$

Assuming $z \neq 0$, we have $\sigma = \frac{1}{z}u \wedge v$ where $u = ze_1 - xe_3$ and $v = ze_2 - ye_3$. Then, by (6.5),

$$W(\sigma) = \frac{1}{z} R_{\varphi(\sigma)}(u) \wedge v + \frac{1}{z} u \wedge R_{\varphi(\sigma)}(v)$$

or, equivalently

$$(6.6) \quad z \cdot W(\sigma) = R_{\varphi(\sigma)}(u) \wedge v + u \wedge R_{\varphi(\sigma)}(v).$$

Substituting the coordinate expressions for σ , u and v we regard both sides of (6.6) as functions of real variables x, y, z . The identity (6.6) holds for all x, y, z such that $z \neq 0$ and hence, by continuity, for all $x, y, z \in \mathbb{R}$. Since σ , u and v are linear functions of (x, y, z) , the right-hand side of (6.6) is a homogeneous polynomial of degree 3 (with values in $\Lambda^2 X$). The left-hand side shows that this polynomial vanishes at $z = 0$, therefore it is divisible by z as a polynomial. Thus W is a quadratic vector field on $\Lambda^2 X$.

Now we verify the properties (1)–(3). Fix $\sigma \in \Lambda^2 X \setminus \{0\}$ and choose $u_0, v_0 \in X$ such that $\sigma = u_0 \wedge v_0$. Let $u = u(t)$ and $v = v(t)$ be the solutions of the o.d.e. system

$$(6.7) \quad \begin{cases} \dot{u} = R_{\varphi(u \wedge v)}(u) \\ \dot{v} = R_{\varphi(u \wedge v)}(v) \end{cases}$$

with initial values $u(0) = u_0$ and $v(0) = v_0$, and let $J \ni 0$ be the largest interval where the solution is defined. (We will show later that $J = \mathbb{R}$). Let $\gamma(t) = u(t) \wedge v(t)$, then

$$\dot{\gamma}(t) = \dot{u}(t) \wedge v(t) + u(t) \wedge \dot{v}(t) = W(\gamma(t))$$

by (6.7) and (6.5). Thus γ is a trajectory of W with $\gamma(0) = \sigma$, in particular $\gamma(t) \neq 0$ for all $t \in J$.

For each $t \in J$, let $L_t: \mathbb{R}^2 \rightarrow X$ be the linear map such that $L_t(e_1) = u(t)$ and $L_t(e_2) = v(t)$ where e_1 and e_2 is the standard basis of $\mathbb{R}^2$. Fix $w = c_1 e_1 + c_2 e_2 \in \mathbb{R}^2$ and consider a curve $\alpha = \alpha(t)$ in X defined by

$$\alpha(t) = L_t(w) = c_1 u(t) + c_2 v(t).$$

By (6.7) and the linearity of $R_{\gamma(t)}$, the velocity of α is given by

$$(6.8) \quad \dot{\alpha}(t) = R_{\varphi(\gamma(t))}(\alpha(t)).$$

Recall that $\varphi(\gamma(t)) = \varphi(u(t) \wedge v(t))$ is a co-vector whose kernel contains $u(t)$ and $v(t)$, hence it also contains their linear combination $\alpha(t)$. This fact, (6.8) and (6.2) imply that $\frac{d}{dt} \Psi(\alpha(t)) = 0$, hence the function $t \mapsto \Psi(\alpha(t))$ is constant. Since $\alpha(t) = L_t(w)$ where w is an arbitrary vector from $\mathbb{R}^2$, we conclude that $\Psi \circ L_t$ is independent of t :

$$(6.9) \quad \Psi \circ L_t = \Psi \circ L_0 \quad \text{for all } t \in J.$$

Note that (6.9) implies that $u(t)$ and $v(t)$ are confined in a compact subset of X . Therefore $u(t)$ and $v(t)$ are defined by (6.7) for all $t \in \mathbb{R}$, thus $J = \mathbb{R}$.

The relation (6.9) implies that the pre-image $L_t^{-1}(K) \subset \mathbb{R}^2$ is independent of t . Observe that

$$\Pi_{\gamma(t)} = \text{LinSpan}\{u(t), v(t)\} = L_t(\mathbb{R}^2),$$

hence $K \cap \Pi_{\gamma(t)}$ is linearly equivalent to $L_t^{-1}(K) = L_0^{-1}(K)$ and (2) follows.

To prove (1), observe that for A defined by (6.3) we have $A(\gamma(t)) = |L_t^{-1}(K)|^{-1}$ where $|\cdot|$ is the standard Euclidean area in $\mathbb{R}^2$. Since $L_t^{-1}(K)$ is independent of t , so is $A(\gamma(t))$. Since γ is an arbitrary trajectory of W , it follows that W is tangent to the level sets of A and in particular to ∂IK .

It remains to prove (3). Assume that $W(\sigma) = 0$, then $\gamma(t) = \sigma$ and hence $\Pi_{\gamma(t)} = \Pi_\sigma$ for all t . Now we may regard L_t as a (bijective) linear map from $\mathbb{R}^2$ to the fixed plane Π_σ , and then (6.9) implies that $\{L_t \circ L_0^{-1}\}_{t \in \mathbb{R}}$ is a one-parameter family of linear self-equivalences of $K \cap \Pi_\sigma$. If this family is not constant then the group $\mathcal{I}(K \cap \Pi_\sigma)$ of self-equivalences is not discrete and hence $K \cap \Pi_\sigma$ is an ellipse, as claimed. Otherwise $\{L_t\}$ and hence $u(t)$ are constants. This and (6.7) imply that $R_{\varphi(\sigma)}(u_0) = 0$, so the map $\lambda \mapsto R_\lambda(u_0)$ is not injective on $u_0^\perp$. This contradicts our standing assumption (6.1) for $p = u_0$. This proves (3) and finishes the proof of Lemma 6.2. $\square$

Now we finish the proof of Proposition 2.5.

Proof of Proposition 2.5. Let X, K, R be as in Proposition 2.5. The case when R is degenerate is covered by Proposition 5.10, so we assume the non-degeneracy condition (6.1). Due to Lemma 2.6 it suffices to show that K is an ellipsoid.

By Lemma 6.1, K is symmetric and ∂K is a real analytic surface. Let $IK \subset \Lambda^2 X$ be the intersection body of K , see (6.4), W a quadratic vector field on $\Lambda^2 X$ constructed in Lemma 6.2, and $S = \partial IK$. Define

$$\mathcal{E} = \{\sigma \in S : K \cap \Pi_\sigma \text{ is an ellipse}\}$$

(recall that Π_σ is the plane associated to a bivector σ). Clearly $\mathcal{E}$ is a closed set. Lemma 6.2(3) implies that $\mathcal{E}$ contains all zeroes of W on S . If $W = 0$ then $\mathcal{E} = S$, which means that all central cross-sections of K are ellipses and hence K is an ellipsoid. We assume that $W \neq 0$ and proceed in several steps.

Step 1: We show that $\mathcal{E}$ contains a non-constant orbit $\{\gamma(t)\}$ of W . As explained before Lemma 6.2, S is a smooth strictly convex surface. Therefore we can apply Lemma 3.7 to S and obtain a non-constant orbit $\{\gamma(t)\} \subset S$ of W whose closure contains a zero of W . Lemma 6.2(2) implies that all cross-sections $K \cap \Pi_{\gamma(t)}$ are linearly equivalent. By continuity they are also linearly equivalent to $K \cap \Pi_\sigma$, which is an ellipse by Lemma 6.2(3). Hence $\gamma(t) \in \mathcal{E}$ for all $t \in \mathbb{R}$.

Step 2: We show that $\mathcal{E}$ contains three linearly independent elements. Suppose to the contrary that $\mathcal{E}$ is contained in some plane $H \in \text{Gr}_2(\Lambda^2 X)$ and hence in the topological circle $S \cap H$. In particular the orbit from Step 1 is contained in H , therefore $W(\gamma(t)) \in H$ for all t . Since W is a quadratic homogeneous polynomial, it follows that $W(q) \in H$ for all $q \in H$. Hence every trajectory of W starting at a point on the circle $S \cap H$ stays on this circle forever and therefore converges to a fixed point of W . As explained above, this implies that the entire trajectory is contained in $\mathcal{E}$. Thus all points of $S \cap H$ belong to $\mathcal{E}$, so in fact $\mathcal{E} = S \cap H$.

Pick $q \in S \setminus H$, consider the orbit of W starting at q , and apply the Poincaré-Bendixson Theorem similarly to the proof of Lemma 3.6. If the closure of the orbit contains a fixed point then we have $q \in \mathcal{E}$ similarly to Step 1. Otherwise the closure contains a closed orbit of W , this closed orbit bounds a disc $D \subset S \setminus H$, which contains a fixed point by Brouwer's fixed-point theorem, and this fixed point belongs to $\mathcal{E}$ by Lemma 6.2(3). In both cases we have found an element of $\mathcal{E}$ outside H . This contradicts our assumptions and thus proves the claim of Step 2.

Step 3: We show that there exists a quadratic form Q on K such that for every $\sigma \in \mathcal{E}$ the cross-section $K \cap \Pi_\sigma$ is compatible with Q in the sense that $Q = 1$ on $\partial K \cap \Pi_\sigma$.

Fix linearly independent $\eta_1, \eta_2, \eta_3 \in \mathcal{E}$ provided by Step 2. The linear independence implies that $\Pi_{\eta_1} \cap \Pi_{\eta_2} \cap \Pi_{\eta_3} = \{0\}$. There exists a quadratic form Q on X compatible with the three cross-sections $K \cap \Pi_{\eta_i}$ in the above sense. Indeed, let $e_1, e_2, e_3 \in \partial K$ be a basis of X such that the planes Π_{η_i} , $i = 1, 2, 3$, are the coordinate planes with respect to this basis: $e_i \in \Pi_{\eta_{i+1}} \cap \Pi_{\eta_{i+2}}$ where the indices are taken modulo 3. The matrix of the desired quadratic form Q in this basis can be obtained as follows: the diagonal entries are all equal to 1 and then each off-diagonal entry is uniquely determined by the compatibility condition on the respective coordinate plane.

Let $\sigma \in \mathcal{E}$ and let Q_σ be the quadratic form on the plane Π_σ corresponding to the ellipse $K \cap \Pi_\sigma$. Our goal is to show that $Q_\sigma = Q|_{\Pi_\sigma}$. In the case when Π_σ does not contain any basis vector, this follows from the fact that a quadratic form on the plane is uniquely determined by its values on the three lines $\Pi_\sigma \cap \Pi_{\eta_i}$. If Π_σ contains a basis vector, say $e_1 \in \Pi_\sigma$, then Q_σ is uniquely determined by its values on the two lines, (e_1) and $\Pi_\sigma \cap \Pi_{\eta_1}$, and the derivative of Q_σ at e_1 . The derivative of Q_σ at e_1 is uniquely determined from the fact that the tangent line of the ellipse $\partial K \cap \Pi_\sigma$ at e_1 belongs to the plane spanned by the tangent lines to $\partial K \cap \Pi_{\eta_2}$ and $\partial K \cap \Pi_{\eta_3}$ at e_1 . In both cases $Q_\sigma = Q|_{\Pi_\sigma}$ and therefore $K \cap \Pi_\sigma$ is compatible with Q .

Step 4: We show that $\Psi^2 = Q$ where Q is the quadratic form from Step 3 (recall that Ψ is the Minkowski norm associated to K). By Step 1 the set $\mathcal{E}$ contains a non-constant smooth curve $\{\gamma(t)\}$, and by Step 3 the equality $\Psi^2 = Q$ holds on each of the planes $\Pi_{\gamma(t)}$. The union of these planes has a nonempty interior, thus $\Psi^2 = Q$ on a nonempty open subset of X . Since ∂K is a real analytic surface, Ψ is real analytic on $X \setminus \{0\}$ and we conclude that $\Psi^2 = Q$ everywhere.

The identity $\Psi^2 = Q$ implies that Q is positive definite and K is an ellipsoid. An application of Lemma 2.6 finishes the proof of Proposition 2.5. $\square$

As shown in §2.3, Theorem 1.3 follows from Proposition 2.4 and Proposition 2.5.

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