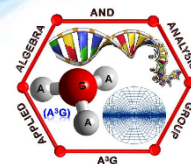




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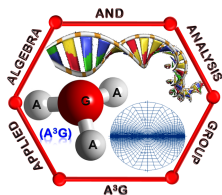
PROCEEDINGS OF THE FOURTH BIENNIAL INTERNATIONAL GROUP THEORY CONFERENCE 2017 (4BIGTC2017)



EXTENDED ABSTRACTS

UTM KUALA LUMPUR, MALAYSIA | 23-26 JANUARY 2017

**Organized by Applied Algebra and Analysis Research Group (AAAG)
in collaboration with Faculty of Science & UTM International**



Proceedings of

The Fourth

Biennial International Group

Theory Conference 2017

(4BIGTC2017)

23-26 January 2017

Extended Abstracts

Universiti Teknologi Malaysia

Malaysia

Preface

The Biennial International Group Theory Conference (BIGTC) is a series of conference that is organized every two years, and has been hosted by three countries - Malaysia, Turkey and Iran. The aim of this conference is to exchange the research ideas amongst the group theorists and postgraduate students. In addition, specialists in the subject of group theory can present their latest research works and encourage interested students to progress and widen their knowledge in group theory. Certainly this provides great opportunities for the graduate students, junior researchers and specialist in the region, which end up with joint research works internationally. The First, Second and Third BIGTC have already taken place in Malaysia (UTM Johor Bahru), Turkey (Istanbul) and Iran (Mashhad), respectively. Once again, this Fourth Biennial International Group Theory Conference 2017 (4BIGTC2017) is hosting by Universiti Teknologi Malaysia (UTM) but this time in UTM Kuala Lumpur. We do hope that others countries besides Malaysia, Turkey and Iran will also join us in organizing BIGTC in the future. This book contains collection of extended abstracts in the broad subject of group theory. All extended abstracts, which will be presented in the plenary, invited or parallel talks in the conference, have been collected in this booklet for the convenience of the participants. We would like to thank all participants and the plenary & invited speakers, who showed interest to attend and give talks in this conference. As the wide range of topics covered by the contributed talks made it necessary to hold different sessions in parallel, so we hope that this collection would in some sense make up the missing talks. This conference is organized by the Applied, Algebra and Analysis Research Group (AAAG) in collaboration with Faculty of Science and UTM International, Universiti Teknologi Malaysia. We are grateful to the Malaysian Mathematical Sciences Society and Centre of Excellence in Analysis on Algebraic Structures (CEAAS) of Ferdowsi University Mashhad, Iran for their generous financial support. We are grateful to the members of Scientific Committee, the members of organizing committee and all plenary and invited speakers of contributed talks for maintaining the stimulating atmosphere and above all, to all of our participants for making this event an unforgettable one. Thanks to the staff of Ferdowsi University of Mashhad for collecting all the abstracts and make this booklet available. Finally, we are thankful to all academic and non-academic staff of UTM International in our Kuala Lumpur Campus, who have helped us in organizing this conference.



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Chair of 4BIGTC2017 Scientific Committee



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Chair of 4BIGTC2017 Organizing Committee

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- Dr. Nor Muhainiah Mohd Ali, Universiti Teknologi Malaysia, Malaysia.

Plenary Speakers

- Prof. Dr. Mark Lewis, Kent State University, USA.
- Prof. Dr. Jamshid Moori, North-West University, Mafikeng, South Africa.

Invited Speakers

- Prof. Dr. Alireza Abdollahi, University of Isfahan, Isfahan, Iran.
- Prof. Dr. Feride Kuzucuoglu, Hacettepe University, Ankara, Turkey.
- Prof. Dr. Ismail Guloglu, Dogus University, Istanbul, Turkey.
- Prof. Dr. Mahmut Kuzucuoglu, Middle East Technical University, Ankara, Turkey.
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- Prof. Dr. Pudji Astuti, Institut Teknologi Bandung, Indonesia.
- Assist. Prof. Dr. Mustafa Gokhan Benli, Middle East Technical University, Ankara, Turkey.
- Dr. Anitha Thillaisundaram, University of Lincoln, UK.
- Dr. Dilber Kocak, Texas A & M University, USA.
- Dr. Wong Kok Bin, University of Malaya, Malaysia.

MONDAY, 23 January 2017

Time	Program	Venue
8:00-09:00	Registration	Lobby (Ground Level)

Time	Title	Plenary Speaker I	Venue
9:00-9:45	Semiextraspecial p -groups	Prof. Dr. Mark Lewis	Hall A

Time	Title	Invited Speaker I	Venue
9:50-10:25	Centralizers in limit monomial groups	Prof. Dr. Mahmut Kuzucuoglu	Hall A

Time	Program	Venue
10:30-11:00	Tea/Coffee Break	Hall B

Time	Title	Invited Speaker II	Venue
11:00-11:35	On branch groups	Dr. Anitha Thillaisundaram	Hall A

Time	Program	Venue
11:40-12:30	Opening Ceremony & Photography Session 11:45 Guests Arrival 11:50 Doa Recitation 11:55 Welcoming Speech: Chair of 4BIGTC2017 (Prof Dr Nor Haniza Sarmin) Welcoming Speech: Chair of Scientific Committee (Prof Dr Ahmad Erfanian) Officiating Speech: Director of UTM KL Campus (Prof Dr Durrishah Idrus) 12:20 Photo Session 12:30 Lunch	Hall A

Time	Program	Venue
12:30-14:00	Lunch	Hall B

Time	Title	Invited Speaker III	Venue
14:00-14:35	Residual finiteness of certain groups	Dr. Wong Kok Bin	Hall A

Time	Title	Invited Speaker IV	Venue
14:40-15:15	On edge-transitivity of Cayley graphs	Prof. Dr. Mohammad Reza Darafsheh	Hall A

MONDAY, 23 January 2017

Parallel Session I

Time	Title	Speaker	Venue
15:20-15:40	The generalization of the homological functors of a Bieberbach group	Tan Yee Ting	Room A
	A note on conjugacy class graphs of p -singular elements	Zohreh Mostaghim	Room B
15:45-16:05	On a group of the form $2^{4+5}:GL(4, 2)$	Ayoub Basheer	Room A
	The connectivity of commuting graphs $\mathcal{C}(G, X)$ in symmetric groups $\text{sym}(n)$	Athirah Nawawi	Room B
16:10-16:30	Cyclic subgroup separability of certain generalized free products	Muhammad Sufi Mohd Asri	Room A
	On the nilpotent conjugacy class graph of groups	Abbas Mohammadian	Room B
16:35-16:55	Some results on central autoisoclinism of groups	Mohammad Javad Sadeghifard	Room A
	On the generalized conjugacy class graph of 3-generator 2-groups	Alia Husna Mohd Noor	Room B

Time	Program	Venue
17:00-17:30	Networking Tea/Coffee Break	Hall B

Time	Program	Venue
19:30-22:00	Dinner and Performance 19:30 Guests Arrival 19:50 VIP's Arrival 20:00 Doa Recitation 20:10 Welcoming Speech: Chair of 4BIGTC2017 Welcoming Speech: Director of UTM KL Campus 20:30 Montage 20:45 Dinner start Performance 22:00 End of Dinner	Hall B

TUESDAY, 24 January 2017

Parallel Session II

Time	Title	Speaker	Venue
8:30-8:50	A homological invariant of a consistent polycyclic group	Siti Afiqah Mohammad	Room A
	The disc structures of commuting involution graphs for certain simple groups	Suzila Mohd Kasim	Room B
8:55-9:15	Invariance of p^m -regular subspaces under derived equivalence	Intan Muchtadi Alamsyah	Room A
	Finite Moufang loops for which the non-commuting graph is a triple complete split-like graph	Karim Ahmadidelir	Room B
9:20-9:40	The nonabelian tensor square of a bieberbach group of dimension four	Nor Fadzilah Abdul Ladi	Room A
	The relative non-nil $(n - 1)$ bipartite graph of a finite group	Muhanizah Abdul Hamid	Room B

Time	Title	Plenary Speaker II	Venue
9:45-10:30	Designs and Codes from Finite Groups	Prof. Dr. Jamshid Moori	Hall A

Time	Program	Venue
10:30-11:00	Tea/Coffee Break	Hall B

Time	Title	Invited Speaker V	Venue
11:00-11:35	Universal Groups of intermediate growth and their invariant random subgroups	Assist. Prof. Dr. Mustafa Gokhan Benli	Hall A

Time	Title	Invited Speaker VI	Venue
11:40-12:05	On the structure of characteristic subgroup lattices of finite abelian p -groups	Prof. Dr. Pudji Astuti	Hall A

Time	Program	Venue
12:30-14:00	Lunch	Hall B

Time	Title	Invited Speaker VII	Venue
14:00-14:35	About the action of Automorphism Groups with TI-centralizers	Prof. Dr. Ismail Guloglu	Hall A

TUESDAY, 24 January 2017

Time	Title	Invited Speaker VIII	Venue
14:40-15:15	Commutative transitive of Lie algebras	Prof. Dr. Mohammad Reza Rajabzadeh Moghaddam	Hall A

Parallel Session III

Time	Title	Speaker	Venue
15:20-15:40	Some properties of CA-nilpotent groups	Samaneh Davoudi Rad	Room A
	On Laplacian eigenvalues of non-commuting graph of dihedral groups	Rabiha Mahmoud	Room B
15:45-16:05	Classification of 2-dimensional evolution algebras, their groups of automorphisms and derivation algebras	Houida Ahmed	Room A
	$\mathbb{R} \times \mathbb{S}$ —additive cyclic codes	Taher Abualrub	Room B
16:10-16:30	On the tensor isoclinism of groups	Shayesteh Pezeshkian	Room A
	On the characterization of bi Γ —ideals of the type $(\in, \in \vee q_k)$ in ordered Γ —semigroups	Ibrahim Gambo	Room B
16:35-16:55	Complete classifications of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras	Ural Bekbaev	Room A
	On the annihilator graph of a commutative semi-group	Kazem Khashyarmansh	Room B

Time	Program	Venue
17:00-17:30	Networking Tea/Coffee Break	Hall B

Time	Program	Venue
18:00-19:00	Dinner	Hall B

WEDNESDAY, 25 January 2017

Time	Title	Invited Speaker IX	Venue
8:30-9:05	Derivations and Jordan Derivations of Some Matrix Rings	Prof. Dr. Feride Kuzucuoglu	Hall A

Parallel Session IV

Time	Title	Speaker	Venue
9:15-9:35	Generating some finite groups using sequential insertion systems	Ahmad Firdaus Yosman	Room A
	The schur multiplier and capability of pairs of groups of order p^4	Adnin Afifi Nawi	Room B
10:05-10:25	g -noncommuting graph of finite groups	Ahmad Erfanian	Room A
	The sub-multiplicative degree for noncyclic subgroups of some non abelian metabelian groups	Fadhilah Abu Bakar	Room B
9:40-10:00	On elliptic curves as cryptographic pairing groups	Elaheh Khamseh	Room A
	Primary ideal in quotient semirings	Dieky Adzkiya	Room B

Time	Program	Venue
10:30-11:00	Tea/Coffee Break	Hall B

Time	Title	Invited Speaker X	Venue
11:00-11:35	Kaplansky Zero Divisor Conjecture on Group Algebras of Torsion-Free	Prof. Dr. Alireza Abdollahi	Hall A

Time	Title	Invited Speaker XI	Venue
11:40-12:05	Finitely presented algebras of intermediate growth	Dr. Dilber Kocak	Hall A

Time	Program	Venue
12:30-14:00	Lunch	Hall B

WEDNESDAY, 25 January 2017

Time	Title	Invited Speaker XII	Venue
14:00-14:35	Recent results on intermediate subgroups of classical groups	Prof. Dr. Nikolai Vavilov	Hall A

Parallel Session V

Time	Title	Speaker	Venue
14:40-15:00	On the order commutativity degree of some finite groups	Suad Saed Alrehaili	Room A
	On the generalized conjugacy class graph of metabelian groups	Nurhidayah Zaid	Room B
15:00-15:20	The commutativity degree in terms of centralizers of metacyclic 2-groups of negative type of nilpotency class two and their centralizer graphs	Siti Norziahidayu Amzee Zamri	Room A
	The relative co-prime graph of a group	Norarida Abd Rhani	Room B
15:20-15:40	Energy of commuting and non-commuting graphs for some metabelian groups	Amira Fadina Ahmad Fadzil	Room A
	Invariants and contractions of low-dimensional leibniz algebras	Abdulkadir Adamu	Room B

Time	Program	Venue
15:45-17:00	Discussion Panel & Closing Ceremony	Hall A

Time	Program	Venue
17:00-17:30	Networking Tea/Coffee Break	Hall B

Time	Program	Venue
18:00-19:00	Dinner	Hall B

THURSDAY, 26 January 2017

Time	Program
8:00-19:00	EXCURSION DAY • Depart from UTMKL • Visit Twin Towers KLCC • Visit National Mosque • Visit Merdeka Square • Visit National Palace • Lunch and prayer break at Putrajaya • Putrajaya Cruise • Return back to UTMKL

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Semiextraspecial p -groups

Mark L. Lewis

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Abstract. In this talk, we will review known results regarding semiextraspecial p -groups. We will show that certain semiextraspecial p -groups can be classified via finite semifields. We will show that the construction used in the classification can be extended to construct other semiextraspecial p -groups.

Semiextraspecial p -groups to have originated with Beisiegel in [1]. Let p be a prime, and let G be a p -group. We say that G is *semiextraspecial* if for every subgroup N having index p in $Z(G)$, then G/N is an extraspecial group. In many ways, semiextraspecial groups are the natural generalization of extraspecial groups. Semiextraspecial groups have occurred in examples in a number of important group theory problems. Recently, there has been research that suggests that semiextraspecial groups may be useful in a certain aspect of coding theory called quantum error correcting codes.

In the first part of our presentation, we summarize some of the known results regarding semiextraspecial groups, most of which were proved by Beisiegel in [1] and Verardi in [3]. One key result is due to MacDonald and Beisiegel ([2] and [1]) who have independently proved that $|G : G'|$ is a square and $|G'| \leq |G : G'|^{1/2}$. Verardi shows that any abelian subgroup of a semiextraspecial group G has order at most $|G : G'|^{1/2}|G'|$ (Theorem 1.8 of [3]). On the other hand, if $x \in G \setminus G'$, then $|C_G(x)| = |G : G'|$ (Proposition 1.7 of [3]). It follows that if $C_G(x)$ is abelian for some $x \in G \setminus G'$, then $|G'|$ must equal $|G : G'|^{1/2}$. Following Beisiegel, we say that a semiextraspecial group G is *ultraspecial* if $|G'| = |G : G'|^{1/2}$.

Following Verardi in [3], we divide ultraspecial groups into three classes:

1. Ultraspecial groups G that have at least two abelian subgroups of order $|G : G'|$.
2. Ultraspecial groups G that have exactly one abelian subgroup of order $|G : G'|$.
3. Ultraspecial groups G that have no abelian subgroups of order $|G : G'|$.

2010 Mathematical Subject Classification. Primary:20D15; Secondary: 20C15, 20E45

Keywords. extraspecial groups, Camina groups, Nilpotence class 2

* Speaker

When p is odd, Verardi has shown that if G is a ultraspecial group that has exponent p and at least two abelian subgroups of order $|G : G'|$, then G can be identified with a finite semifield. We will define finite semifields, and we will survey the results regarding finite semifields in the context of finite geometries. Using these known results, we determine when two semifields yield isomorphic groups, and we will show that this can give a classification of the ultraspecial groups G with at least two subgroups of order $|G : G'|$ in terms of semifields.

Then we will generalize a construction of Verardi to construct all of the ultraspecial groups G with exponent p and exactly one abelian subgroup of order $|G : G'|$. Finally, we will generalize the construction of Verardi to construct all semiextraspecial groups G with exponent p and two abelian subgroups of order $|G : G'|^{1/2}|G'|$ that intersect in the derived subgroup, and we will show that we can similarly generalize our construction to obtain all all semiextraspecial groups G with exactly one subgroup of order $|G : G'|^{1/2}|G'|$ and exponent p .

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- [1] B. Beisiegel, Semi-extraspezielle p -Gruppen, *Math. Z.* **156** (1977), 247-254.
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- [3] L. Verardi, Gruppi semiextraseziali di esponente p , *Ann. Mat. Pura Appl.* **148** (1987), 131-171.



Designs and codes from finite groups

Jamshid Moori

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Abstract. We will discuss two methods for constructing codes and designs from finite groups, mostly simple finite groups. This is a survey of the collaborative work by the author with J D Key and B Rodrigues (and including our research groups at UKZN and NWU). The background material and results required from finite groups, permutation groups and representation theory will be presented in details in the main presentation.

1 Introduction

Error-correcting codes that have large automorphism groups can be useful in applications as the group can help in determining the code's properties, and can be useful in decoding algorithms: see Huffman [1] for a discussion of possibilities, including the question of the use of permutation decoding by searching for PD-sets.

We will discuss two methods for constructing codes and designs for finite groups (mostly simple finite groups). In the *first method* we discuss construction of symmetric 1-designs and binary codes obtained from the primitive permutation representations, that is from the action on the maximal subgroups, of a finite group G . This method, which was introduced in [2], has been applied to several sporadic simple groups by various authors.

The *second method* introduces a technique from which a large number of non-symmetric 1-designs could be constructed. Let G be a finite group, M be a maximal subgroup of G and $C_g = [g] = nX$ be the conjugacy class of G containing g . We construct $1 - (v, k, \lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = nX$ and $\mathcal{B} = \{(M \cap nX)^y | y \in G\}$. The parameters v, k, λ and further properties of $\mathcal{D}$ are determined. Both these methods are fully discussed in [4] and they

2010 Mathematical Subject Classification. Primary:20D05; Secondary: 05B05.

Keywords. Designs, codes, simple groups, maximal subgroups, conjugacy classes.

* Speaker

have been applied to several finite simple groups, for which results have appeared in various papers. We also study codes associated with these Designs.

2 Main Results

Our construction for the symmetric 1-designs (using method 1) is based on the following result, mainly Theorem 2.1 below, which is the Proposition 1 of [2] with its corrected version in [3]:

Theorem 1. Let G be a finite primitive permutation group acting on the set Ω of size n . Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α . If $\mathcal{B} = \{\Delta^g : g \in G\}$ and, given $\delta \in \Delta$, $\mathcal{E} = \{\{\alpha, \delta\}^g : g \in G\}$, then $\mathcal{D} = (\Omega, \mathcal{B})$ forms a $1-(n, |\Delta|, |\Delta|)$ design with n blocks. Further, if Δ is a self-paired orbit of G_α , then $\Gamma = (\Omega, \mathcal{E})$ is a regular connected graph of valency $|\Delta|$, $\mathcal{D}$ is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.

In the following we assume G is a finite simple group, M is a maximal subgroup of G , nX is a conjugacy class of elements of order n in G and $g \in nX$. Thus $C_g = [g] = nX$ and $|nX| = |G : C_G(g)|$. Also let $\chi_M = \chi(G|M)$ be the permutation character afforded by the action of G on Ω , the set of all conjugates of M in G . Clearly if g is not conjugate to any element in M , then $\chi_M(g) = 0$. The construction of our 1-designs (using method 2) is based on the following theorem, Theorem 2.2.

Theorem 2. Let G be a finite simple group, M a maximal subgroup of G and nX a conjugacy class of elements of order n in G such that $M \cap nX \neq \emptyset$. Let $\mathcal{B} = \{(M \cap nX)^y | y \in G\}$ and $\mathcal{P} = nX$. Then we have a $1 - (|nX|, |M \cap nX|, \chi_M(g))$ design $\mathcal{D}$, where $g \in nX$. The group G acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive (not necessarily primitive) on points of $\mathcal{D}$.

Note that since in a $1 - (v, k, \lambda)$ design $\mathcal{D}$ we have $kb = \lambda v$, we deduce that

$$k = |M \cap nX| = \frac{\chi_M(g) \times |nX|}{[G : M]}.$$

If $\lambda = 1$, then $\mathcal{D}$ is a $1 - (|nX|, k, 1)$ design. Since nX is the disjoint union of b blocks each of size k , we have $\text{Aut}(\mathcal{D}) = S_k \wr S_b = (S_k)^b : S_b$. Clearly in this case for all p , if C is our associated code over the prime field F_p , then we have $C = [|nX|, b, k]_p$, that is a code of length $|nX|$ with dimension b and minimum weight k . Furthermore $\text{Aut}(C) = \text{Aut}(\mathcal{D})$.

The designs $\mathcal{D}$ constructed by using Theorem 2.2 are not symmetric in general. In fact $\mathcal{D}$ is symmetric if and only if

$$b = |\mathcal{B}| = v = |\mathcal{P}| \Leftrightarrow [G : M] = |nX| \Leftrightarrow$$

$$[G : M] = [G : C_G(g)] \Leftrightarrow |M| = |C_G(g)|.$$

3 Conclusion

As we mentioned in previous sections, both methods 1 and 2 have been applied to various sporadic simple groups and to some groups of Lie type. Currently, we aim to provide and prove some general results regarding the structure of the $\text{Aut}(\mathcal{D})$ and its relation with $\text{Aut}(G)$, where $\mathcal{D}$ is constructed from a finite simple group by Method 1 and Method 2. This is an ongoing research and for more details see a recent paper by Le-Moori appeared in *Designs, Codes and Cryptography* [5].

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Direct limits of monomial groups

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Joint work with B. V. Oliynyk, V. I. Sushchanskii.

Abstract. We give the construction of homogenous monomial groups as a direct limit of monomial groups. Then we find the structure of the centralizers of elements and conjugacy of two elements in homogenous monomial groups. Moreover isomorphisms of two homogenous monomial groups will be discussed.

1 Introduction

Let H be an arbitrary group and $\mathbb{Z}H$ denote the integral group ring of H . Monomial group over $\mathbb{Z}H$ can be described as $m \times m$ invertible matrices over $\mathbb{Z}H$ in which every row and every column has only one nonzero entry from $\mathbb{Z}H$. These groups are studied by O. Ore in [3]. Denote the complete monomial group of degree m over H by $\Sigma_m(H)$.

Let $\xi = (p_1, p_2, \dots)$ be an infinite sequence of not necessarily distinct primes. The homogeneous monomial group with respect to the above sequence is denoted by $\Sigma_\xi(H)$ and it is obtained as a direct limit of the groups $\Sigma_{n_i}(H)$ embedded into $\Sigma_{n_{i+1}}(H)$ by strictly diagonal embeddings where $n_i = p_1 p_2 \dots p_i$.

Here we are interested in conjugacy of elements, the structure of centralizers of elements in direct limits of monomial groups and classification of such direct limits of monomial groups. The case $H \cong \mathbb{Z}_2$ is studied in [2].

For the centralizers of elements we prove the following.

2010 Mathematical Subject Classification. Primary: 20B22; Secondary: 20E18.

Keywords. .

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2 Main Results

Theorem 1. (Kuzucuoğlu, Oliynyk, Sushchanskii) Let ρ' be an element of $\Sigma_\xi(H)$ with its normal form $\rho = \lambda_1 \dots \lambda_l$, where $\lambda_i = \gamma_{i1} \dots \gamma_{ir_i}$ and for a fixed i , the γ_{ij} are the normalized cycles of the same length m_i and the determinant class a_i with principal beginning ρ_0 is in $\Sigma_{n_k}(H)$. Then the centralizer $C_{\Sigma_\xi(H)}(\rho')$ is isomorphic to $C_{\Sigma_\xi(H)}(\rho)$ and

$$C_{\Sigma_\xi(H)}(\rho) \cong (\Sigma_{\xi_1}(C_{a_1})) \times (\Sigma_{\xi_2}(C_{a_2})) \times \dots \times (\Sigma_{\xi_l}(C_{a_l}))$$

where C_{a_i} is the centralizer of a single element $\gamma_{ij} \in \Sigma_{m_i}(H)$.

The group C_{a_i} consists of elements of the form $\kappa = [c_i] \gamma_{i1}^j$ where $c_i \in C_H(a_i)$ and $Char(\xi_i) = \frac{Char(\xi)}{n_k} r_i$.

Observe that homogenous monomial group over H becomes homogenous symmetric group when $H = \{1\}$. Therefore our results are compatible with centralizers of elements in homogenous symmetric groups see [1].

Theorem 2. (Kuzucuoğlu, Oliynyk, Sushchanskii) Let H be any finite group. The groups $\Sigma_{\xi_1}(H)$ and $\Sigma_{\xi_2}(H)$ are isomorphic if and only if $Char(\xi_1) = Char(\xi_2)$

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On branch groups

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Abstract. In 1902, William Burnside posed the following question: Do there exist infinite finitely generated torsion groups? Some of the early examples of such infinite finitely generated p -groups were groups acting on trees, the so-called branch groups. This important class of groups also provided the first example of a finitely generated group of intermediate word growth.

We will consider a certain family of branch groups, which generalise the early examples, and show that their maximal subgroups cannot have infinite index in the group. We link this result to a conjecture of Passman on group rings. This is joint work with Benjamin Klopsch and Theofanis Alexoudas.

1 Introduction

Branch groups are groups acting spherically transitively on a spherically homogeneous infinite rooted tree and having subnormal subgroup structure similar to the corresponding structure in the full group of automorphisms of the tree. Early constructions were produced by Grigorchuk [2] and Gupta and Sidki [3], and they were generalised to GGS-groups.

The class of branch groups have provided important and more geometric examples of finitely generated groups of intermediate word growth and of finitely generated infinite torsion groups (in response to the general Burnside problem).

Pervova [5, 6] proved that the Grigorchuk group and the torsion GGS-groups do not possess maximal subgroups of infinite index. On the other hand, Bondarenko [1] proved that there exist finitely generated branch groups which

2010 Mathematical Subject Classification. Primary:20D15; Secondary: 20XX

Keywords. branch groups, maximal subgroups, group rings

* Speaker

do possess maximal subgroups of infinite index. So one may ask: Which finitely generated branch groups possess maximal subgroups of infinite index?

2 Main Result

We extend Pervova's results to generalised multi-edge spinal groups, which are a generalisation of GGS-groups:

Theorem 1. Let G be a torsion generalised multi-edge spinal group acting on the regular p -adic rooted tree, for p an odd prime. Then every maximal subgroup of G is normal of finite index p .

One motivation for our investigation comes from a conjecture of Passman concerning the group algebra $K[G]$ of a finitely generated group G over a field K of characteristic p . The conjecture states that if the Jacobson radical coincides with the augmentation ideal, then G is a finite p -group. In [4], Passman has shown that if the Jacobson radical coincides with the augmentation ideal then G is a p -group and every maximal subgroup of G is normal of index p .

Hence torsion generalised multi-edge spinal groups form natural candidates for testing Passman's conjecture. It is important to widen this class of candidates, as various GGS-groups within this subclass do not satisfy the condition of Passman's conjecture.

3 Conclusion

We remark that most parts of the proof of the theorem work under the assumption that the group is just infinite and not necessarily torsion. Therefore one might ask if in fact every just infinite generalised multi-edge spinal group has the property that all maximal subgroups are of finite index.

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Residual finiteness of certain groups

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Abstract. In this talk, we will discuss some results on residually finiteness of outer automorphism groups of certain HNN extensions. We will also discuss some results on residually finiteness of certain one-relator groups.

1 Introduction

A group G is said to be *residually finite* if for each element $x \in G$ such that $x \neq 1$, there exists $N \triangleleft_f G$, such that $Nx \neq N$ in G/N ($N \triangleleft_f G$ means N is a normal subgroup of finite index in G). Equivalently, G is residually finite if for each element $x \in G$ such that $x \neq 1$, there exists $N \triangleleft_f G$ such that $x \notin N$.

If $u \in G$, then ρ_u shall denote the inner automorphism where $\rho_u(g) = u^{-1}gu$ for all $g \in G$. Let $\text{Aut}(G)$ and $\text{Inn}(G)$ denote the automorphism group and inner automorphism group of G , respectively. Let $\text{Out}(G)$ denotes the outer automorphism group, i.e., $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$.

Baumslag [1] proved that if G is a finitely generated group and G is residually finite then $\text{Aut}(G)$ is also residually finite. A natural question to ask is whether the following statement is true: “If G is a finitely generated group and G is residually finite then $\text{Out}(G)$ is also residually”. Unfortunately, the previous statement is not true in general, as Wise [6] has given a counterexample. Nevertheless, it is still interesting to find classes of groups for which the statement is true.

Let $F(X)$ be the free group on X and R be a word on $X^{\pm 1}$. A *one-relator group* has the form

$$\mathcal{P} = \langle X ; R \rangle.$$

2010 Mathematical Subject Classification. Primary: 20E26, 20F05; Secondary: 20E06

Keywords. Residual finiteness, Outer automorphisms, One-relator group

* Speaker

In fact,

$$\mathcal{P} = F(X)/N(R),$$

where $N(R)$ is the smallest normal subgroup of $F(X)$ generated by R . When is a one-relator group residually finite?

Wise [5] showed that every one-relator group with a positive relator satisfying small cancelation condition is residually finite. This can be considered the strongest non-probabilistic result. Sapir and Špakulová [4] have shown that almost all (that is with probability tending to 1) one-relator groups with at least three generators and the relator of length $n \gg 1$ are residually finite. Pride [3] gave a sufficient condition for a one-relator group to be residually finite. We shall describe Pride's result in the next section.

2 Main Results

Let G be a group. A *conjugating endomorphism / automorphism*, α , of G is an endomorphism / automorphism such that for each $g \in G$, there exists a $w_g \in G$ depending on g such that $\alpha(g) = w_g^{-1} g w_g$. A group G is said to have property $\mathcal{A}$ if each conjugating automorphism of G is an inner automorphism of G . A group G is said to be *conjugacy separable* if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G , there exists $N \triangleleft_f G$, such that Nx and Ny are not conjugate in G/N .

Theorem 1. (Grossman [2]) Let G be finitely generated, conjugacy separable and have property $\mathcal{A}$. Then $\text{Out}(G)$ is residually finite.

By using Theorem 1, Wong and Wong [7] proved the following theorem.

Theorem 2. Let $G = \langle t, A; t^{-1} H t = K, \varphi \rangle$ be an HNN extension where A is a polycyclic-by-finite group. Suppose H and K are subgroups in the center of A and $H \neq A \neq K$. If G is residually finite then $\text{Out}(G)$ is residually finite.

Let W be a word on $X^{\pm 1}$, say $W = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ ($x_i \in X, \epsilon_i = \pm 1, 1 \leq i \leq n$). The *length* of W is n . For $x \in X$, the *exponent sum* $\sigma_x(W)$ of W is

$$\sum_{x_i=x} \epsilon_i.$$

The exponent sum *vector* is

$$\sigma(W) = (\sigma_x(W))_{x \in X}.$$

A *proper subword* of W is $x_p^{\epsilon_p} \dots x_q^{\epsilon_q}$ where $1 \leq p \leq q \leq n$ and $(p, q) \neq (1, n)$. The *initial segments* $W^{(i)}$ ($0 \leq i \leq n$) of W are: the empty word if $i = 0$, and $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_i^{\epsilon_i}$ if $1 \leq i \leq n$. A *weight function* on X is a vector $\vec{m} = (m_x)_{x \in X}$, where each $m_x \in \mathbb{Q} \setminus \{0\}$.

A word W is said to have the *unique min property* if there is a weight function $\vec{m}$ and a $1 \leq k \leq n$ such that (i) $\sigma(W) \cdot \vec{m} = 0$ and (ii) $\sigma(W^{(k)}) \cdot \vec{m} < \sigma(W^{(i)}) \cdot \vec{m}$ for $1 \leq i \leq n, i \neq k$. A word W is said to have the *unique max property* if there is a weight function $\vec{m}$ and a $1 \leq k \leq n$ such that (i) $\sigma(W) \cdot \vec{m} = 0$ and (ii) $\sigma(W^{(k)}) \cdot \vec{m} > \sigma(W^{(i)}) \cdot \vec{m}$ for $1 \leq i \leq n, i \neq k$. We say that W has the *unique max-min property* if there is a weight function $\vec{m}$ which *simultaneously* fulfils the criteria for being both the *unique min property* and the *unique max property*.

Theorem 3. (Pride [3]) Let $\mathcal{P} = \langle X ; R \rangle$. Suppose there is a $Y \subseteq X$ such that R^o is a word on $Y^{\pm 1}$ and is a subword of R . If R^o has the unique max-min property, then $\mathcal{P}$ is residually finite.

Let $y \in X$. A word W on $X^{\pm 1}$ is said to be *y-positive* if $\sigma_y(W) > 0$ and W does not contain the subword y^{-1} . A cyclically reduced word W on $X^{\pm 1}$ is said to be *nice* if (i) W is *y-positive* for some $y \in X$, (ii) there is no proper non-trivial subword V of W with $\sigma(V) = \mathbf{0}$, (iii) for any non-trivial subword V of W , $\sigma(V) = \sigma(W)$ implies that $V = W$.

By using Theorem 3, Pride, Vernitski, Wong, and Wong [8] proved the following theorem.

Theorem 4. Let $\mathcal{P} = \langle X ; R \rangle$. Suppose there is a $Y \subseteq X$ such that R^o is a word on $Y^{\pm 1}$ and is a subword of R . If R^o is nice and there are $x_1, x_2, \dots, x_m \in Y$ such that

$$\gcd(\sigma_{x_1}(R^o), \sigma_{x_2}(R^o), \dots, \sigma_{x_m}(R^o)) = 1,$$

then $\mathcal{P}$ is residually finite.

3 Conclusion

In general, it is difficult to determine whether the outer automorphism group of a group is residually finite. At the moment, one can try to find classes of groups for which its outer automorphism group is residually finite.

In general, it is difficult to determine whether $\mathcal{P} = \langle X ; R \rangle$ is residually finite. At the moment, one can try to find sufficient condition on R so that $\mathcal{P}$ is residually finite.

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On edge-transitivity of Cayley graphs

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Abstract. Let G be a finite group and S be an inverse closed subset of G such that $1 \notin S$. We will investigate the Cayley graph $\text{Cay}(G, S) = \Gamma$ and the action of its automorphism group on the set of edges of Γ . For certain groups G we find normal edge-transitive Cayley graphs and obtain their automorphism group.

1 Introduction

Let $\Gamma = (V, E)$ be a simple graph with vertex set V and edge set E , where no loops or multiple edges are allowed in Γ . If σ is a permutation on V preserving the edges of Γ , then σ is called an automorphism of Γ . The set of all automorphisms of Γ is denoted by A , and it is obvious that it is a subgroup of the symmetric group on V . The graph Γ is called vertex-transitive if A acts transitively on V and it is called edge-transitive if A acts transitively on E . An arc is an ordered pair (u, v) of vertices of Γ and Γ is called arc-transitive if A acts transitively on the set of all arcs in Γ .

Let G be a finite group and S be an inverse closed subset of G , i. e. $S^{-1} = S$, such that $1 \notin S$. The Cayley graph of G on S , denoted by $\text{Cay}(G, S)$, is a graph with vertex set G where distinct vertices x and y are joined by an edge if and only if $y = sx$ for some $s \in S$. It is easy to verify that $\Gamma = \text{Cay}(G, S)$ is a regular graph of valency $|S|$ and it is connected if and only if G is generated by S .

For $g \in G$, we define $\rho_g : G \rightarrow G$ by $\rho_g(x) = xg$ for all $x \in G$. Then ρ_g is a permutation of G which preserves edges of Γ . Therefore ρ_g is an automorphism of the Cayley graph $\text{Cay}(G, S)$. The right regular representation of G , is denoted by $R(G)$ is a subset of A isomorphic to G which acts regularly on the vertices of Γ , forcing Γ to be a vertex transitive graph.

2010 Mathematical Subject Classification. Primary:20D60; Secondary: 05B25.

Keywords. Cayley graph, Automorphism groups, Edge-transitive Cayley graphs.

* Speaker

For the Cayley graph $\Gamma = \text{Cay}(G, S)$ we define $\mathbb{A}ut(G, S) = \{\alpha \in \mathbb{A}ut(G) | \alpha(S) = S\}$, and it can be verified that $\mathbb{A}ut(G, S)$ is a subgroup of $\mathbb{A}ut(\Gamma)$, which acts on $R(G)$ by $(\rho_g)^\sigma = \rho_{\sigma^{-1}(g)}$, where $\sigma \in \mathbb{A}ut(G, S)$ and $\rho_g \in R(G)$. Therefore we can form the semi-direct product $R(G) \rtimes \mathbb{A}ut(G, S)$ with respect to this action. Obviously $R(G) \rtimes \mathbb{A}ut(G, S)$ is a subgroup of $A = \mathbb{A}ut(\Gamma)$ and in [4] it is shown that $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$, where $N_A(R(G))$ denotes the normalizer of $R(G)$ in A . In [9] the concept of normality of a graph is defined when $R(G)$ is a normal subgroup of A , and in this case we have $A = R(G) \rtimes \mathbb{A}ut(G, S)$.

The edge transitivity of graphs, and in particular Cayley graphs of small valency has been the subject of research by several authors. In [6] the edge-transitive tetravalent Cayley graphs on groups of square-free order are determined. In [5] all the tetravalent edge-transitive Cayley graphs on the group $PSL_2(3)$ are determined. Normal edge-transitivity of Cayley graphs are a rather special case of edge-transitive ones in which the full automorphism group is known. In fact in the general case of determining edge-transitive Cayley graphs the special case reduces to determination of the normal edge-transitive Cayley graphs.

In [1] the author found normal edge-transitive Cayley graphs of abelian groups, and in [2] normal edge-transitive Cayley graphs of Frobenius groups of order pq where p and q are different prime numbers are determined. In [3] the authors consider groups of order $4p$, where p is an odd prime and study normal edge-transitive Cayley graphs on these groups.

2 Preliminary Results

In [7] the following result is proved which gives a criteria for normality of a Cayley graph.

Lemma 1. Let $\Gamma = \text{Cay}(G, S)$ be the Cayley graph of the group G with respect to S and let $A = \mathbb{A}ut(\Gamma)$. Then the following hold:

- (a) $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$
- (b) $R(G) \triangleleft A$ if and only if $A = R(G) \rtimes \mathbb{A}ut(G, S)$
- (c) Γ is normal if and only if $A_1 = \mathbb{A}ut(G, S)$, where A_1 denotes the stabilizer of the vertex 1 under A . We set $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$ and remark that for normal edge-transitivity of $\text{Cay}(G, S)$ the group N need only be transitive on edges of Γ , from [8] we have the following criteria:

Lemma 2. Let $\Gamma = \text{Cay}(G, S)$ be the Cayley graph of G with respect to S . Then the following are equivalent:

- (a) Γ is normal edge-transitive.
- (b) $S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G, S)$ on S .
- (c) There exists a subgroup H of $\mathbb{A}ut(G)$ and $g \in G$ such that $S = g^H \cup (g^{-1})^H$, where $g^H = \{g^h | h \in H\}$

3 Main Results

We study normal edge-transitive Cayley graphs on different groups.

(a) The group U_{6n}

This group has the following presentation

$$U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$$

The automorphism group of this group has order $6\varphi(n)$, where φ denotes the Euler function.

Proposition 1. If $U_{6n} = \langle S \rangle$, $|S| = 4$, $S = S^{-1}$, and each element S of has order $2n$, then S is equivalent to $\{a, a^{-1}, a^j b, a^{-j} b\}$ where $1 \leq j \leq 2n$ and $(j, 2n) = 1$.

Proposition 2. Let $S = \{a, a^{-1}, a^j b, a^{-j} b\}$, $1 \leq j \leq 2n$, $(j, 2n) = 1$. If $j^2 \equiv \pm 1 \pmod{2n}$, then $\text{Aut}(U_{6n}, S)$ acts transitively on S , otherwise $\text{Aut}(U_{6n}, S)$ has two orbits on S .

Proposition 3. Let $S = \{a, a^{-1}, a^j b, a^{-j} b\}$, $1 \leq j \leq 2n$, $(j, 2n) = 1$.

(i) If $j^2 \equiv \pm 1 \pmod{2n}$, then $\text{Cay}(U_{6n}, S)$ is not normal edge-transitive.

(ii) If $j^2 \equiv \pm 1 \pmod{2n}$, then $\text{Cay}(U_{6n}, S)$ is normal edge transitive with automorphism group isomorphic to $U_{6n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

(iii) If $j^2 \equiv -1 \pmod{2n}$, then $\text{Cay}(U_{6n}, S)$ is normal edge-transitive with automorphism group isomorphic to $U_{6n} \rtimes D_8$.

(b) The generalized quaternion group

This group has order $4n$ with the following presentation:

$$Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$$

The automorphism group of Q_{4n} has order $2n\varphi(n)$.

Proposition 4. Let S be a subset of Q_{4n} such that $Q_{4n} = \langle S \rangle$, $|S| = 4$, $S = S^{-1}$, and each element of S has order 4. Then there is only one normal edge-transitive Cayley graph on S with automorphism group $Q_{4n} \rtimes D_8$ if n is even and $Q_{4n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ if n is odd.

(c) The group V_{8n} of order $8n$

This group has the following presentation:

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, (ab)^2 = (a^{-1}b)^2 = 1 \rangle$$

The automorphism group of V_{8n} has order $4n\varphi(2n)$.

Proposition 5. There is no normal edge-transitive Cayley graph on V_{8n} .

4 Conclusion

By studying normal edge-transitivity and edge-transitivity of Cayley graphs on the groups U_{6n} , Q_{4n} , and V_{8n} , we conclude that depending on n there are tetravalent edge-transitive Cayley graphs on the above groups except the group V_{8n} .

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Universal groups of intermediate growth and their invariant random subgroups

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Joint work with Rostislav Grigorchuk and Tatiana Nagnibeda

Abstract. We exhibit examples of groups of intermediate growth with $2^{\aleph_0}$ ergodic, continuous, invariant random subgroups. The examples come from the class of universal groups associated with a family of groups of intermediate growth.

1 Introduction

Let G be a finitely generated group and S a finite generating set. The *growth function* $\gamma(n)$ of G counts the number of elements of length (with respect to S) at most n . The asymptotic behaviour of this function does not depend on the generating set S . The relation between this function and the structure of the group was studied extensively in the past. Among many achievements one can single out two: First, Gromov's celebrated theorem which states that the class of groups of polynomial growth coincides with the class of virtually nilpotent groups. Second Grigorchuk's [Gri84] construction of groups of *intermediate growth*, i.e., groups for which the growth function grows faster than any polynomial and strictly slower than the exponential function.

Let G be a group and $S(G)$ the space of subgroups of G equipped with the Chabauty topology. An *Invariant Random Subgroup* (IRS) of G is a Borel probability measure on $S(G)$ which is invariant under the action of G on $S(G)$ by conjugation. The delta mass corresponding to a normal subgroup is a trivial example of an IRS, as well as the average over a finite orbit of delta masses associated with groups in a finite conjugacy class. Hence, one is rather

2010 Mathematical Subject Classification. Primary:20F69 ; Secondary: 20E08.

Keywords. Groups of Intermediate Growth, Invariant Random Subgroups.

* Speaker

interested in continuous invariant probability measures on $S(G)$. Clearly, such a measure does not necessarily exist, for example if the group only has countably many subgroups. Given a countable group G , it is a basic question whether there exists a continuous *IRS*. Ultimately, one wishes to describe the structure of the simplex of invariant probability measures of the topological dynamical system $(\text{Inn}(G), S(G))$ where $\text{Inn}(G)$ is the group of inner automorphisms of G . The extreme points of the simplex, are of particular interest.

A very fruitful idea in the subject belongs to Vershik [Ver12] who introduced the notion of a totally non free action of a locally compact group G on a space X with invariant measure μ , i.e., an action with the property that different points $x \in X$ have different stabilizers $St_G(x)$ μ -almost surely. Then the map $St : X \rightarrow S(G)$ defined by $x \mapsto St_G(x)$ is injective μ -almost surely and the image of μ under this map is the law of an *IRS* on G which is continuous and ergodic whenever μ is.

2 Main Results

It follows from Gromov's theorem that a group of polynomial growth does not possess continuous *IRS*. On the other extreme, Bowen [Bow15] shows that non-abelian free groups have many continuous *IRS*. In relation of growth and *IRS* our main theorem is the following:

Theorem 1. There exists a finitely generated group of intermediate growth with $2^{\aleph_0}$ distinct continuous ergodic invariant random subgroups.

To any family of marked groups we associate a marked group which we call the *universal group* of the family. We explore main properties of this construction. In order to prove the main theorem, we investigate universal groups associated to Grigorchuk's examples of groups with intermediate growth. We make use of the fact that the natural action of these groups on the boundary of the binary rooted tree is totally non free.

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On the structure of characteristic subgroup lattices of finite abelian p -groups

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Abstract. This paper gives explicit descriptions of characteristic and fully invariant subgroups of a finite abelian p -group in term of its cyclic decomposition. The results are then utilized to identify the structure of the lattice of characteristic subgroups.

1 Introduction

Kerby and Rode [4] addressed the question when two finite abelian groups have isomorphic lattices of characteristic subgroups. By utilizing an explicit description of the structure of the groups, the problem can be reduced to both groups of being primary. Further, a complete answer of the question for the class of the groups having odd order was obtained. The case of 2-groups is more complicated as a result of the present of irregular characteristic subgroups.

In spirit to be able to complete the above investigations, in this paper we will explain an explicit description of the structure of finite abelian 2-groups. Then we will utilize it to characterize the lattice of characteristic subgroups. For this investigation we are using some notions and notation used in [4]. Let G be a finite abelian p -group for some prime element p . It is a well known fact that G can be decomposed as

$$G = Z_{p^{\beta_1}}^{\alpha_1} \times Z_{p^{\beta_2}}^{\alpha_2} \times \dots \text{ times } Z_{p^{\beta_n}}^{\alpha_n} \quad (1)$$

where Z_{p^λ} is a cyclic group of order p^λ , $1 \leq \beta_1 < \beta_2 < \dots < \beta_n$ and $\alpha_1, \alpha_2, \dots, \alpha_n \geq 1$. Related to that decomposition of the group G , let $\alpha_0 = 0, \lambda_0 = 0, m = \sum_{i=1}^n \alpha_i$ and the exponent of G , $\lambda(G) = (\lambda_1, \lambda_2, \dots, \lambda_m)$, be an m -tuples of integers with $\lambda_j = \beta_i$ for $\sum_{\ell=0}^{i-1} \alpha_\ell < j \leq \sum_{\ell=1}^i \alpha_\ell$. For any two tuple of integers $\mathbf{a} = (a_1, \dots, a_m)$

2010 Mathematical Subject Classification. 20K27; Secondary: 20E15.

Keywords. Abelian groups, Characteristic subgroups, Fully invariant subgroups, Lattices.

* Speaker

$\mathbf{b} = (b_1, \dots, b_m)$ we define

$$\begin{aligned} \mathbf{a} &\leq \mathbf{b} \text{ if } a_i \leq b_i \text{ for all } i \in \{1, \dots, m\}; \\ \mathbf{a} \wedge \mathbf{b} &= (\min\{a_1, b_1\}, \min\{a_2, b_2\}, \dots, \min\{a_m, b_m\}) \\ \mathbf{a} \vee \mathbf{b} &= (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \dots, \max\{a_m, b_m\}). \end{aligned}$$

Then, the set $\Lambda(G) = \{\mathbf{a} : \mathbf{a} \leq \lambda(G)\}$, equipped with the above partially order and operations, forms a finite lattice. Referring to [2] the sublattice $\mathcal{C}(G) = \{(a_1, \dots, a_m) \in \Lambda(G) : 0 \leq a_i - a_{i-1} \leq \lambda_i - \lambda_{i-1} \text{ } i = 2, \dots, m\}$ is distributive and self-dual with an anti-isomorphism $(a_1, \dots, a_m) \mapsto (\lambda_1 - a_1, \dots, \lambda_m - a_m)$. Any element of $\mathcal{C}(G)$ is called canonical.

A subgroup of G which is invariant with respect to all automorphisms (endomorphisms) of G is called characteristic (fully invariant) [3]. The set of all characteristic (fully invariant) subgroups of G , denoted by $\text{Char}(G)$ ($\text{FI}(G)$), forms lattice. Let $t_1, t_2, \dots, t_m$ be generators of G with respect to the decomposition (1). Referring to [4], if G is a p -group with p is an odd prime then the lattices $\mathcal{C}(G)$ and $\text{Char}(G)$ are isomorphic; that is there exists a lattice isomorphism $\theta : \mathcal{C}(G) \rightarrow \text{Char}(G)$ with $\theta(\mathbf{a}) = \prod_{i=1}^m \langle t_i^{p^{\lambda_i - a_i}} \rangle$ for all $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathcal{C}(G)$.

2 Main Results

Before we explain main results of this paper, we gather some facts about the structures of G , a finite abelian p -group with a decomposition of the form (1). It is trivial that any fully invariant subgroup of G is characteristic. A characteristic subgroup $S \subset G$ is fully invariant if and only if there exists $\mathbf{a} \in \mathcal{C}(G)$, $\mathbf{a} = (a_1, \dots, a_m)$ such that $S = \prod_{i=1}^m \langle t_i^{p^{\lambda_i - a_i}} \rangle$. If p is odd or $\alpha_i \geq 2$ except at most one $\alpha_i = 1$ then $\text{Char}(G) = \text{FI}(G)$. In the language of Kerby and Rode [4], a fully invariant subgroup is a regular characteristic subgroup and a characteristic but not fully invariant subgroup is an irregular characteristic subgroup.

In general, for any prime number p , the lattices $\mathcal{C}(G)$ and $\text{FI}(G)$ are isomorphic; that is there exists a lattice isomorphism $\theta : \mathcal{C}(G) \rightarrow \text{FI}(G)$ with $\theta(\mathbf{a}) = \prod_{i=1}^m \langle t_i^{p^{\lambda_i - a_i}} \rangle$ for all $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathcal{C}(G)$. In connection with the decomposition (1) and for any $i \in \{1, \dots, m\}$, one can define the projection mapping on the i -th coordinate $\pi_i : G \rightarrow G$ with $\pi_i(\prod_{\ell=1}^m g_\ell) = g_i$ for all $\prod_{\ell=1}^m g_\ell \in G$. Then, a characteristic subgroup of G , say S , is fully invariant if and only if for every $i \in \{1, \dots, m\}$, $\pi_i(S) = S \cap \langle t_i \rangle$.

For the class of finite abelian 2-groups we obtain particular results which are adapted from [1] and its references. Let S be a characteristic subgroup of 2-group G . For every $i \in \{1, \dots, m\}$, we can obtain a_i, b_i integers such that $S \cap \langle t_i \rangle = \langle t_i^{2^{\lambda_i - a_i}} \rangle$ and $\pi_i(S) = \langle t_i^{2^{\lambda_i - b_i}} \rangle$. Then S is fully invariant if and only if $a_i = b_i$ for all $i = 1, \dots, m$. If S is not fully invariant then there exist at least two integers i_1, i_2 such that $b_{i_\ell} = a_{i_\ell} + 1$ for $\ell = 1, 2$. Further, there exists j_1, j_2 such that $i_\ell = 1 + \sum_{k=0}^{j_\ell-1} \alpha_k$ and $\alpha_{j_\ell} = 1$ for $\ell = 1, 2$. Furthermore $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_m) \in \mathcal{C}(G)$ and $\theta(\mathbf{a})$ ($\theta(\mathbf{b})$) is the largest (smallest) fully invariant subgroup contained in (contains) S .

Inspired by the above observation, for any $\mathbf{a}, \mathbf{b} \in \mathcal{C}(G)$ with $\mathbf{a} \leq \mathbf{b}$, we define an interval $[\theta(\mathbf{a}), \theta(\mathbf{b})]$ as the set of all characteristic subgroups of G in between $\theta(\mathbf{a})$ and $\theta(\mathbf{b})$. It is obvious that the interval $[\theta(\mathbf{a}), \theta(\mathbf{b})]$ is a sublattice of $\text{Char}(G)$. The main results of this paper are as follows.

Theorem 1. Let G be a finite abelian 2-group with a decomposition (1) and $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_m) \in \mathcal{C}(G)$. If for every $i \in \{1, 2, \dots, m\}$, $b_i = a_i$ or $b_i = a_i + 1$ holds then there exists an anti-isomorphism from $[\theta(\mathbf{a}), \theta(\mathbf{b})]$ onto $[\theta(\lambda(G) - \mathbf{b}), \theta(\lambda(G) - \mathbf{a})]$.

Theorem 2. Let G be a finite abelian 2-group with a decomposition (1). If there exists at least two elements $i, j \in \{1, 2, \dots, n\}$, $i \neq j$ such that $\alpha_i = \alpha_j = 1$ and $\beta_i + 1 < \beta_j$ then $\text{IF}(G) \subsetneq \text{Char}(G)$ and there exists anti-isomorphism of $\text{Char}(G)$ with the restriction on $\text{IF}(G)$ is an anti-isomorphism of $\text{IF}(G)$.

3 Conclusion

According to facts above we may read that results of Kerby and Rode [4] addressed the question: when do two finite abelian groups of odd order have isomorphic lattices of fully invariant subgroups? In connection with this view we are quite certain that the approach in [4] can be extended to obtain a complete answer of the question for any two finite abelian groups, including 2-groups. With this finding and Theorem 2.2, we expect the original considered question, that is when do two finite abelian 2-groups have isomorphic lattices of characteristic subgroups?, can be investigated through the following open question: given G, H two finite abelian 2-groups having both lattices $\text{Char}(G)$ and $\text{Char}(H)$ are isomorphic, are both the sublattices $\text{FI}(G)$ and $\text{FI}(H)$ isomorphic too?

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About the action of Automorphism Groups with TI-centralizers

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Abstract. A subgroup H of a group G is called a TI-subgroup if $H \cap H^g \neq 1$ for some $g \in G$ implies that $g \in N_G(H)$ and H is called an STI-subgroup of G if for any normal subgroup N of G the subgroup HN/N is TI in G/N . In this paper the following result is proven: If A is of prime order and acts coprimely on a finite group G in such a way that $C_G(A)$ is a STI-subgroup of G then $C_G(A)$ is solvable if and only if G is solvable.

This work has been supported by the Research Project TUBITAK 114F223.

1 Introduction

This note is the first report on a recently started, ongoing project about the structure of groups G admitting a group A as a group acting on G by automorphisms in such a way that the subgroup $C_G(A)$ of fixed points of A in G is a trivial intersection subgroup of G . Here the subgroup H of G is said to be a trivial intersection subgroup, or in short a TI-subgroup of G if $1 \neq H^g \cap H$ for some $g \in G$ implies that $H^g = H$. Clearly any normal subgroup of G is a TI-subgroup, also any subgroup of prime order is a TI-subgroup.

This problem is clearly a generalization of the so-called fixed-point-free action which assumes in the above context that $C_G(A) = 1$. In this case we know that G has to be solvable and if additionally $(|G|, |A|) = 1$ and A is a group acting with regular orbits on every irreducible A -module then the Fitting length of G is bounded by the length of the longest chain of subgroups of A (see [2] for the research on this topic until then). The main conjecture in the area of the influence of the fixed-point-free acting group of automorphisms on the structure of the group is

2010 Mathematical Subject Classification. Primary: 20D10; Secondary: 20D15.

Keywords. automorphism, solvable, TI-subgroup.

* Speaker

Conjecture *If A acts on the group G by automorphisms fixedpointfreely and either $(|G|, |A|) = 1$ or A is nilpotent then G is solvable and the Fitting length of G is bounded by the length of the longest chain of subgroups of A .*

This conjecture is still open even in the coprime case. So it seems rather out of place to look at the much more general situation where $C_G(A)$ is a TI-subgroup, much more so because this property of being TI does not behave well with respect to taking subgroups or homomorphic images, so that one cannot hope to apply induction arguments. We have made us the life a bit easier by assuming, somewhat unnaturally, that $C_G(A)$ is an STI-subgroup of G , i.e. $C_G(A)N/N$ is a TI-subgroup of G/N for any normal subgroup N of G . For example every subgroup of prime order is an STI-subgroup. As a first result along these lines we have obtained the following theorem

Theorem 1. Let A be a coprime automorphism of G of prime order. If $C_G(A)$ is an STI-subgroup of G then the group G is solvable if and only if $C_G(A)$ is solvable.

All the groups throughout the paper are finite, notation and terminology are standard.

2 Proof of the Theorem

In this section we give a proof of our theorem.

Proof. Set $A = \langle a \rangle$. We proceed by induction on the order of G to prove that G is solvable if $C_G(A)$ is solvable. Let N be a nontrivial proper normal A -invariant subgroup of G . Since the action is coprime we have $C_{G/N}(A) = C_G(A)N/N$ and hence $C_{G/N}(A)$ is an STI-subgroup of G/N . Applying induction to the action of A on N and G/N we see that N and G/N are both solvable. This forces that G is solvable. Therefore we may assume that G has no nontrivial proper A -invariant normal subgroup. As a consequence $[G, A] = G$. Notice also that G is characteristically simple and hence $G = G_1 \times \dots \times G_k$ where $G_1, \dots, G_k$ are nonabelian simple groups permuted by A transitively. Suppose $k > 1$. Since A is of prime order we have $C_G(A) = \{\sum_{a \in A} g^a : g \in G_1\} \cong G_1$ and so G_1 is also solvable. This yields that $k = 1$, that is, G is simple. Suppose that $|A| = r$. Since r does not divide the order of G , G is not sporadic or alternating. It follows that G is a universal Lie type group and A is a field automorphism of G . By Theorem 4.9.1 (a) and (c) of [1] we deduce that $G \cong^d \Sigma(q^r)$ and $C_G(A) \cong^d \Sigma(q)$ for some root system Σ and some prime power q . As $C_G(A)$ is solvable, it is isomorphic to one of the groups $A_1(2), {}^2A_2(2), {}^2B_2(2), A_1(3)$. Then the group G is isomorphic to one of $A_1(2^r), {}^2A_2(2^r), {}^2B_2(2^r), A_1(3^r)$. All of these simple groups have a single conjugacy class of involutions. Therefore there exists a Sylow 2-subgroup T of G which contains an involution of $C_G(A)$ in its center and hence normalizes $C_G(A)$. In the first three cases T normalizes the nontrivial subgroup $S = O_{2'}(C_G(A))$. Then ST is a subgroup of G which is contained in a maximal subgroup of G containing a Sylow 2-subgroup. Since G contains a strongly embedded subgroup, the unique maximal subgroup of G containing a Sylow 2-subgroup of G is the normalizer of this Sylow subgroup. This implies that $[S, T] = 1$, which is impossible. If $G \cong A_1(3^r)$ then a Sylow 3-subgroup of G is abelian. Therefore there exists a Sylow 3-subgroup T of G which contains an element of order 3 of $C_G(A)$ and hence normalizes $C_G(A)$. It follows that T normalizes $S = O_2(C_G(A))$ too. This implies that a maximal subgroup of T centralizes S which is not the case. $\square$

Remark 1. (1) The following example shows that under the hypothesis of the above theorem G need not be nilpotent, if $C_G(A)$ is nilpotent. Set $T = S_3 \wr \mathbb{Z}_5$ and let $M = O_3(T)$. Then $M = \langle a_1 \rangle \times \cdots \times \langle a_5 \rangle$ where $|a_i| = 3$ for $i = 1, \dots, 5$. Let $Q = [\langle b_1 \rangle \dots \langle b_5 \rangle, a]$ where $Z_5 = \langle a \rangle$. Then $Q = \langle b_1^{-1}b_2, b_2^{-1}b_3, b_3^{-1}b_4, b_4^{-1}b_5 \rangle$. Set $A = \langle a \rangle$ and $G = MQ$. Now $Q = [Q, A]$ implies $G = [G, A]$, and $C_G(A) = C_M(A) = \langle a_1 a_2 a_3 a_4 a_5 \rangle \cong \mathbb{Z}_3$.

(2) The following example shows that the above result is not true if the coprimeness condition is dropped:

Set $G = A_5$ and let $a \in G$ with $|a| = 5$. Let α be the inner automorphism of G induced by a . Let $A = \langle a \rangle$. Now $C_G(A) (\cong \mathbb{Z}_5)$ is a TI-subgroup of G .

3 Conclusion

The research on this topic will go on by trying to see whether the condition of being STI is really necessary, and whether one can extend this result to any acting group A of coprime order. The next step would be to find the best possible bound for the Fitting height of G .

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Some properties of commutative transitive of groups and Lie algebras

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Abstract. In 2007 a general notion of χ -transitive groups was introduced by C. Delizia, P. Moravec and C. Nicotera, where χ is a class of groups. In 2013, L. Ciobanu, B. Fine and G. Rosenberger studied the relationship among the notions of conjugately separated abelian, commutative transitive (CT) and fully residually χ -groups.

We introduce and discuss the concept of n -commutative transitive groups (n -CT) and under some conditions it is shown the three notions n -commutative transitivity, conjugately separated n -central, and fully residually χ -groups are equivalent. Also the notion of 2-Engel transitive group (2-ET) will be discussed and give its relationship with conjugately separated 2-Engel group and fully residually χ -groups.

Finally, the concept of commutative transitive Lie algebras are discussed.

1 Introduction

Residual properties have played a major role in infinite group theory. Let χ be a class of groups. Then a group G is *residually* χ if given any non-trivial element $g \in G$ there is a homomorphism $\phi : G \rightarrow H$ where H is a group in χ

2010 Mathematical Subject Classification. Primary:15A27; Secondary: 16U80.

Keywords. CT groups, 2-ET groups, CT Lie algebras.

* Speaker

such that $\phi(g) \neq 1$. A group G is *fully residually χ* if given finitely many non-trivial elements $g_1, \dots, g_n$ in G there is a homomorphism $\phi : G \rightarrow H$, where H is a group in χ , such that $\phi(g_i) \neq 1$ for all $i = 1, \dots, n$. Fully residually free groups have played a crucial role in the study of equations and first order formulas over free groups.

Definition 1. A subgroup H of a group G is called conjugately separated, if $H \cap H^x = 1$ for all $x \in G \setminus H$.

It is clear that the intersection of a family of conjugately separated subgroups is again conjugately separated.

Definition 2. Given a non-zero integer n , a group G is said to be *n -central* if $[x, y^n] = 1$, for all $x, y \in G$. Hence n -central property is equivalent to $G/Z(G)$ having finite exponent dividing n .

Here $[x, y] = x^{-1}y^{-1}xy$ is the usual commutator of the elements x and y of the group G and $x^y = y^{-1}xy$ is the conjugate of x by y .

Also a group G is called *Conjugately Separated n -Central* or CSC^n -group, if all of its maximal n -central subgroups are conjugately separated.

A group G is said to be a *CSA-group*, if all of its maximal abelian subgroups are conjugately separated and it is commutative transitive or CT group if commutativity is transitive on the set of non-trivial elements of G . L. Ciobanu et. al. [4] examined the relationships among the classes of non-abelian CSA, CT and fully residually χ -groups.

We will extend the definition of CT groups and we introduce the notion of *n -commutative transitive* groups, then we study its relationship with CSC^n and fully residually χ -groups.

Definition 3. A group G is *n -commutative transitive* (henceforth n -CT), if $[x, y^n] = 1$ and $[y, z^n] = 1$ imply that $[x, z^n] = 1$, for non-trivial elements x, y, z in G and $n \geq 1$.

Clearly n -CT groups are the usual CT groups, for $n = 1$. Also 1-central groups are abelian and CSC^1 groups are CSA.

Benjamin Baumslag proved that a group is fully residually free is equivalent to being residually free and commutative transitive.

We show that under some conditions the three notions n -commutative transitivity, CSC^n , and fully residually χ -groups are equivalent for many important classes of groups, including those of free products of cyclics not containing the infinite dihedral group, torsion-free hyperbolic groups and one-relator groups with only odd torsion.

So, the main theorem of this section reads as follows;

Theorem 1. Let χ be a class of groups such that each of its n -noncentral group is CSC^n and G be n -noncentral residually χ -group. Then the following statements are equivalent:

- (i) G is fully residually χ ;
- (ii) G is CSC^n ;
- (iii) G is n -CT.

Corollary 1. If G is a residually free group. Then G is fully residually free if and only if G is n -commutative transitive.

2 2-Engel transitive groups

A group G is called a *conjugately separated 2-Engel group* (henceforth CSE²-group), if all of its maximal 2-Engel subgroups are conjugately separated.

In the following, we discuss the notion of 2-Engel transitive group and then give its relationship with CSE²-group and fully residually χ -groups.

Definition 4. (a) A group G is *2-Engel transitive* (henceforth 2-ET), whenever $[x, y, y] = 1$ and $[y, z, z] = 1$ imply that $[x, z, z] = 1$, for every non-trivial elements x, y, z in G .

(b) For a given element x of G , we call

$$E_G^2(x) = \{y \in G : [x, y, y] = 1, [y, x, x] = 1\}$$

to be the set of 2-Engelizer of x in G . The family of all 2-Engelizers in G is denoted by $E^2(G)$ and $|E^2(G)|$ denotes the number of distinct 2-Engelizers in G .

As an example consider $Q_{16} = \langle a, b : a^8 = 1, a^4 = b^4, b^{-1}ab = a^{-1} \rangle$, the Quaternion group of order 16 and take the element b in Q_{16} . Then one can easily check that the 2-Engelizer set of b is as follows:

$$E_{Q_{16}}^2(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}.$$

Clearly in general, the 2-Engelizer of each non-trivial element of an arbitrary group G does not form a subgroup. The following example shows our claim.

Example 1. Let G be a finitely presented group of the following form:

$$G = \langle a_1, a_2, a_3, a_4 : a_3^3 = a_4^3 = 1, [a_1, a_2] = 1, [a_1, a_3] = a_4, \\$$

$$[a_1, a_4] = 1, [a_2, a_3] = 1, [a_2, a_4] = a_2, [a_3, a_4] = 1 \rangle.$$

It is clear that G is an infinite group. One can easily check that G is not 2-ET, as $[a_2, a_1, a_1] = 1$ and $[a_1, a_4, a_4] = 1$, while $[a_2, a_4, a_4] = a_2$. Moreover, $E_G^2(a_1)$ is not a subgroup of G , since it is easily calculated that $a_2, a_3 \in E_G^2(a_1)$ but $a_2a_3 \notin E_G^2(a_1)$.

Now, we give a condition under which the 2-Engelizer of each non-trivial element of a group G forms a subgroup.

Theorem 2. Let G be an arbitrary group, then the set of each 2-Engelizer of a non-trivial element in G forms a subgroup if and only if the group $x^{E_G^2(x)}$ is abelian for all non-trivial elements x of G .

Here we show that Baumslag's Theorem is also true in the case of 2-Engel transitive groups.

Theorem 3. Let G be a residually free group. Then G is fully residually free if and only if G is 2-Engel transitive.

3 Commutative transitive Lie algebras

A commutative transitive groups was defined and studied by Weisner [6] in 1925.

Definition 5. A Lie algebra L is *commutative transitive* (henceforth CT), if $[x, y] = 0$ and $[y, z] = 0$ imply that $[x, z] = 0$, for any non-zero elements x, y, z in L .

The property of CT is clearly subalgebra closed, yet it is not quotient closed, as every free Lie algebra is CT (see [3], Example 4.4 for more detail).

The following lemma of [1] is needed to prove the main theorem of this section.

Lemma 1. (Bokut and Kukin [1], Lemma 4.16.2) A Lie algebra L is fully residually free if and only if, for every two linearly independent elements x_1 and x_2 in L , there exists a homomorphism ϕ from the Lie algebra L into a free Lie algebra $\mathcal{F}$ such that the elements $\phi(x_1)$ and $\phi(x_2)$ are linearly independent in $\mathcal{F}$.

Now, using the above lemma, it can be shown that Baumslag's Theorem is also held for free Lie algebras.

Theorem 4. Let L be a residually free Lie algebra. Then L is fully residually free, if and only if L is CT.

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Derivations and Jordan Derivations of Some Matrix Rings

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Joint work with U. Sayın.

Abstract. Let K be any associative ring with identity and J be an ideal of K . Let $NT_n(K)$ be the ring of all (lower) niltriangular $n \times n$ matrices over K which are all zeros on and above the main diagonal and let $M_n(J)$ be the ring of all $n \times n$ matrices over an ideal J . We describe all derivations of the matrix ring $R = R_n(K, J) = NT_n(K) + M_n(J)$.

1 Introduction

Let K be an associative ring with identity. Recall that a derivation of a ring K is an additive map $\Psi : K \rightarrow K$ so that for all $a, b \in K$, $\Psi(ab) = \Psi(a)b + a\Psi(b)$. Let $NT_n(K)$ be the ring of all (lower) niltriangular $n \times n$ matrices over K which are all zeros on and above the main diagonal. Levchuk characterized in [2] the automorphisms of the ring $NT_n(K)$ over any associative ring K with identity. In 2006, Chun and Park [1] determined derivations of the niltriangular matrix ring $NT_n(K)$. Let $M_n(J)$ be the ring of all $n \times n$ matrices over an ideal J of K and $R = R_n(K, J) = NT_n(K) + M_n(J)$.

Now we define fundamental derivations.

(A) Let $\lambda_n : J \rightarrow \text{Ann}_K J$, $\lambda_n(J^2) = 0$ and $\lambda_i : K \rightarrow \text{Ann}_K J$, $\lambda_i(J) = 0$ be additive maps for $i = 1, 2, \dots, n-1$. Then the map

$$\begin{aligned} \Omega : R_n(K, J) &\longrightarrow R_n(K, J) \\ X = [x_{i,j}] &\longrightarrow \left(\lambda_n(x_{1,n}) + \sum_{i=1}^{n-1} \lambda_i(x_{i+1,i}) \right) e_{n,1} \end{aligned}$$

2010 Mathematical Subject Classification. Primary: 16W25; Secondary: 16S50.

Keywords. Niltriangular matrix, Derivation, Jordan derivation.

* Speaker

determines a derivation of the ring R which will be called an annihilator derivation.

(B) If additive group homomorphisms $\sigma : J \longrightarrow K$ and

$\lambda, \mu : J \longrightarrow J$ satisfy the following relations

$$\begin{aligned} i) \quad \lambda(xy) &= x\lambda(y) \\ ii) \quad \mu(yx) &= \mu(y)x \\ iii) \quad \sigma(y)z &= \sigma(yz) = y\sigma(z) = 0 \\ iv) \quad \lambda(y)z + y\mu(z) &= 0 \end{aligned}$$

then the map

$$\begin{aligned} \Delta : R_n(K, J) &\longrightarrow R_n(K, J) \\ ye_{1,n} &\longrightarrow \lambda(y)e_{1,1} + \mu(y)e_{n,n} + \sigma(y)e_{n,1} \\ ye_{i,n} &\longrightarrow \lambda(y)e_{i,1} \quad , \quad 1 < i \leq n \\ ye_{1,j} &\longrightarrow \mu(y)e_{n,j} \quad , \quad 1 \leq j < n \end{aligned}$$

determines a derivation of the ring R . This derivation will be called almost annihilator.

(C) Every derivation $\theta_{i,j}^{i,j}$ of the coefficient ring K or J induces a derivation $\bar{\theta}$ of the ring $R_n(K, J)$ by the rule $\bar{\theta}(X) := \left(\sum_{i,j=1}^n \theta_{i,j}^{i,j} \right) (X) = \sum_{i,j=1}^n \theta_{i,j}^{i,j}(x_{i,j})e_{i,j}$ for $X = [x_{i,j}] \in R_n(K, J)$. This derivation will be called a ring derivation as usual.

(D) For any ring R and any element a of this ring, the map $\Psi_a(x) = ax - xa$, $x \in R$ is easily seen to be a derivation on R . Such derivations are called inner derivations induced by the element a .

(E) Let $d = \sum d_i e_{i,i}$ ($d_i \in K$). Then the derivation $\delta_d(X) = dX - Xd$ of R is called a diagonal derivation, where $X = [x_{i,j}] \in R_n(K, J)$.

We describe all derivations of the matrix ring $R_n(K, J) = NT_n(K) + M_n(J)$. If time permits we will discuss Jordan derivations of the matrix ring $R_n(K, J)$.

2 Main Results

Theorem 1. Let K be an associative ring with identity and J be an ideal of K . Then every derivation of $\phi : R = R_n(K, J) \longrightarrow R_n(K, J)$, for $n > 2$ is a sum of certain diagonal, inner, almost annihilator, annihilator and ring derivations.

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Kaplansky zero divisor conjecture on group algebras of torsion-free groups

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Abstract. Kaplansky's zero divisor conjecture (unit conjecture, respectively) states that for a torsion-free group G and a field $\mathbb{F}$, the group ring $\mathbb{F}[G]$ has no zero divisors (has no unit with support size greater than 1). In this paper, we study possible zero divisors and units in $\mathbb{F}[G]$ whose supports have size 3. For any field $\mathbb{F}$ and all torsion-free groups G , we prove that if $\alpha\beta = 0$ for some non-zero $\alpha, \beta \in \mathbb{F}[G]$ such that $|supp(\alpha)| = 3$, then $|supp(\beta)| \geq 10$. If $\mathbb{F} = \mathbb{F}_2$ is the field with 2 elements, the latter result can be improved so that $|supp(\beta)| \geq 20$. This improves a result in [J. Group Theory, 16 (2013), no. 5, 667-693]. Concerning the unit conjecture we prove that if $\alpha\beta = 1$ for some $\alpha, \beta \in \mathbb{F}[G]$ such that $|supp(\alpha)| = 3$, then $|supp(\beta)| \geq 8$. The latter improves a part of a result in [Experimental Mathematics, 24 (2015), 326-338] to arbitrary fields.

This is a joint work with Zahra Taheri.

1 Introduction and Results

Let R be a ring. A non-zero element α of R is called a zero divisor if $\alpha\beta = 0$ or $\beta\alpha = 0$ for some non-zero element $\beta \in R$. Let G be a group. Denote by $R[G]$ the group ring of G over R . If R contains a zero divisor, then clearly

2010 Mathematical Subject Classification. Primary: 20K15; Secondary: 20K20, 20C07.

Keywords. Kaplansky's zero divisor conjecture, Kaplansky's unit conjecture, group ring, torsion-free group, zero divisor.

* Speaker

so does $R[G]$. Also, if G contains a non-identity torsion element x of finite order n , then $R[G]$ contains zero divisors $\alpha = 1 - x$ and $\beta = 1 + x + \cdots + x^{n-1}$, since $\alpha\beta = 0$. Around 1950, Irving Kaplansky conjectured that existence of a zero divisor in a group ring depends only on the existence of such elements in the ring or non-trivial torsions in the group by stating one of the most challenging problems in the field of group rings [11].

Conjecture 1.1 (Kaplansky's zero divisor conjecture). Let $\mathbb{F}$ be a field and G be a torsion-free group. Then $\mathbb{F}[G]$ does not contain a zero divisor.

Over the years, some partial results have been obtained on Conjecture 1.1 and it has been confirmed for special classes of groups which are torsion free. One of the first known special families which satisfy Conjecture 1.1 are unique product groups [16, Chapter 13], in particular ordered groups. Furthermore, by the fact that Conjecture 1.1 is known to hold valid for amalgamated free products when the group ring of the subgroup over which the amalgam is formed satisfies the Ore condition [15], it is proved by Formanek [8] that supersolvable groups are another families which satisfy Conjecture 1.1. Another result, concerning large major sorts of groups for which Conjecture 1.1 holds in the affirmative, is obtained for elementary amenable groups [13]. The latter result covers the cases in which the group is polycyclic-by-finite, which was firstly studied in [2] and [6], and then extended in [20]. Some other affirmative results are obtained on congruence subgroups in [14] and [7], and certain hyperbolic groups [3]. Nevertheless, Conjecture 1.1 has not been confirmed for any fixed field and it seems that confirming the conjecture even for the smallest finite field $\mathbb{F}_2$ with two elements is still out of reach.

The support of an element $\alpha = \sum_{x \in G} a_x x$ of $R[G]$, denoted by $\text{supp}(\alpha)$, is the set $\{x \in G \mid a_x \neq 0\}$. For any division ring $\mathbb{K}$ and all torsion-free group G , it is known that $\mathbb{K}[G]$ does not contain a zero divisor whose support is of size at most 2 (see [4, Proposition 2.6] and also [19, Theorem 2.1] when $\mathbb{K}$ is assumed to be a field), but it is not known a similar result for group ring elements with the support of size 3. By describing a combinatorial structure, named matched rectangles, Schweitzer [19] showed that if $\alpha\beta = 0$ for $\alpha, \beta \in \mathbb{F}_2[G] \setminus \{0\}$ when $|\text{supp}(\alpha)| = 3$, then $|\text{supp}(\beta)| > 6$. Also, with a computer-assisted approach, he showed that if $|\text{supp}(\alpha)| = 3$, then $|\text{supp}(\beta)| > 16$.

Let G be an arbitrary torsion-free group and let $\alpha \in \mathbb{F}[G]$ be a possible zero divisor such that $|\text{supp}(\alpha)| = 3$ and $\alpha\beta = 0$ for some non-zero $\beta \in \mathbb{F}[G]$. In this paper, we study the minimum possible size of the support of such an element β . Let β have minimum possible support size and $\mathbb{F} = \mathbb{F}_2$. In [19, Definition 4.1] a graph is associated to the non-degenerate $3 \times |\text{supp}(\beta)|$ matched rectangle corresponding to α and β and it is proved in [19, Theorem 4.2] that the graph is a simple cubic one without triangles. We call the graph Kaplansky graph of (α, β) over $\mathbb{F}_2$ and it is denoted by $K_{\mathbb{F}_2}(\alpha, \beta)$. We extend such definition to the case that $\mathbb{F}$ is an arbitrary field and the corresponding Kaplansky graph is denoted by $K_{\mathbb{F}}(\alpha, \beta)$. So, any Kaplansky graph is derived from a possible zero divisor with support of size 3 in the group algebra of a torsion-free group over the field $\mathbb{F}$. In fact $K_{\mathbb{F}}(\alpha, \beta)$ is the induced subgraph on the set $\text{supp}(\beta)$ of the Cayley graph $\text{Cay}(G, S)$, where $S = \{h^{-1}h' \mid h, h' \in \text{supp}(\alpha), h \neq h'\}$. Here we study forbidden subgraphs of Kaplansky graphs. Our main results on Conjecture 1.1 are the followings.

Theorem 1.2. None of the graphs in Figure ?? can be isomorphic to a subgraph of any Kaplansky graph over any field $\mathbb{F}$.

By a Cayley graph $\text{Cay}(G, S)$ for a group G and a subset S of G with $1 \notin S = S^{-1}$, is the graph whose vertex set is G and two vertices g_1, g_2 are adjacent if $g_1 g_2^{-1} \in S$.

Theorem 1.3. Let α and β be non-zero elements of the group algebra of any torsion-free group over an arbitrary field. If $|supp(\alpha)| = 3$ and $\alpha\beta = 0$ then $|supp(\beta)| \geq 10$.

Theorem 1.4. None of the graphs in Table ?? can be isomorphic to a subgraph of any Kaplansky graph over $\mathbb{F}_2$.

The following result improves a result in [19].

Theorem 1.5. Let α and β be non-zero elements of the group algebra of any torsion-free group over the field with two elements. If $|supp(\alpha)| = 3$ and $\alpha\beta = 0$ then $|supp(\beta)| \geq 20$.

It is known that $\mathbb{F}[G]$ contains a zero divisor if and only if it contains a non-zero element whose square is zero (see [16]). Using the latter fact it is mentioned in [19, p. 691] that it is sufficient to check Conjecture 1.1 only for the case that $|supp(\alpha)| = |supp(\beta)|$, but in the construction that, given a zero divisor produces an element of square zero, it is not clear how the length changes. We clarify the latter by the following.

Proposition 1.6. If $\mathbb{F}[G]$ has no non-zero element α with $|supp(\alpha)| \leq k$ such that $\alpha^2 = 0$, then there exist no non-zero elements $\alpha_1, \alpha_2 \in \mathbb{F}[G]$ such that $\alpha_1\alpha_2 = 0$ and $|supp(\alpha_1)||supp(\alpha_2)| \leq k$.

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Finitely presented algebras of intermediate growth

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Abstract. For any integer $d \geq 1$ we construct examples of finitely presented algebras with intermediate growth of type $[e^{n^{d/(d+1)}}]$. We produce these examples by computing the growth types of some finitely presented metabelian Lie algebras.

1 Introduction

Let A be an (not necessarily associative) algebra over a field k generated by a finite set X , and X^n denote the subspace of A spanned by all monomials on X of length at most n . The *growth function* of A with respect to X is defined by

$$\gamma_{X,A}(n) = \dim_k(X^n)$$

The asymptotic behaviour of this function does not depend on the generating set X and it is called the *growth rate* of A . The growth rate is a widely studied invariant for finitely generated algebraic structures such as groups, semigroups and algebras. There are mainly three types of growth rate for algebraic objects: *exponential*, *polynomial* and *intermediate* (i. e. faster than any polynomial and slower than the exponential function). It is still an open problem whether there exist finitely presented groups of intermediate growth. In contrast, there are examples of finitely presented algebras of intermediate growth. The first such example is the universal enveloping algebra of the Lie algebra W with basis $\{w_{-1}, w_0, w_1, w_2, \dots\}$ and brackets defined by $[w_i, w_j] = (i - j)w_{i+j}$. W is a subalgebra of the generalized Witt algebra $W_{\mathbb{Z}}$. It was proven in [Ste75] that W has a finite presentation with two generators and six relations. The growth rate of W is equivalent to $e^{\sqrt{n}}$.

2010 Mathematical Subject Classification. Primary: 20F69 ; Secondary: 20E08.

Keywords. Finitely presented algebras, Intermediate growth, metabelian Lie algebras.

* Speaker

2 Main Results

All the known examples of finitely presented algebras of intermediate growth have growth types $e^{\sqrt{n}}$ (which is equivalent to the growth rate of the partition function). We construct examples of finitely presented associative algebras of intermediate growth having different growth types. Specifically, our main result is the following:

Theorem 1. For any positive integer d , there exists a finitely presented associative algebra with intermediate growth of type $[e^{n^{d/(d+1)}}]$.

In [Bau77], Baumslag established the fact that every finitely generated metabelian Lie algebra can be embedded in a finitely presented metabelian Lie algebra. Using ideas of [Bau77] (and clarifying some arguments thereof), we embed the free metabelian Lie algebra M (with d generators) into a finitely presented metabelian Lie algebra W^+ and show that universal enveloping algebra of W^+ is a finitely presented associative algebra of growth type $[e^{n^{d/(d+1)}}]$.

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Recent results on intermediate subgroups of classical groups

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Abstract. My talk is devoted to several recent results concerning the description of subgroups of classical groups over commutative rings, containing large subgroups, such as groups of Aschbacher classes, and groups of points of almost simple algebraic groups in irreducible representations.

This talk is based on research within the framework of the Russian Science Foundation project “Decomposition of unipotents in reductive groups” 14-11-00297.

1 Introduction

Recall the context of *intermediate subgroups* discussed in [10, 11, 14]. Let G be a group, D be its subgroup. We are interested in the description of the lattice

$$\mathcal{L}(D, G) = \{H \mid D \leq H \leq G\}$$

of subgroups in G which contain D . We call $H \in \mathcal{L}(D, G)$ a subgroup *intermediate* between D and G , or an *overgroup* of D in G . We are primarily interested in parametrisation of intermediate subgroups and their behaviour with respect

2010 Mathematical Subject Classification. Primary:20G35.

Keywords. Classical groups, Aschbacher classes.

* Speaker

to set-theoretic and group-theoretic operations like intersection, conjugation, passage to the normaliser of H , to the normal closure of D in H , etc.

A natural context to specify many natural classes of *large subgroups* is provided by Michael Aschbacher Subgroup Structure Theorem [2, 5], and its generalisations to exceptional groups established by Martin Liebeck and Gary Seitz.

2 Main Results

Recall, that over arbitrary commutative rings such subgroups are essentially described in the following cases

- Overgroups of subsystem subgroups, classes $\mathcal{C}_1 + \mathcal{C}_2$, Borewicz, Vavilov, Stepanov, Shchegolev, and others, see, for instance, [4, 10, 7] and references there.
- Overgroups of tensored subgroups, classes $\mathcal{C}_4 + \mathcal{C}_5$, by Li Shangzhi, Ananievsky, Vavilov and Sinchuk, see references in [1].
- Overgroups of subring subgroups, class $\mathcal{C}_5$. Following the initial partial results by many authors, finally solved recently by Stepanov and Bak, [8, 9, 3].
- Classical subgroups. class $\mathcal{C}_8$, in full generality solved by You Hong, Petrov and Vavilov, see [12, 13]. Similar result for the overgroups of the Chevalley group of type F_4 inside the Chevalley group of type E_6 was obtained by Luzgarev, [6].

The main new results of this talk, are related to the description of overgroups for maximal classical embeddings, in the last remaining case, odd orthogonal group inside the even orthogonal group. The paper with the full proofs of the following results, and further related results is presently under way.

Let $\mathrm{SO}(n, R)$ be the split orthogonal group of degree n over a commutative ring R , let $\mathrm{GO}(n, R)$ be the corresponding similarity group, and let $\mathrm{EO}(n, R)$ be the corresponding elementary subgroup. Further, let $A \trianglelefteq R$ be an ideal of R , and let $\mathrm{EO}(n, R, A)$ be the corresponding relative elementary subgroup.

For an odd $n = 2l - 1$ we consider the embedding $\mathrm{SO}(2l - 1, R)$ into $\mathrm{GO}(2l, R)$ as a twisted group, and set

$$\mathrm{EOO}(n + 1, R, A) = \mathrm{EO}(n, R) \mathrm{EO}(n + 1, R, A).$$

Further, consider the reduction homomorphism $\rho_A : \mathrm{GO}(n + 1, R) \rightarrow \mathrm{GO}(n + 1, R/A)$ and denote by $\mathrm{CGO}(n + 1, R, A)$ the full preimage of the group $\mathrm{GO}(n, R)$ with respect to ρ_A .

Now, our main result can be stated as follows. Together with the papers [12, 13, 6] this provides a complete solution of the following problem: describe overgroups of subgroups defined by an order 2 folding of root systems.

Theorem 1. Let R be an arbitrary commutative ring and $l \geq 3$. Then for every subgroup H , $\mathrm{EO}(2l - 1, R) \leq H \leq \mathrm{GO}(2l, R)$, there exists a unique ideal $A \trianglelefteq R$ such that

$$\mathrm{EOO}(2l, R, A) \leq H \leq N_{\mathrm{GO}(2l, R)}(\mathrm{EOO}(2l, R, A)).$$

Let us state the following important step towards the proof of the above theorem, providing a description of the normaliser in the right hand side by congruences.

Theorem 2. Under the above assumptions, for any ideal $A \trianglelefteq R$

$$N_{\text{GO}(2l, R)}(\text{EOO}(2l, R, A)) = \text{CGO}(2l, R).$$

Also, we mention some other related results, such as description of subgroups of $\text{GL}(n(n-1)/2, R)$ containing $\bigwedge^2(E(n, R))$, $n \geq 3$, by Roman Lubkov and Ilya Nekrasov.

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The generalization of the homological functors of a bieberbach group

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Abstract. Homological functors are useful tools to reveal the property of a group. Meanwhile, Bieberbach groups are torsion free crystallographic groups. In this paper, some homological functors of a Bieberbach group with symmetric point group of order six are constructed and generalized up to finite dimension. For abelian homological functors, their formulas are presented in the form of product of cyclic groups. For nonabelian homological functors, their generalized presentations are given.

1 Introduction

The homological functors of a group G such as the nonabelian tensor square, $G \otimes G$, the central subgroup of the nonabelian tensor square, $\nabla(G)$ and the nonabelian exterior square, $G \wedge G$, are originated from Algebraic K-theory. The $G \otimes G$ is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g, h, k \in G$ where $g^h = h^{-1}gh$ [1]. It is a specialization of the nonabelian tensor product which is defined as two groups that are acting on each other and their actions are taken to be conjugation.

2010 Mathematical Subject Classification. Primary: 20E05; Secondary: 20J05.

Keywords. Homological functors, Bieberbach group.

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Moreover, $\nabla(G)$ is generated by the elements $x \otimes x$ for $x \in G$ while the $G \wedge G$ is the quotient subgroup of the $G \otimes G$ by the $\nabla(G)$ [1]. The determination of the nonabelian tensor squares of free nilpotent groups of finite rank [2], of polycyclic groups [3], and of Bieberbach groups with dihedral point group of order eight [4] have been done in the past. In addition to that, the exterior squares of some crystallographic groups [5] and of a Bieberbach group with cyclic point group of order two [6] have also been computed.

This paper aims to generalize some homological functors of a Bieberbach group with symmetric point group of order six, denoted as $B_3(n)$, up to dimension n . The consistent polycyclic presentation of $B_3(n)$ is first constructed followed by the generalizations of its homological functors such as its central subgroup of the nonabelian tensor square, $\nabla(B_3(n))$, its nonabelian tensor square, $B_3(n) \otimes B_3(n)$ and its nonabelian exterior square, $B_3(n) \wedge B_3(n)$.

2 Main Results

In this section, the generalization of some homological functors of $B_3(n)$ are presented. First, the consistent polycyclic presentation of $B_3(n)$ is constructed as follows.

Theorem 1. The polycyclic presentation of $B_3(n)$ is consistent where

$$\begin{aligned} B_3(n) = \langle a, b, l_1, l_2, l_3, \dots, l_n \mid & a^2 = l_1 l_4^{-1} l_5, b^3 = l_4^{-2} l_5, b^a = b^2 l_4, \\ & l_1^a = l_1, l_2^a = l_2 l_3, l_3^a = l_3^{-1}, l_4^a = l_5^{-1}, l_5^a = l_4^{-1}, l_p^a = l_p, \\ & l_1^b = l_1, l_2^b = l_3, l_3^b = l_2^{-1} l_3^{-1}, l_4^b = l_4, l_5^b = l_5, l_p^b = l_p, \\ & l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \quad \text{for } 1 \leq i < j \leq n \text{ and } 6 \leq p \leq n \rangle. \end{aligned}$$

Next, the generalizations of $\nabla(B_3(n))$ followed by $B_3(n) \otimes B_3(n)$ and $B_3(n) \wedge B_3(n)$ are presented.

Theorem 2. $\nabla(B_3(n))$

The central subgroup of the nonabelian tensor square of $B_3(n)$, $\nabla(B_3(n))$ is given as

$$\begin{aligned} \nabla(B_3(n)) = \langle & [a, a^\varphi], [b, b^\varphi], [l_p, l_p^\varphi], [a, b^\varphi][b, a^\varphi], [a, l_p^\varphi][l_p, a^\varphi], \\ & [b, l_p^\varphi][l_p, b^\varphi], [l_p, l_q^\varphi][l_q, l_p^\varphi] \text{ for } 6 \leq p < q \leq n \rangle \\ & \cong C_0^{\frac{(n-3)(n-2)}{2}}. \end{aligned}$$

Theorem 3. $B_3(n) \otimes B_3(n)$

The nonabelian tensor square of $B_3(n)$ is nonabelian and is given as follows:

$$\begin{aligned}
B_3(n) \otimes B_3(n) = \langle g_1, g_2, \dots, g_{(n-3)^2+6} \mid & g_4^2 = g_5^2 = [g_8, g_9] = [g_8, g_{10}] = 1, \\
& [g_7, g_8] = g_9^{-2} g_4^{-1}, [g_7, g_9] = g_9^2 g_{10}, [g_7, g_{10}] = g_4 g_{10} g_9^{-1}, \\
& = [g_w, g_k][g_9, g_{10}] = g_4, [g_i, g_j] = [g_t, g_k] = [g_u, g_k] \\
& = [g_v, g_k] = [g_x, g_k] = [g_y, g_k] = [g_z, g_k] = 1, \text{ for} \\
& 1 \leq i \leq 6, 1 \leq j \leq (n-3)^2+6, j > i, \\
& 11 \leq t, u, v, w, x, y, z < k \leq (n-3)^2+6, \text{ and} \\
& k, t, u, v, w, x, y, z \in \mathbb{Z}^+ \rangle
\end{aligned}$$

where

$$\begin{aligned}
g_1 &= a \otimes a, & g_2 &= b \otimes b, & g_3 &= (a \otimes b)(b \otimes a), \\
g_4 &= l_2 \otimes l_3, & g_5 &= l_4 \otimes (l_4^{-1} b^{-1}), & g_6 &= a \otimes (l_4 l_5^{-1}), \\
g_7 &= a \otimes b, & g_8 &= a \otimes l_3, & g_9 &= b \otimes l_2 \\
g_{10} &= b \otimes l_3, & g_t &= l_p \otimes l_p, & g_u &= a \otimes l_p, \\
g_v &= b \otimes l_p, & g_w &= l_p \otimes l_q, & g_x &= (a \otimes l_p)(l_p \otimes a), \\
g_y &= (b \otimes l_p)(l_p \otimes b), & g_z &= (l_p \otimes l_q)(l_q \otimes l_p) & \text{for } 5 \leq p < q \leq n.
\end{aligned}$$

Theorem 4. $B_3(n) \wedge B_3(n)$

The nonabelian exterior square of $B_3(n)$ is nonabelian and is given as follows:

$$\begin{aligned}
B_3(n) \wedge B_3(n) = \langle g_1, g_2, \dots, g_{\frac{(n-3)(n-4)}{2}+6} \mid & g_1^2 = g_2^2 = [g_5, g_6] = [g_5, g_7] = 1, \\
& [g_4, g_5] = g_6^{-2} g_1^{-1}, [g_4, g_6] = g_6^2 g_7, [g_4, g_7] = g_1 g_7 g_6^{-1}, \\
& [g_6, g_7] = g_1, [g_i, g_j] = [g_x, g_k] = [g_y, g_k] = [g_z, g_k] = 1, \text{ for} \\
& 1 \leq i \leq 3, 1 \leq j \leq \frac{(n-3)(n-2)}{2} + 6, j > i, \\
& 8 \leq x, y, z < k \leq \frac{(n-3)(n-2)}{2} + 4, \text{ and} \\
& k, x, y, z \in \mathbb{Z}^+ \rangle,
\end{aligned}$$

where

$$\begin{aligned}
g_1 &= l_2 \wedge l_3, & g_2 &= l_4 \wedge (l_4^{-1} b^{-1}), & g_3 &= a \wedge (l_4 l_5^{-1}), \\
g_4 &= a \wedge b, & g_5 &= a \wedge l_3, & g_6 &= b \wedge l_2 \\
g_7 &= b \wedge l_3, & g_x &= a \wedge l_p, & g_y &= b \wedge l_p, \\
g_z &= l_p \wedge l_q & \text{for } 6 \leq p < q \leq n.
\end{aligned}$$

3 Conclusion

In this paper, the generalization of some homological functors of a Bieberbach group with symmetric point group of order six, denoted as $B_3(n)$ are constructed up to finite dimension. Since the central subgroup of the nonabelian tensor square of $B_3(n)$, denoted as $\nabla(B_3(n))$ is always abelian, its formula is presented in the form of product of cyclic groups. Moreover, both the nonabelian tensor square of $B_3(n)$, $B_3(n) \otimes B_3(n)$ and the nonabelian exterior square of $B_3(n)$, $B_3(n) \wedge B_3(n)$ are found to be nonabelian and their generalized presentations are constructed. These findings can be used to generalize other homological functors of the group such as the Schur multipliers.

Acknowledgement

The first author is indebted to her MyPhD Scholarship.

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A note on conjugacy class graphs of p -singular elements

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Abstract. In this paper we consider the conjugacy class sizes of p -singular elements of a finite group and study the conjugacy class graph associated with these elements in some classes of finite groups.

1 Introduction

It is well known in finite group theory that there is a strong relation between properties such as solvability or nilpotency of a group and the sizes of its conjugacy classes. There exist several known results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. For example, if one knows that there is only one conjugacy class size, then the group is abelian. However if you know the collection of conjugacy class sizes, that is the multiplicities then the order of the group is also known. However it would still not be possible to identify the group. In [3], Burnside proved that if a group G has a conjugacy class with prime power size, then G is not simple. Since then, many authors have investigated the relationship between the structure of a group and its conjugacy class sizes. A classic result by Itô[6] asserts that a group with two conjugacy class sizes is nilpotent. On the other hand, the structure of a normal subgroup N of a group G was given if N is the union of some G -conjugacy classes. It is natural to wonder what information on the structure of N can be obtained from its G -class sizes. The authors in [1] proved the nilpotency of a normal subgroup with exactly two G -conjugacy class sizes. Some efforts have been made in constructing various graphs from the sets of conjugacy class sizes. One of this graphs is conjugacy class graph related to a finite group G that its vertex set is the set of non-central classes of G , and two distinct vertices C and D are connected by an edge if and only if their class lengths have a non-trivial common divisor. This graph is denoted by $\Gamma(G)$. This graph has been

2010 Mathematical Subject Classification. Primary:20E45; Secondary: 05C25

Keywords. conjugacy class, p -singular element, size

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widely studied.(see [2],[4]).

The notation we use is standard. All groups considered in this paper are finite. Let G be a finite group, x an element of G . x^G denotes the conjugacy class containing x and $|x^G|$ denotes the size of x^G . We denote the center of G by $Z(G)$. Let p be a prime number. We say that x is a p -singular element if p divides $o(x)$, the order of the element x . Also we denote the graph related to the set of conjugacy class sizes of non-central p -singular elements of G by $\Gamma(G_p)$. In this paper we are interested in conjugacy class graphs of p -singular elements.

2 Main Results

We list some lemmas which are useful in the sequel.

Lemma 2.1.[5] Let N be a normal subgroup of a group G and x an element of G . Then

- (a) $|x^N|$ divides $|x^G|$;
- (b) $|(Nx)^{G/N}|$ divides $|x^G|$.

Lemma 2.2. [5] Let N be a normal subgroup of a group G and $N \cap Z(G) = 1$. If $|x^G| = 1$ or m for every element x of N , then N is abelian.

Theorem 2.3. Let G be one of the following groups and p a prime divisor of $|G|$:

- (i) $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$; (n odd or $4 \mid n$)
- (ii) $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$; (n even or n and p both odds)
- (iii) $V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle$; (n odd)
- (iv) $U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$ (n odd and $p \nmid n$).

Then $\Gamma(G_p)$ is complete.

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On a group of the form $2^{4+5}:GL(4, 2)$

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Abstract. The affine general linear group $2^5:GL(5, 2)$ of $GL(6, 2)$ has a maximal subgroup, of index 31, of the form $2^{4+5}:GL(4, 2) := \overline{G}$. In this paper we use the coset analysis method and Clifford-Fischer theory to determine the conjugacy classes and the ordinary character table of $\overline{G}$. For more details on the techniques of the coset analysis and Clifford-Fischer theory, see [2, 3, 4]. This talk is based on [1] and the full work will be in [1].

1 Introduction

With the help of GAP [7], we were able to determine the structures of all the maximal subgroups of the affine general linear group $2^5:GL(5, 2)$ of $GL(6, 2)$. This affine group has 6 classes of maximal subgroups and the largest two maximal subgroups are of the forms $2_+^{1+8}:GL(4, 2)$ and $2^{4+5}:GL(4, 2)$. In this article we consider the group $2^{4+5}:GL(4, 2) := \overline{G}$, where we firstly determine its conjugacy classes using the coset analysis technique. The structures of inertia factor groups are also determined. These are the groups $H_1 = H_2 = GL(4, 2)$, $H_3 = H_4 = H_5 = 2^3:GL(3, 2)$ and $H_6 = 2^4:D_{12}$. The determination of the Fischer matrices is in progress, where these matrices have sizes range between 2 and 12. The character table of $\overline{G}$ is a 75×75 complex matrix and is partitioned into 84 blocks correspond to the six inertia factor groups and the 14 classes of $GL(4, 2)$. This is a very good example for the applications of Clifford-Fischer theory since the kernel of the extension is a non-abelian group. Not many examples of this type have been studied via

2010 Mathematical Subject Classification. Primary: 20C15; Secondary: 20C40

Keywords. character table, inertia groups, Fischer matrices

* Speaker

Clifford-Fischer theory. For the notation used in this paper and the description of Clifford-Fischer theory, we follow [2, 3, 4, 5].

In [1] two generators α and β of $\overline{G}$ (in terms of 6×6 matrices over $\mathbb{F}_2$) were supplied, where $o(\alpha) = 4$ and $o(\beta) = 6$ with $o(\alpha\beta) = 10$. We found that $\overline{G}$ has 5 non-trivial proper normal subgroups of orders 16, 32, 256, 512 and 5160960. Let N be the normal subgroup of order 512. One can check that $Z(N) = \Phi(N) = N' \cong 2^4$, the elementary abelian group of order 16, where $Z(N)$, $\Phi(N)$ and N' are the center, Frattini and derived subgroups of N respectively. Further the quotient group $N/Z(N)$ is an elementary abelian group of order 32. Thus N is a special 2-group of order 512 of the form 2^{4+5} . Using GAP, it will also be easy to see that $Z(N)$ has no complement in N and thus N can also be written in the form $N = 2^4 \cdot 2^5$ (non-split). It has 271 involutions and 240 elements of order 4. It also has 152 conjugacy classes as follows: singleton conjugacy class consisting of the identity element; 15 singleton classes, each of which consists of a central involution; one conjugacy class consists of 16 non-central involutions; 120 conjugacy classes, each of which consists of two non-central involutions and 15 conjugacy classes, each of which consists only of 16 elements of order 4. The *coset analysis* has been used in [1] to determine the classes of $\overline{G}$. As an example of the coset analysis, consider the identity coset $N1_G$, where $G = GL(4, 2)$. The action of N on $N1_G$ produces the above classes of N . Now the action of $C_G(1_G)$ on these 152 orbits leaves invariant the identity class and the class consists of 16 non-central involutions, while fuses the 15 singleton classes (each of which consists of a central involution) into one orbit; out of 120 classes (each of which consists of two non-central involutions) 15 classes fuse together to form a new orbit, and the remaining 105 classes fuse together to form another orbit; and finally the 15 conjugacy classes (each of which consists only of 16 elements of order 4) into a single orbit. Thus in $\overline{G}$, we get 6 conjugacy classes of sizes 1, 15, 16, 30, 210 and 240 correspond to the identity coset. In Table 1 we give an example of the conjugacy classes of $\overline{G}$ correspond to the classes 1A and 7A of G . Correspond to the 14 conjugacy classes of $G = GL(4, 2)$, we obtained 75 classes of $\overline{G}$, listed in Table

Table 1: Some conjugacy classes of $\overline{G}$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$ [g_{ij}]_{\overline{G}} $	$ C_{\overline{G}}(g_{ij}) $
$g_1 = 1A$	$k_1 = 152$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	10321920
		$f_{12} = 15$	$m_{12} = 15$	g_{12}	2	15	688128
		$f_{13} = 1$	$m_{13} = 16$	g_{13}	2	16	645120
		$f_{14} = 15$	$m_{14} = 30$	g_{14}	2	30	344064
		$f_{15} = 105$	$m_{15} = 210$	g_{15}	2	210	49152
		$f_{16} = 15$	$m_{16} = 240$	g_{16}	4	240	43008
$g_{11} = 7A$	$k_{11} = 5$	$f_{11,1} = 1$	$m_{11,1} = 64$	$g_{11,1}$	7	184320	56
		$f_{11,2} = 1$	$m_{11,2} = 64$	$g_{11,2}$	14	184320	56
		$f_{11,3} = 1$	$m_{11,3} = 128$	$g_{11,3}$	14	368640	28
		$f_{11,4} = 1$	$m_{11,4} = 128$	$g_{11,4}$	14	368640	28
		$f_{11,5} = 1$	$m_{11,5} = 128$	$g_{11,5}$	28	368640	28

1 of [1]. The number of $\overline{G}$ -classes that correspond to a class of G is ranging between 2 and 12.

2 Inertia factors and Fischer matrices of $\overline{G}$

We have $|\text{Irr}(N)| = 152$, where 32 characters are linear, while all the other 120 characters have degrees 2. We have seen that the action of G on N produced 6 orbits (of lengths 1, 15, 16, 30, 210 and 240). By a theorem of Brauer (see for example Theorem 5.1.1 of [5]), it follows that the action of G on $\text{Irr}(N)$ will also produce 6 orbits. With the

help of GAP, we found that the orbit lengths of G on $\text{Irr}(N)$ are 1, 1, 15, 15, 15 and 105. By checking the maximal subgroups of $GL(4, 2)$ (see the ATLAS [6]) and also with the help of GAP, we infer that the inertia factor groups are: $H_1 = H_2 = GL(4, 2)$, $H_3 = H_4 = H_5 = 2^3:GL(3, 2)$ and $H_6 = 2^4:D_{12}$. By application of Theorem 5.1.8 of [5] we know that all the linear characters are extendible to their respective inertia groups. Thus for the construction of the character table of $\overline{G}$ we will use the ordinary characters of H_1 , H_2 , H_3 and H_4 , while for H_5 and H_6 we have to use projective character tables. Using GAP, we computed the *Schur multipliers* $M(H_5)$ and $M(H_6)$ and we found both to be 2^3 . The next step is to form the *covering groups* $M(H_5) \cdot H_5$ and $M(H_6) \cdot H_6$ (central extension). To find all the projective characters of H_5 and H_6 , it is sufficient to find $\text{Irr}(2^3 \cdot H_5)$ and $\text{Irr}(2^3 \cdot H_6)$. Since $|\text{Irr}(H_1)| = |\text{Irr}(H_2)| = 14$ and $|\text{Irr}(H_3)| = |\text{Irr}(H_4)| = 11$, we must have

$$|\text{IrrProj}(H_5, \alpha^{-1})| + |\text{IrrProj}(H_6, \beta^{-1})| = |\text{Irr}(\overline{G})| - \sum_{i=1}^4 |\text{Irr}(H_i)| = 25, \quad (1)$$

where α and β are factor sets of $M(H_5)$ and $M(H_6)$ respectively. Since $|\text{Irr}(H_5)| = 11$ and $|\text{Irr}(H_6)| = 14$, Equation (1) suggests that we can use the ordinary character tables of H_5 and H_6 , but there is no reason to let us assume this. We have to determine all the projective character tables of H_5 and H_6 and eliminate those cases that lead to contradictions.

We used the arithmetical properties of Fischer matrices (see Proposition 2 of [3]) to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. With the help of the symbolic mathematical package Maxima [8], we were able to solve these systems of equations for many of the Fischer matrices and hence we have computed most of the Fischer matrices of $\overline{G}$. The computation of the few remaining Fischer matrices is in progress. Below is an example of the Fischer matrices of $\overline{G}$ correspond to the classes $1A$ and $7A$ of G .

$\mathcal{F}_1$						
$g_1 = 1A$	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
$o(g_{1j})$	1	2	2	2	2	4
$ C_{\overline{G}}(g_{1j}) $	10321920	688128	645120	344064	49152	43008
(k, m)	$ C_{H_k}(g_{1km}) $					
$(1, 1)$	20160	1	1	1	1	1
$(2, 1)$	20160	1	1	-1	1	-1
$(3, 1)$	1344	15	15	15	-1	-1
$(4, 1)$	1344	15	15	-15	-1	1
$(5, 1)$	1344	30	-2	0	14	-2
$(6, 1)$	192	210	-14	0	-14	2
m_{1j}	1	1	15	15	15	105

$\mathcal{F}_{11}$					
$g_{11} = 7A$	$g_{11,1}$	$g_{11,2}$	$g_{11,3}$	$g_{11,4}$	$g_{11,5}$
$o(g_{11j})$	7	14	14	14	28
$ C_{\overline{G}}(g_{11j}) $	56	56	28	28	28
(k, m)	$ C_{H_k}(g_{11km}) $				
$(1, 1)$	7	1	1	1	1
$(2, 1)$	7	1	1	-1	-1
$(3, 1)$	7	1	1	1	-1
$(4, 1)$	7	1	1	-1	1
$(5, 1)$	7	2	-2	0	0
m_{11j}	64	64	128	128	128

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The connectivity of commuting graphs $\mathcal{C}(G, X)$ in symmetric groups $\text{sym}(n)$

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Abstract. Commuting graph is a graph denoted by $\mathcal{C}(G, X)$ where G is any group and X , a subset of a group G , is a set of vertices for $\mathcal{C}(G, X)$. Two distinct vertices, $x, y \in X$ will be connected by an edge if the commutativity property is satisfied or $xy = yx$. This study presents results for the connectivity of $\mathcal{C}(G, X)$ when G is a symmetric group of degree n , $\text{Sym}(n)$, and X is a conjugacy class of elements of order three in G .

1 Introduction

We assume $G = \text{Sym}(\Omega) = \text{Sym}(n)$ acts on the set $\Omega = \{1, \dots, n\}$ in the usual manner. Write, without loss of generality,

$$t = (1, 2, 3)(4, 5, 6)(7, 8, 9) \dots (3r - 2, 3r - 1, 3r).$$

Thus t has order 3 and is of cycle type $1^{n-3r}3^r$. Set $X = t^G$, the G -conjugacy class of t . Evidently, the centralizer of t in G is

$$C_G(t) \cong (3^r : \text{Sym}(r)) \times \text{Sym}(n - 3r).$$

For $g \in G$ we can write g as a product of disjoint cycles $g_1, \dots, g_r$ and we denote the cycle type of g by a string $c_1^{m_1} \dots c_s^{m_s}$ of distinct positive integers c_i and positive integers m_i such that m_i is number of cycles of length c_i

2010 Mathematical Subject Classification. Primary: 20F65; Secondary: 20D05.

Keywords. Commuting graph, Symmetric group, Order 3 elements.

* Speaker

occurring in the decomposition $g_1 \dots g_r$ (including fixed points as cycles of length 1). Let S be a set of positive integers. Then $\text{lcm } S$ will denote the smallest positive integer which is divisible by all $s \in S$ and $\text{gcd } S$ will denote the greatest positive integer which divides all $s \in S$. If P is a set of subsets of Ω and S a subset of Ω , then $P \cap S$ will denote the set of those elements of P contained in S .

We now define a graph which depends on the cycle type of an element $a \in G$. Let $a \in \text{Sym}(n)$ of cycle type $e_1^{f_1} \dots e_m^{f_m}$. Then let Γ be a graph with vertices $\{1, \dots, m\}$ and (i, j) (for $i \neq j$) is an edge if and only if $e_i h_i = e_j h_j$, for some $1 \leq h_i \leq f_i$ and $1 \leq h_j \leq f_j$. An edge of Γ is *exact* if this is only possible with $h_i = f_i$ and $h_j = f_j$. For $1 \leq i \leq m$, let

$$b(i) := \frac{e_i}{\text{lcm}\{d : d \mid e_i, d \leq f_i\}}.$$

An edge (i, j) of Γ is *special* with source i if $e_j f_j = e_i$ and $b(i) = e_i$. If V is a set of vertices of Γ , then $\Gamma \setminus V$ will denote the subgraph of Γ with all vertices of Γ not contained in V and all edges of Γ between vertices not in V . Furthermore, if E is a set of edges of Γ , then $\Gamma \setminus E$ will denote the subgraph of Γ with the same vertices as Γ and all edges of Γ except those in E .

Given this notation, it is worth noting the following two results in [3] as these are extremely useful to prove the connectivity of commuting graphs in symmetric groups for elements of order three.

Theorem 1 (Bundy). Let $G = \text{Sym}(n)$, $a \in G$ be of cycle type e^f , and $X = a^G$. Then $\mathcal{C}(G, X)$ is connected if and only if $b(1) = 1$, or $e \leq 3$ and $f = 1$.

Theorem 2 (Bundy). Let $G = \text{Sym}(n)$, $a \in G$ be of cycle type $e_1^{f_1} \dots e_m^{f_m}$ with $m > 1$. Let $X = a^G$. Then $\mathcal{C}(G, X)$ is connected if and only if the following hold:

1. Γ is connected.
2. $\text{gcd } \{b(i) : 1 \leq i \leq m\} = 1$.
3. Γ has at least one edge which is not exact.
4. The vertex set of Γ is **not** of the form $E \cup Y$, with $E \cap Y = \emptyset$ and $E, Y \neq \emptyset$, such that the following hold:
 - (a) for all $i, j \in E$ with $i \neq j$, (i, j) is an exact edge,
 - (b) there exists a vertex $y \in Y$ such that for all $i \in E$, (i, y) is a special edge with source y ,
 - (c) no vertex of E is joined to a vertex of $Y \setminus \{y\}$, and
 - (d) $\text{gcd } \{b(i) : i \in Y\} = e_y$.

Evidently, there are certain cases where $\mathcal{C}(G, X)$ is not connected or disconnected. Consequently, there will be some questions arisen when $\mathcal{C}(G, X)$ is not connected: for example, how many connected components are there? What is the size or diameter of the connected components? One thing for sure is that the connected components will be isomorphic graphs. Therefore, in this study, we also attempt to give some information about the number of connected components of some specific disconnected graphs of symmetric groups and also the size of each connected components.

2 Main Results

By using Theorem 1 and 2 where appropriate, we can now put these to good use to partially prove the following result (note that this observation is based on results in Table 1 of [1] and Table 1 of [4]):

Theorem 3. Let $G = \text{Sym}(n)$ and $t \in G$ be of cycle type 3^r with $r \geq 1$. Let $X = t^G$. Then $\mathcal{C}(G, X)$ is disconnected if and only if one of the following holds:

1. $n = 3r$ and $r = 2$,
2. $n = 3r + 1$ or $n = 3r + 2$ and $r \geq 1$,
3. $n = 3r + q$; $r = 1$ or $r = 2$ and $q = 3$.

Based on the calculations which were carried out by hand and also with the help of Magma [2], we give a general formula of size and number of connected components for some disconnected $\mathcal{C}(G, X)$ as follows:

Theorem 4. If t has cycle type 3^r and $n = 3r + 1$ or $n = 3r + 2$, then the size of each connected components is 2^r . Moreover, the number of connected components is $\frac{|X|}{2^r}$.

3 Conclusion

We can extend the investigation of finding the connectivity of commuting graphs $\mathcal{C}(G, X)$ for varieties of G , such as dihedral groups or simple groups listed in Atlas [5]. Different subset X can also be considered, for instance, involutions class of G or any class of other prime order elements.

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Cyclic subgroup separability of certain generalized free products

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Abstract. In this paper we show that certain generalized free products amalgamating the subgroup of the form $\langle h \rangle \times D$, where D is a finite central subgroup of the factor groups, to be cyclic subgroup separable, and hence residually finite. Furthermore, we will show that the tree products of certain cyclic subgroup separable groups amalgamating this type of subgroups to be cyclic subgroup separable. We then apply our results to free-by-finite groups, polycyclic-by-finite groups and finitely generated nilpotent groups.

1 Introduction

In [2], Kim and Tang have found some conditions that the generalised free products of conjugacy separable groups amalgamating $\langle h \rangle \times D$, where D is in the centre of both factors to be conjugacy separable. Hence, by using this, they proved that the generalised free products of finitely generated nilpotent groups amalgamating $\langle h \rangle \times D$, where D is in the centre of both factors, are conjugacy separable.

Note that Zhou and Kim in [3] have actually shown the cyclic subgroup separability and subgroup separability of generalized free product of subgroup separable groups amalgamating subgroup of the form $\langle h \rangle \times D$, where D is in the center of both factors. They used this to show that the generalized free product of nilpotent-by-finite groups amalgamating this type of subgroup to be cyclic subgroup separable and subgroup separable.

In this paper, we show that the generalized free products of subgroup separable groups to be π_c by using the criterion (Theorem 1.1). Furthermore, we show that their tree product is also π_c . Then, we apply our results to free-by-finite groups, polycyclic-by-finite groups and finitely generated nilpotent groups.

2010 Mathematical Subject Classification. Primary: 20E06, 20E26; Secondary: 20F10

Keywords. generalized free product, tree product, residually finite

* Speaker

In [1], Kim has proved the following theorem (Theorem 1.1) to show the cyclic subgroup separability of generalized free product of groups. Since then it has been used as criterion by many researchers to show the cyclic subgroup separability of generalized free product. Thus we shall use it to prove our results.

Theorem 1. [1, Proposition 1.2] Let $G = A *_H B$. Suppose that,

- (i) A and B are both π_c and H -separable;
- (ii) for each $R \triangleleft_f H$, there exists $M_A \triangleleft_f A$ and $M_B \triangleleft_f B$ such that $M_A \cap H = M_B \cap H \subset R$

Then G is π_c .

2 Main Results

The following are our main results.

Theorem 2. Let $G = A *_H B$, where A and B are subgroup separable groups. Let $H = \langle h \rangle \times D$ such that $|h| = \infty$ and $D \subset Z(A) \cap Z(B)$ is finite. Suppose A and B are $\langle h \rangle$ -wpot. Then G is π_c , and hence residually finite.

Then we extend the above Theorem 2.1 to tree product in the following theorem.

Theorem 3. Let $G = \langle G_1, G_2, \dots, G_n \mid H_{ij} = H_{ji} \rangle$ be a tree product of $G_1, G_2, \dots, G_n$ amalgamating the subgroups $H_{ij} < G_i$ and $H_{ji} < G_j$. Suppose each $H_{ij} = \langle h_{ij} \rangle \times D_{ij}$ where $|h_{ij}| = \infty$ and $D_{ij} \subset Z(G_i) \cap Z(G_{i+1})$ is finite. Suppose each G_i is subgroup separable and $\langle h_{ij} \rangle$ -wpot. Then G is π_c .

3 Conclusion

Now we know that the generalized free product of subgroup separable groups amalgamating subgroup of the form $\langle h \rangle \times D$, where D is in the centre of both factors are subgroup separable and π_c , hence residually finite. We also seen that their tree products are π_c . Now the natural questions arise: Does this generalized free product is still π_c and hence residually finite when we have different types of D ? Can we extends this result to polygonal products or even to fundamental groups of graphs of groups?

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On the nilpotent conjugacy class graph of groups

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Abstract. For a group G , let $\Gamma(G)$ be the nilpotent conjugacy class graph of G associated with the non-trivial conjugacy classes of G in such a way that the vertices of $\Gamma(G)$ are the non-trivial conjugacy classes of G , and two distinct vertices C and D are adjacent when $\langle x, y \rangle$ is nilpotent for some $x \in C$ and $y \in D$. The aim of this talk is twofold. First, we proved that if G is a periodic solvable group, then $\Gamma(G)$ has at most two components, each of diameter at most 7. If G is any locally finite group, then $\Gamma(G)$ has at most 6 components, each of diameter at most 10. Second, we investigate the possible structure of a (supersolvable) solvable group G , whenever $\Gamma(G)$ is disconnected.

1 Introduction

There are many ways a graph is associated with conjugacy classes of a group. In 2009, Herzog, Longobardi and Maj [1] introduced the *commuting conjugacy class graph* $\Delta(G)$ of G associated with the non-trivial conjugacy classes of G . The vertices of $\Delta(G)$ are the non-trivial conjugacy classes of G and two distinct vertices C and D are adjacent whenever there exist two elements $x \in C$ and $y \in D$ such that $xy = yx$. They investigate connectivity of $\Delta(G)$ for periodic and locally finite group.

In this talk, we define new graph on the non-trivial conjugacy classes of G . The vertices of $\Gamma(G)$ are the non-trivial conjugacy classes of G and two distinct vertices C and D are adjacent whenever there exist two elements $x \in C$ and $y \in$

2010 Mathematical Subject Classification. Primary:05C25; Secondary: 20E45.

Keywords. Conjugacy classes, Graph, Locally finite group, Periodic group.

* Speaker

D such that $\langle x, y \rangle$ is nilpotent. We prove that if G is a periodic solvable group, then $\Gamma(G)$ has at most two components, each of diameter at most 7. More generally, we show that if G is any locally finite group (not necessarily solvable), then $\Gamma(G)$ has at most 6 components, each of diameter at most 10. Finally, we characterize finite (supersolvable) solvable groups G with a disconnected graph $\Gamma(G)$.

2 Main Results

Lemma 1. Let G be a locally finite group and p is a prime number. Then the following statements hold:

- (1) If $x, y \in G \setminus \{1\}$ are p -elements, then $d(x, y) \leq 1$.
- (2) If $x, y \in G \setminus \{1\}$ are of noncoprime orders, then $d(x, y) \leq 3$. Moreover $d(x, y) \leq 2$, whenever either x or y is of prime power order.

Lemma 2. Let G be a locally finite group and $x, y \in G$. Suppose that $p \parallel |x|$ and $q \parallel |y|$, where p, q are distinct primes and G has an element, say z , of order pq . Then we have:

- (1) $d(x, y) \leq 5$ and moreover, $d(x, y) \leq 4$ if either x or y is of prime power order.
- (2) If either a Sylow p -subgroup or a Sylow q -subgroup of G is a cyclic or generalized quaternion finite group, then $d(x, y) \leq 4$. Moreover, $d(x, y) \leq 3$ if either x or y is of prime order.
- (3) If both a Sylow p -subgroup and a Sylow q -subgroup of G are either cyclic or generalized quaternion finite groups, then $d(x, y) \leq 3$. Moreover, $d(x, y) \leq 2$ if either x or y is of prime order.

Lemma 3. Let G be a finite group with a normal subgroup H and a subgroup K such that $\Gamma(H)$ and $\Gamma(K)$ are connected. If there exist two elements $h \in H \setminus \{1\}$ and $x \in HK \setminus H$ such that h^{HK} and x^{HK} are connected in $\Gamma(HK)$, then $\Gamma(HK)$ is connected.

If G is a finite group, then the graph $\Pi(G)$ is obviously related to the so-called prime graph G , whose vertices are the primes dividing the order of G and two vertices p and q are joined by an edge if and only if G contains an element of order pq . This graph was first investigated by Gruenberg and Kegel in an unpublished manuscript, and then by many authors (see for instance [2, 3]). We show in following Theorem that given a finite group G , the graphs $\Pi(G)$ and $\Gamma(G)$ have the same number of connected components. Thus known results about the graph $\Pi(G)$ can be applied for the investigation of the graph $\Gamma(G)$.

Theorem 1. Let G be a finite group and π_i , $1 \leq i \leq r$, be the connected components of the prime graph $\Pi(G)$. Then the set $C_i = \{x^G \mid x \text{ is a } \pi_i\text{-element}\}$, $1 \leq i \leq r$, are the connected components of the graph $\Gamma(G)$.

Theorem 2. Let G be a finite solvable group and p, q, r are distinct prime divisors of $|G|$. Then $\pi(G) \cap \{pq, pr, qr\} \neq \emptyset$

Theorem 3. Let G be a finite solvable group. Then $\Gamma(G)$ has at most two connected components and diameter of each is at most 7.

Theorem 4. Let G be a periodic solvable group. Then $\Gamma(G)$ has at most two connected components and diameter of each is at most 7.

Theorem 5. Let G be a finite group. Then $\Gamma(G)$ has at most six connected components and diameter of each is at most 10.

Theorem 6. Let G be a locally finite group. Then $\Gamma(G)$ has at most six connected components and diameter of each is at most 10.

Lemma 4. Let G be a group and H a nilpotent subgroup of G in which $C_G(x) \leq H$ for every $x \in H \setminus \{1\}$. Then $\text{nil}_G(x) = H$ for every $x \in H \setminus \{1\}$.

Theorem 7. Let G be a supersolvable group. Then the graph $\Gamma(G)$ is disconnected if and only if G has one of the following structure:

- (1) If G is an infinite group, then $G = H \rtimes \langle x \rangle$, where $x \in G$, $|x| = 2$ and H is a subgroup of G on which x acts fixed point freely.
- (2) If G is a finite group, then $G = H \rtimes K$ is a Frobenius group with kernel H and a cyclic complement K .

Theorem 8. Let G be a finite solvable group with a disconnected graph $\Gamma(G)$. Then there exists a nilpotent normal subgroup H of G such that one of the following holds:

- (1) $G = H \rtimes K$ is a Frobenius group with the kernel H and a complement K ,
- (2) $G = (H \rtimes L) \rtimes \langle x \rangle$, where L is a nontrivial cyclic subgroup of G of odd order which acts fixed point freely on H , $x \in N_G(L)$ is such that $\langle x \rangle$ acts fixed point freely on L , and there exist $h_1 \in H \setminus \{1\}$ and $i \in \mathbb{N}$ such that $x^i \neq 1$, which satisfy $[x^i, h_1] = 1$.

Conversely, if either (1) or (2) holds, then $\Gamma(G)$ is disconnected.

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Some results on central autoisoclinism of groups

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Abstract. The concept of isoclinism of groups introduced by P. Hall in 1940, and extended to n -isoclinism by N. Hekster in 1986. Also the first author et. al. in 2015 introduced the notion of autoisoclinism and studied some of its properties.

In this talk we introduce and discuss a new concept of central autoisoclinism. Note that the concept of central autoisoclinisms yields an equivalence relation on the class of all groups and we call each equivalence class a central autoisoclinism family of groups.

Also, we investigate the internal structure of central autoisoclinism families of groups.

1 Introduction

Let G be a finite group then the *autocommutator* of the element g in G and the automorphism α in $\text{Aut}(G)$ is defined to be

$$[g, \alpha] = g^{-1}g^\alpha = g^{-1}\alpha(g).$$

2010 Mathematical Subject Classification. Primary: 20E10; Secondary: 20E12.

Keywords. Central autoisoclinism, autoisoclinism, isoclinism.

* Speaker

Using this definition, the subgroup

$$K(G) = \langle [x, \alpha] : x \in G, \alpha \in \text{Aut}(G) \rangle$$

is called the *autocommutator subgroup* of G . The concept of autocommutator subgroups has been already studied in [3]. Also

$$L(G) = \{g \in G : [g, \alpha] = 1, \forall \alpha \in \text{Aut}(G)\},$$

is called the *autocentre* of G . Clearly if α runs over the inner automorphisms of G , then $K(G)$ and $L(G)$ will be the commutator subgroup, G' , and the centre of G , $Z(G)$, respectively. One notes that, $K(G)$ and $L(G)$ are characteristic subgroups of G (see [3, 6] for more information).

An automorphism α of G is called *central* if $x^{-1}\alpha(x)$ is in the centre of G , for all $x \in G$. The set of all central automorphisms of G is a normal subgroup of $\text{Aut}(G)$, denoted by $\text{Aut}_c(G)$. Note that $\text{Aut}_c(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$, the centralizer of the inner automorphisms in the automorphisms group of G .

In 1955, Franklin Haimo [6] introduced the following subgroup of a given group G , which we denote it by $L_c(G)$,

$$L_c(G) = \{x \in G : [x, \alpha] = x^{-1}x^\alpha = 1, \forall \alpha \in \text{Aut}_c(G)\}.$$

Clearly, $L_c(G)$ is a normal subgroup of G and contains $L(G)$.

One may also define the *autocentral commutator subgroup* of G as follows:

$$K_c(G) = \langle [x, \alpha] : x \in G, \alpha \in \text{Aut}_c(G) \rangle.$$

It is easy to see that $K_c(G)$ is a central subgroup of G , which is contained in $K(G)$.

One can easily check that the central automorphisms of a group G fix the commutator subgroup pointwise, and hence $G' \subseteq L_c(G)$. This shows that $G/L_c(G)$ is abelian and if $L_c(G)$ is trivial, then G is abelian.

In the following lemmas we provide some preliminary properties of the new notions $L_c(G)$ and $K_c(G)$.

Lemma 1. Let G be any group, then $L_c(G) \subseteq C_G(Z_2(G))$.

The following lemma is very useful for our further investigations.

Lemma 2. Let x and y be elements of a group G and $\alpha, \beta \in \text{Aut}(G)$. Then the following identities hold:

- (a) $[xy, \alpha] = [x, \alpha]^y[y, \alpha]$;
- (b) $[x, \alpha^{-1}] = ([x, \alpha]^{-1})^{\alpha^{-1}}$;
- (c) $[x^{-1}, \alpha] = ([x, \alpha]^{-1})^{x^{-1}}$;
- (d) $[x, \alpha\beta] = [x, \alpha][x, \beta]^\alpha = [x, \alpha][x, \beta][x, \beta, \alpha]$;
- (e) $[x, \alpha]^\beta = [x^\beta, \alpha^\beta]$;
- (f) $[x, \alpha^{-1}, \beta]^\alpha [\alpha, \beta^{-1}, x]^\beta [\beta, x^{-1}, \alpha]^\alpha = 1$.

The above lemma gives the following result.

Lemma 3. Let G be any group, then $L_c(G)$ and $K_c(G)$ are both characteristic subgroups.

Definition 1. For a normal subgroup N of a given group G , we define a subgroup of G generated by the following set

$$\{[gn, \alpha][g, \alpha]^{-1} : \text{for all } g \in G, n \in N, \alpha \in \text{Aut}_c(G)\},$$

and denoted by T_N .

The following lemma gives more properties of $L_c(G)$ and $K_c(G)$ of a group G .

Lemma 4. Let N be a normal subgroup of a group G , then the following properties hold.

- (a) $K_c(G) = 1 \Leftrightarrow L_c(G) = G$;
- (b) $T_N = 1 \Leftrightarrow N \subseteq L_c(G)$;
- (c) $[K_c(G), L_c(G)] = 1$;
- (d) If N is characteristic in G and $N \cap K_c(G) = 1$, then $N \subseteq L_c(G)$.

2 Main results

In this section we introduce a new concept of central autoisoclinism between two groups, which is vital in our investigations.

Definition 2. A *central autohomoclinism* between two groups G and H is a binary homomorphisms $(\alpha \times \gamma, \beta)$, where $\alpha : G/L_c(G) \rightarrow H/L_c(H)$, $\beta : K_c(G) \rightarrow K_c(H)$ and $\gamma : \text{Aut}_c(G) \rightarrow \text{Aut}_c(H)$ are homomorphisms such that the following diagram is commutative:

$$\begin{array}{ccc} \frac{G}{L_c(G)} \times \text{Aut}_c(G) & \xrightarrow{\alpha \times \gamma} & \frac{H}{L_c(H)} \times \text{Aut}_c(H) \\ \downarrow & & \downarrow \\ K_c(G) & \xrightarrow{\beta} & K_c(H). \end{array}$$

We also say that $(\alpha \times \gamma, \beta)$ is a *central autoisoclinism* between G and H if $\alpha \times \gamma$ is surjective and β and γ are injectives. In this case, we say that G and H are *central autoisoclinic* and denoted by $G \stackrel{\sim}{\sim} H$.

Note that, if one replaces the automorphisms by inner automorphisms of the group then the notion of homoisoclinism is obtained (see P. Hall [2]). Moreover, if one replaces central automorphisms by automorphisms, one obtains the notion of autoisoclinism (see [6] for more detail).

In the following we give an example of two groups, which are central autoisoclinic, but they are neither isoclinic nor autoisoclinic.

Example 1. Let $D_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ be the Dihedral group of order 16 and the Quaternion group of order 8, respectively. One can calculate that

$$\text{Aut}_c(D_{16}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2, L_c(D_{16}) = \{1, a^2, a^4, a^6\} \text{ and } K_c(D_{16}) = \{1, a^4\}.$$

Moreover,

$$\text{Aut}_c(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2, L_c(Q_8) = K_c(Q_8) = \{1, a^2\}.$$

Using Definition 2, one can see that $D_{16} \sim Q_8$. It is easily checked that these two groups are neither isoclinic nor autoisoclinic.

In the following under some conditions, we show that each group G is a central autoisoclinic with its factor group.

Lemma 5. Let N be a characteristic subgroup of a group G with $N \cap K_c(G) = \langle 1 \rangle$. Then there exists a monomorphism between $Aut_c(G)$ and $Aut_c(G/N)$.

Theorem 1. [7, Theorem 1] Let G be a p -group with no abelian direct factor. Then

$$|Aut_c(G)| = \prod_{i=1}^e |Z_i|^{r_i},$$

where e is the exponent of G/G' , and $r_1, r_2, \dots, r_e$ are the invariants of G/G' , and Z_i is the subgroup of $Z(G)$, whose orders divide p^i , $i = 1, 2, \dots, e$.

Using the above theorem, we have the next useful result.

Lemma 6. Let N be a characteristic subgroup of finite group G with G and G/N are both purely non-abelian. If $N \cap K_c(G) = \langle 1 \rangle$, then

- (a) $Aut_c(G) \cong Aut_c(G/N)$;
- (b) $K_c(G/N) = K_c(G)N/N$;
- (c) $L_c(G/N) \cong L_c(G)/N$.

Now, using the above lemma we are able to deduce the following corollaries.

Corollary 1. Let N be a characteristic subgroup of a finite group G with G and G/N are both purely non-abelian and N has trivial intersection with the subgroup $K_c(G)$. Then $G \sim G/N$.

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On the generalized conjugacy class graph of 3-generator 2-groups

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Abstract. The generalized conjugacy class graph is defined as a graph whose vertices are non-central conjugacy classes where two vertices are connected if they are not coprime. In this paper, the generalized conjugacy class graph of some 3-generator 2-groups is determined. Furthermore, some graph properties including the independent number, chromatic number, clique number and dominating number are also found.

1 Introduction

Graphs are widely used in our real life, for example in network problem. Jasmine in [1] studied on the connectivity and coverage in hybrid wireless sensor networks using dynamic random geometric graph model while Baranidharan *et al.* [2] relate graph theory with the based routing protocol for wireless sensor networks.

A graph consists of points, which are called vertices and connections, which are called edges and which are indicated by line segments of curves joining certain pairs of vertices [3]. Omer *et al.* [4] started to find the generalized conjugacy class graph of some finite non-abelian group in 2015. In the same year, El-Sanfaz *et al.* [5] have done their research on the probability that a group element fixes a set and its generalized conjugacy class graph. The generalized conjugacy class graph in this research is defined as : Let G be a finite group. Let Ω be the set of all ordered pairs (a,b) in $G \times G$ such that $\text{lcm}(|a|,|b|)=2$, $ab = ba$, and $a \neq b$. If G acts on Ω by conjugation, then the vertices of generalized conjugacy

2010 Mathematical Subject Classification. Primary:20D99; Secondary:20E45.

Keywords. Generalized conjugacy class graph, orbit, properties of graph.

* Speaker

class graph are $V(\Gamma_G^\Omega) = K(\Omega) - |A|$ where $K(\Omega)$ is non-central conjugacy classes under group action on Ω and $A = \{\omega \in \Omega, g\omega = \omega g, g \in G\}$. Two vertices ω_1 and ω_2 are adjacent if their cardinalities are not coprime, i.e $\gcd(|\omega_1|, |\omega_2|) \neq 1$ [4].

Some properties of the generalized conjugacy class graph that are discussed in this paper are the independent number, chromatic number, clique number and dominating number.

One of the classification of 3-generator 2-groups which are used in this research has been given by Kim [6] is stated in the following :

$$H = \langle x, y, z | x^2 = y^2 = z^4 = 1, [x, z] = [y, z] = 1, [x, y] = z^2 \rangle.$$

2 Main Results

In this section, we provide our main results on the generalized conjugacy class graph of one of the 3-generator 2-groups. Besides, some graph properties are presented.

Theorem 1. Let H be a 3-generator 2-group of order 16 such that

$$H = \langle x, y, z | x^2 = y^2 = z^4 = 1, [x, z] = [y, z] = 1, [x, y] = z^2 \rangle.$$

Let Ω be the set of all ordered pairs (a, b) in $H \times H$ such that $\text{lcm}(|a|, |b|) = 2$, $ab = ba$, and $a \neq b$. If H acts on Ω by conjugation, then the generalized conjugacy class graph Γ_H^Ω is the complete graph of eight vertices, K_8 .

Proposition 1. The independent number of H , $\alpha(\Gamma_H^\Omega) = 1$.

Proposition 2. The chromatic number of H , $\chi(\Gamma_H^\Omega) = 8$.

Proposition 3. The clique number of H , $\omega(\Gamma_H^\Omega) = 8$.

Proposition 4. The dominating number of H , $\gamma(\Gamma_H^\Omega) = 1$.

3 Conclusion

In this research, the generalized conjugacy class graph of one of the 3-generator 2-groups is shown to be the complete graph of eight vertices, K_8 . Some properties of the graphs have also been determined which include the independent number, chromatic number, clique number and dominating number.

Acknowledgement The authors would like to acknowledge Universiti Teknologi Malaysia, Johor Bahru for the Research University Grant (GUP) Vote No 13H79.

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A homological invariant of a consistent polycyclic group

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Abstract. Bieberbach group with quaternion extension is a consistent polycyclic group. It is also known as a torsion free crystallographic group which has an extension of a free abelian group of finite rank by a quaternion group. It has a description on the symmetrical pattern of a crystal that will reveal its algebraic properties including its homological invariants. One of the homological invariants is the subgroup of the nonabelian tensor square of the group. In this paper, the computation of G -trivial subgroup of the nonabelian tensor square of a Bieberbach group with quaternion point group of order eight will be shown.

1 Introduction

A G -trivial subgroup of the nonabelian tensor square of a group G , denoted as $J(G)$, is one of the homological invariants of the group which is closely related to the nonabelian tensor square of the group. The nonabelian tensor square of a group $G \otimes G$, which was introduced by Brown and Loday in [1], is vital in the determination of its homological invariants. It is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to the defining relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^k \otimes h^k)$ for all $g, h, k \in G$, where $g^h = h^{-1}gh$ and there exists a homomorphism mapping κ , where $\kappa : G \otimes G \rightarrow G'$ defined by $\kappa(g \otimes h) = [g, h]$ and $g^{-1}h^{-1}gh = [g, h]$. Here, $J(G) = \ker(\kappa)$ and $J(G)$ is a G -trivial subgroup of $G \otimes G$ contained in its center. In this research, the group being considered is taken from Crystallographic, Algorithms and Table (CARAT) package [2]. By using the technique developed by Blyth and Morse

2010 Mathematical Subject Classification. Primary: 20J05; Secondary: 20E05.

Keywords. Bieberbach group, homological invariant.

* Speaker

[3], this group is transformed from matrix representation to polycyclic presentation. Next, in order to proceed with the computations of its homological invariant, we must first check that this group satisfies its consistency relations.

Some basic definitions and preliminary results that are used in the computations G -trivial subgroup of the nonabelian tensor square are presented.

Definition 1. [4] Polycyclic Presentation

Let F_n be a free group on generators $g_1, \dots, g_n$ and R be a set of relations of group F_n . The relations of a polycyclic presentation F_n/R have the form:

$$\begin{aligned} g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} && \text{for } i \in I, \\ g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \cdots g_n^{y_{i,j,n}} && \text{for } j < i, \\ g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \cdots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I \end{aligned}$$

for some $I \subseteq \{1, \dots, n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \mathbb{Z}$ for all i, j and k .

Definition 2. [4] Consistent Polycyclic Presentation

Let G be a group generated by $g_1, \dots, g_n$ and the consistency relations in G can be evaluated in the polycyclic presentation of G using the collection from the left as in the following:

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j-1} (g_j g_i) && \text{for } j > i, j \in I, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i-1} && \text{for } j > i, i \in I, \\ (g_i^{e_i}) g_i &= g_i(g_i^{e_i}) && \text{for } i \in I, \\ g_j &= (g_j g_i^{-1}) g_i && \text{for } j > i, i \notin I \end{aligned}$$

for some $I \subseteq \{1, \dots, n\}$, $e^i \in \mathbb{N}$. Then, G is said to be given by a consistent polycyclic presentation.

Based on Definition 1, a Bieberbach group with quaternion point group of order eight that is considered is found to be polycyclic as shown in the following theorem.

Theorem 1. [5] Let G be a Bieberbach group with quaternion point group of order eight, then its polycyclic presentations is established as:

$$\begin{aligned} G = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \mid & a^2 = cl_6, b^2 = cl_5l_6^{-1}, b^a = bcl_5^{-2}l_6^2, \\ & c^2 = l_5l_6^{-1}, c^a = cl_5l_6^{-1}, c^b = c, l_1^a = l_4^{-1}, l_1^b = l_3^{-1}, l_1^c = l_1^{-1}, \\ & l_2^a = l_3, l_2^b = l_4^{-1}, l_2^c = l_2^{-1}, l_3^a = l_2^{-1}, l_3^b = l_1, l_3^c = l_3^{-1}, \\ & l_4^a = l_1, l_4^b = l_2, l_4^c = l_4^{-1}, l_5^a = l_6, l_5^b = l_5, l_5^c = l_5, l_6^a = l_5, \\ & l_6^b = l_6, l_6^c = l_6, l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \text{ for } j > i, 1 \leq i, j \leq 6 \rangle. \end{aligned}$$

The presentation in Theorem 1.1 is proven to be consistent satisfying Definition 2 [6].

Theorem 2. Let G be a Bieberbach group with quaternion point group of order eight, then its nonabelian tensor square is established as:

$$\begin{aligned} G \otimes G = \langle a \otimes a, b \otimes b, l_1 \otimes l_1, a \otimes b, a \otimes c, a \otimes l_1, a \otimes l_2, b \otimes l_1, b \otimes l_2, \\ (a \otimes b)(b \otimes a), (a \otimes c)(c \otimes a), (a \otimes l_1)(l_1 \otimes a), (a \otimes l_2)(l_2 \otimes a), \\ (a \otimes l_6)(l_6 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b) \rangle. \end{aligned}$$

The nonabelian tensor square of a Bieberbach group with quaternion point group of order eight has been found. Therefore, in this research, the G -trivial subgroup of the nonabelian tensor square of this group, denoted as $J(G)$, will be explicated.

2 Main Result

In this section, the main result of this research which is the G -trivial subgroup of $G \otimes G$ of a Bieberbach group with quaternion point group of order eight is presented as follows:

Theorem 3. The G -trivial subgroup of the nonabelian tensor square of a Bieberbach group with quaternion point group of order eight is given as

$$\begin{aligned} J(G) = \langle a \otimes a, b \otimes b, l_1 \otimes l_1, (a \otimes b)(b \otimes a), (a \otimes c)(c \otimes a), (a \otimes l_1)(l_1 \otimes a), \\ (a \otimes l_2)(l_2 \otimes a), (a \otimes l_6)(l_6 \otimes a), (b \otimes l_1)(l_1 \otimes b), (b \otimes l_2)(l_2 \otimes b) \rangle \\ \cong C_0^4 \times C_2^5 \times C_4. \end{aligned}$$

3 Conclusion

The G -trivial subgroup of the nonabelian tensor square of a Bieberbach group with quaternion point group of order eight is isomorphic to $C_0^4 \times C_2^5 \times C_4$.

Acknowledgment

The authors would like to express their appreciation for the support of the sponsor; Ministry of Higher Education (MOHE) Malaysia for the financial funding for this research through Research University Grant (GUP), Vote No: 11J96 and Fundamental Research Grant Scheme (FRGS), Vote No: 4F898 from Research Management Centre (RMC) Universiti Teknologi Malaysia (UTM) Johor Bahru. The first author is also indebted to UTM for her Zamalah Scholarship. The third author is also grateful to UTM and MOHE for her postdoctoral scholarship in University of Leeds, United Kingdom.

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The disc structures of commuting involution graphs for certain simple groups

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Abstract. Suppose G is a finite group and X is a subset of G . The commuting graph on the set X which is denoted by $\mathcal{C}(G, X)$ whose vertex set X contains a pair of vertices joined by an edge if and only if they commute. In this paper, we consider G as the Mathieu groups and Symplectic groups and X are conjugacy classes of involutions. Here we investigate the orbits under the action of $C_G(t)$ and find the subgroups generated by elements of t and x where x is $C_G(t)$ -orbit representative. We also highlight some observations about $\langle t, x \rangle$.

1 Introduction

Let G be any finite group and X is any G -conjugacy class. Note that $X = t^G$ and t as an arbitrary involution in X to be a fixed point of $\mathcal{C}(G, X)$. We form undirected graph with vertex set X , denoted as $\mathcal{C}(G, X)$, such that any two distinct vertices $x, y \in X$ being joined whenever $x \neq y$ and $xy = yx$. Such a graph is known as a commuting graph of G on X . There is a large literature which is devoted to study the ways in which one can associate a group with a graph, for the purpose of investigating the algebraic structure using properties of the associated graph. Most authors with a number of different perspectives had studied $\mathcal{C}(G, X)$ for various kind of G and X . The case when G has even order

2010 Mathematical Subject Classification. Primary:20F65; Secondary: 20D05.

Keywords. Commuting involution graph, Sporadic simple group, Classical simple group.

* Speaker

and $X = G \setminus Z(G)$, first studied in [4], is one of the earliest investigations. When X is specifically a G -conjugacy class of involution, we call $\mathcal{C}(G, X)$ a commuting involution graph. Commuting involution graphs first arose in [8] which led to the construction of 3-transposition groups. Amount of work of commuting involution graphs can be found in [1, 2, 6, 7, 5, 10].

For $x \in X$, we use the stabilizer $G_x = C_G(x)$ so called the centralizer of x in G . Let $d(x, y)$ be the usual distance function on the commuting involution graphs. When $\mathcal{C}(G, X)$ is connected, the i th disc of $\mathcal{C}(G, X)$ around t is defined as

$$\Delta_i(t) = \{x \in X \mid d(t, x) = i\}.$$

We then define the diameter of $\mathcal{C}(G, X)$, $\text{Diam } \mathcal{C}(G, X)$, to be the maximal distance between two of its vertices. From now on, the commuting involution graphs for groups G are $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ (Mathieu groups) and $S_4(2)', S_4(3), S_6(2)$ (Symplectic groups). Also, X is a G -conjugacy class of involution. We will use standard ATLAS [11] notations for all conjugacy classes.

Throughout this research, we investigate the orbits under $C_G(t)$ action and determine the subgroups generated by elements of t and $x \in X$. In the following section we summarize the elementary results. Finally, the future work arising from these studies is discussed in section 3.

2 Preliminary Results

There are some basic results which are fundamental in our study of commuting involution graphs. These are presented below.

Theorem 1. [1] Let G be any Mathieu groups - $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$, and X be the conjugacy classes of involutions in G . Then $\mathcal{C}(G, X)$ is connected with $\text{Diam } \mathcal{C}(G, X) = 3$ For $G = M_{12}$ and $X = 2A$ has $\text{Diam } \mathcal{C}(G, X) = 2$.

Theorem 2. Let G be any Symplectic groups - $S_4(2)', S_4(3), S_6(2)$, and X be the conjugacy classes of involutions in G .

- (i) $\text{Diam } \mathcal{C}(G, X) = 2$ for $(S_4(3), 2A), (S_6(2), 2A)$ and $(S_6(2), 2B)$.
- (ii) $\text{Diam } \mathcal{C}(G, X) = 3$ for $(S_4(3), 2B)$ and $(S_6(2), 2C)$.
- (iii) $\text{Diam } \mathcal{C}(G, X) = 3$ for $(S_4(2)', 2A)$ and $(S_6(2), 2D)$.

Theorem 3. The number of $C_G(t)$ -orbits and their orbit sizes and the subgroup generated by t and x , $\langle t, x \rangle$ are given in Table 1 - Table 14, according to the groups considered.

Remark 1. Let $x \in X$ and assume that m be the order of tx .

- (i) If $x \in \Delta_1(t)$ then $\langle t, x \rangle \cong 2^2$, elementary abelian group of order 4.
- (ii) If $d(t, x) \geq 2$ then $\langle t, x \rangle \cong D_{2m}$, dihedral group of order $2m$.

3 Conclusion

In this section, we recommend the direction for future research. The commuting graphs have been employed for some groups with conjugacy classes of involutions. However, this research deserves further consideration on conjugacy classes with larger order.

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Invariance of p^m -regular subspaces under derived equivalence

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Abstract. Let p be a prime, G, H be groups where p divides the order of G and H , and F be a field with characteristic p . The p^m -regular subspace $Z_{p'}^{(m)}FG$ is the subspace of the center $Z(FG)$ spanned by class sums of p^m -regular elements. Given a derived equivalence between FG and FH , in this paper we show that the p^m -regular subspace is not always invariant under derived equivalence, and the invariance depends on some central unit in $Z(FH)$.

1 Introduction

Let F be a field of characteristic p and G be a finite group. The group algebra FG is the F -vector space with basis elements of G . We denote by ZFG the center of the group algebra FG , it has a basis consists of the set of all class sums in FG . We define the p^m -regular subspace $Z_{p'}^{(m)}FG$ as the subspace of ZFG with basis p^m -regular class sums.

Some invariances of derived equivalence have been proved, for example, for Hochschild homology and Hochschild cohomology by Rickard [3], cyclic homology and cyclic cohomology by Keller [1] and for generalized Reynold ideals by Zimmermann [4]. Given a derived equivalence between FG and FH , in this paper we show that the p^m -regular subspace is not always invariant under derived equivalence, and the invariance depends on some central unit in $Z(FH)$. We will also consider the p^m -regular subspaces of some path algebras.

2010 Mathematical Subject Classification. Primary:20C05; Secondary: 16G20

Keywords. group algebra, derived equivalence, p-regular subspace

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2 Main Results

The center of FG is $ZFG := \{a \in FG \mid ab = ba, b \in FG\}$. Let $C_1, \dots, C_n$ be conjugacy classes in G . Define

$$C_i^+ = \sum_{c \in C_i} c$$

as a class sum in FG . The set of all class sums, $\{C_1^+, C_2^+, \dots, C_n^+\}$ forms a basis of ZFG .

The p^m -regular subspace $Z_{p'}^{(m)}(FG)$ is the subspace of ZFG with basis p^m -regular class sums:

$$Z_{p'}^{(m)}(FG) = \langle C_{h^{p^s}}^+ : o(h) = p^s t, p \nmid t, s = 0, \dots, m \rangle_F.$$

Let A be a symmetric algebra, the p -power map $\mu_p : A/[A, A] \rightarrow A/[A, A]$ defined as

$$\mu_p(a) = a^p \text{ for all } a \in A/[A, A].$$

The following theorem gives the identification of $Z_{p'}^{(m)}(FG)$ as some intersection of images of p -power maps and the image of $Z_{p'}^{(m)}(FG)$ under derived equivalence.

Theorem 1. 1. Let F be a field with characteristics $p > 0$ and G be a finite group. Let $h \in G$, then

$$\left(h^{p^s} + [FG, FG] \in \bigcap_{t > m} im(\mu_p^t) \right) \Leftrightarrow (o(h) = p^s t, p \nmid t), s = 0, 1, \dots, m.$$

2. Let $\delta : Z(FG) \rightarrow FG/[FG, FG]$ an isomorphism. We have

$$\delta(Z_{p'}^{(m)}(FG)) = \bigcap_{t > m} im(\mu_p^t).$$

3. Let G, H be finite groups. If $\mathcal{F} : D^b(FG) \rightarrow D^b(FH)$ is an equivalence, then

$$\mathcal{F}(Z_{p'}^{(m)}(FG)) = v \cdot Z_{p'}^{(m)}(FH)$$

for some central unit v in FH .

4. If $Z_{p'}^{(m)}(FG)$ is an algebra then

$$Z_{p'}^{(m)}(FG) = v \cdot Z_{p'}^{(m)}(FG) \Leftrightarrow v \in Z_{p'}^{(m)}(FG)$$

for $m = 1, 2, \dots, n$

2.1 The p^m -regular Subspaces of Some Symmetric Path Algebras

For any symmetric algebra A , the p^m -regular subspace of A , denoted as $Z_{p'}^{(m)}A$, is the δ -preimage of $\bigcap_{t > m} im(\mu_p^t)$ where δ is an isomorphism between centre ZA and commutator quotient space $A/[A, A]$.

We made the restriction to path algebras with quiver Γ satisfying these conditions:

1. every vertex has at least a cycle,
2. and if there is two or more cycles in one vertex, there is a relation that makes the cycles equal or 0.

In [2] we have given the p -regular subspaces of some Nakayama algebra, algebras of dihedral type and algebras of semi dihedral type. The following theorem generalizes the result in [2].

Theorem 2. Let A be a symmetric path algebra over a field F with quiver Γ . Basis of p^m -regular subspace of A is all cycles of longest length in quiver Γ .

Theorem 3. Let A be a path algebra over a field F with quiver Γ and B a symmetric algebra. If A and B derived equivalent under functor $\mathcal{F}$, then image of p^m -regular subspace $Z_{p'}^{(m)}(A)$ under this equivalence is $Z_{p'}^{(m)}(B)$.

3 Conclusion

We have shown that the invariance of p^m -regular spaces depends on some central units. For further research one can classify all units in ZFG to classify all invariances.

Acknowledgement

This research is funded by Hibah Kerjasama Luar Negeri DIKTI 2016.

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Finite Moufang loops for which the non–commuting graph is a triple complete split–like graph

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Abstract. The non–commuting graph associated to a non–abelian group G , $\Gamma(G)$, is a graph with vertex set $G \setminus Z(G)$ where distinct non–central elements x and y of G are joined by an edge if and only if $xy \neq yx$. Recently, Akbari and Moghaddamfar have determined the structure of any finite non–abelian group G whose non–commuting graph is a complete split graph, that is, a graph whose vertex set can be partitioned into two sets such that the induced subgraph on one of them is a complete graph and the induced subgraph on the other is an independent set. They have proved that the non–commuting graph of a group G is a split graph if and only if it is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n .

In this paper, we are going to generalize the above theorem and determine those finite Moufang loops that whose non–commuting graphs are triple complete split–like graphs or generalized triple complete split–like graphs. We will also try to compute the number of their spanning trees and the eigenvalues of the adjacency and Laplacian matrices of them. We show that these very special graphs are hi–energetic with few distinct eigenvalues.

1 Introduction

The *non-commuting graph* associated to a non-abelian group G , $\Gamma(G)$, is a graph with vertex set $G \setminus Z(G)$ where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. Recently, many authors have studied the non-commuting graph associated to a non-abelian group [2, 5, 7]. Also in a paper, Akbari and Moghadamfar have determined the structure of any finite non-abelian group G whose non-commuting graph is a *complete split* graph, that is, a graph whose vertex set can be partitioned into two sets such that the induced subgraph on one of them is a complete graph and the induced subgraph on the other is an independent set, [4].

A set Q with one binary operation is a *quasigroup* if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. *Loop* is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x . *Moufang loops* are loops in which any of the (equivalent) Moufang identities $((xy)x)z = x(y(xz))$, $x(y(zx)) = ((xy)z)x$, $(xy)(zx) = x((yz)x)$, $(xy)(zx) = (x(yz))x$ holds.

The author, have been generalized the notion of non-commuting graph to the finite loops and then characterized some small Moufang loops by their non-commuting graphs [3]. *Commutant* (or *Moufang center* or *centrum*) of a loop L is defined by $\{x \in L \mid xy = yx, \forall y \in L\}$ and is denoted by $C(L)$. So we define $L \setminus C(L)$, as the vertex set of the non-commuting graph of L , with two vertices x and y joined by an edge whenever the commutator of x and y is not the identity. In this paper, we are going to generalize the above theorem and determine those finite Moufang loops that whose non-commuting graphs are generalized triple complete split graphs or generalized triple complete split-like graphs.

The *complete product* (or *join*) $G_1 \nabla G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ (*union* of G_1 and G_2) by joining every vertex of G_1 with every vertex of G_2 . For $a, b, n \in \mathbb{N}$, the following classes of graphs have been defined and studied in [6] and [1]:

- the *complete split graph*: $CS_b^a \cong \bar{K}_a \nabla K_b$;
- the *multiple complete split-like graph*: $MCS_{b,n}^a \cong \bar{K}_a \nabla (nK_b)$;

where, K_b denotes the complete graph on b vertices, the notation nK_b is short for the union of n copy of K_b and $\bar{K}_a$ denotes the graph with a vertices and no edges. Now, we generalize these definitions as follows:

- the *generalized complete split graph*: $GCS_m^a \cong \bar{K}_a \nabla S_m$;
- the *generalized multiple complete split-like graph*: $GMCS_{m,n}^a \cong \bar{K}_a \nabla (nS_m)$; where, S_m denotes the *strongly regular graph* on m vertices with parameters (m, k, λ, μ) , i.e., S_m is regular with degree k such that every two adjacent vertices have λ common neighbours and every two non-adjacent vertices have μ common neighbours.

2 Main Results

A *perfect graph* is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. A *chordal graph* is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Equivalently, every induced cycle in the graph should have at most three vertices. Chordal graphs are a subset of the perfect graphs and split graphs are a subset of the Chordal graphs.

Theorem 1. Every multiple complete split-like graph $MCS_{b,n}^a \cong \bar{K}_a \nabla (nK_b)$ is perfect and also chordal but not split. Particularly, every complete split graph, $CS_b^a \cong \bar{K}_a \nabla K_b$, is perfect and also chordal.

Theorem 2. Every generalized multiple complete split-like graph, $GMCS_{m,n}^a \cong \bar{K}_a \nabla (nS_m)$, and so every generalized complete split graph, $GCS_m^a \cong \bar{K}_a \nabla S_m$, is perfect but not chordal.

There is an important class of non-associative Moufang loops, called *Chein loops*, that is well understood. Let G be a group of order n , and let u be a new element. Define multiplication $\circ$ on $G \cup Gu$ by $g \circ h = gh$, $g \circ hu = (hg)u$, $gu \circ h = (gh^{-1})u$, $gu \circ hu = h^{-1}g$, where $g, h \in G$. The resulting loop $(G \cup Gu, \circ) = M(G, 2)$ is a Moufang loop. It is non-associative if and only if G is non-abelian. It has been proved that $M(G, 2)$ is isomorphic to $M(H, 2)$ if and only if G is isomorphic to H . Thus, we obtain as many non-associative Moufang loops of order $2n$ as there are non-abelian groups of order n .

In [4], Akbari and Moghaddamfar have shown that non-commuting graph of the dihedral group D_{2n} , with odd n , is a complete split graph and then they have determined the structure of any finite non-abelian group G whose non-commuting graph is a complete split graph. They have proved that the non-commuting graph of a group G is a complete split graph if and only if it is isomorphic to a *Frobenius group* of order $2n$, n is odd, whose *Frobenius kernel* is abelian of order n . Now, we generalize these results to the class of Chein loops.

Theorem 3. Let $D = D_{2n}$ be the dihedral group of order $2n$, with odd n , and $M = M(D, 2)$ be the associated Chein loop. Then the non-commuting graph of M is a triple complete split-like graph, $GMCS_{n,3}^{n-1} \cong \bar{K}_{n-1} \nabla (3K_n)$. Moreover, this graph is perfect and also chordal.

Theorem 4. Let $D = D_{2n}$ be the dihedral group of order $2n$, with even n , and $M = M(D, 2)$ be the associated Chein loop. Then the non-commuting graph of M is a generalized triple complete split-like graph, $GMCS_{n,3}^{n-2} \cong \bar{K}_{n-2} \nabla (3S_n)$, where S_n is a strongly regular graph on n vertices with parameters $(n, n-2, n-4, n-2)$. Moreover, this graph is perfect but not chordal.

Theorem 5. Let G be a finite group and $M = M(G, 2)$ be the associated Chein loop. Then the non-commuting graph of M is a triple complete split-like graph if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n . Moreover, this graph is perfect and also chordal.

3 Conclusion

We have characterized finite Moufang loops with triple complete split non-commuting graph and have computed the number of their spanning trees and the eigenvalues of the adjacency and Laplacian matrices of them. We have shown that these very special graphs are hi-energetic with few distinct eigenvalues. Finally we propose a problem:

Problem. Determine all finite groups with generalized complete split-like graphs and all Chein loops with generalized triple complete split-like graphs.

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The nonabelian tensor square of a Bieberbach group of dimension four

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Abstract. A Bieberbach group is a torsion free crystallographic group. This paper focuses on a Bieberbach group of dimension four with elementary abelian 2-group point group. The consistent polycyclic presentation of this group is first constructed. Based on the method developed for polycyclic groups, the nonabelian tensor square of the Bieberbach group is computed. It is found that the nonabelian tensor square of the group is nonabelian and its presentation is constructed.

1 Introduction

A Bieberbach group is also known as a crystallographic group. This group is an extension of a finite point group P and free abelian group L which satisfy the short exact sequence $1 \rightarrow L \rightarrow G \rightarrow P \rightarrow 1$. The main objective of this study is to compute the nonabelian tensor square of a Bieberbach group with an elementary abelian 2-group point group, $C_2 \times C_2$. The nonabelian tensor square of a group G , $G \otimes G$ is generated by $g \otimes h$ for all $g, h \in G$, subject to relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$$

2010 Mathematical Subject Classification. Primary:20E05; Secondary:20J05

Keywords. Bieberbach group, polycyclic group, nonabelian tensor square

* Speaker

for all $g, g', h, h' \in G$, where ${}^g g' = gg'g^{-1}$.

Brown et. al [1] are among the mathematicians who first studied the nonabelian tensor square of groups. Following to that, other researchers such as Ellis and Leonard [2] and Mat Hassim et. al [3] started to study the properties of nonabelian tensor square of 2-generator Burnside groups of exponent 4 and of 2-Engel groups of order at most 16, respectively. Beginning in 2009, some other researchers started to study the properties of Bieberbach groups with certain point groups such as the nonabelian tensor squares of Bieberbach groups with cyclic point groups of order two [4], of Bieberbach groups with dihedral point group [5, 6] and of Bieberbach group with symmetric point group [7].

In this paper, the focused group is a Bieberbach group an elementary abelian 2-group point group which is denoted as $S_3(4)$. Based on the matrix group from CARAT website, the polycyclic presentation of $S_3(4)$ is obtained as in the following:

$$\begin{aligned} S_3(4) = \langle a_0, a_1, l_1, l_2, l_3, l_4 | & a_0^2 = l_2^{-1} l_3^{-1}, a_1^2 = l_1^{-1}, {}^{a_0} a_1 = a_1 l_3^{-1} l_4, \\ & {}^{a_0} l_1 = l_1, {}^{a_0} l_2 = l_2, {}^{a_0} l_3 = l_3, {}^{a_0} l_4 = l_4^{-1}, {}^{a_1} l_1 = l_1, {}^{a_1} l_2 = l_2, \\ & {}^{a_1} l_3 = l_3^{-1}, {}^{a_1} l_4 = l_4^{-1}, {}^{l_1} l_2 = l_2, {}^{l_1} l_3 = l_3, {}^{l_1} l_4 = l_4, {}^{l_2} l_3 = l_3, \\ & {}^{l_2} l_4 = l_4, {}^{l_3} l_4 = l_4 \rangle \quad (1) \end{aligned}$$

2 Main Results

In this section, the presentation of the nonabelian tensor square of $S_3(4)$, denoted as $S_3(4) \otimes S_3(4)$ is constructed.

Theorem 1. The nonabelian tensor square of $S_3(4)$, $S_3(4) \otimes S_3(4)$ is nonabelian and the presentation is given as follows:

$$\begin{aligned} S_3(4) \otimes S_3(4) = \langle g_1, g_2, \dots, g_{11} | & g_6^4 = g_8^2 = g_9^2 = g_{10}^2 = g_{11}^4 = 1, [g_1, g_2] = \\ & g_{11}^{-8}, [g_1, g_3] = g_{11}^{-8}, [g_2, g_3] = g_6^{16}, [g_1, g_j] = 1, [g_2, g_j] = \\ & 1, [g_i, g_j] = 1 \rangle \end{aligned}$$

for $3 \leq i \leq 11, 4 \leq j \leq 11$ where,

$$\begin{aligned} g_1 &= [a_1, l_3^{-2\varphi}], g_2 = [a_0, l_4^{-2\varphi}], g_3 = [a_1, l_4^{-2\varphi}], g_4 = [a_0, a_0^\varphi], g_5 = [l_2, l_2^\varphi], g_6 = [l_4, l_4^\varphi], g_7 = [a_1, l_2^\varphi][l_2, a_1^\varphi], \\ g_8 &= [a_1, l_4^\varphi][l_4, a_1^\varphi], g_9 = [l_2, l_4^\varphi][l_4, l_2^\varphi], g_{10} = [a_1, l_2^\varphi] \text{ and } g_{11} = [l_3, l_4^{-2\varphi}]. \end{aligned}$$

3 Conclusion

In this paper, the nonabelian tensor square of a Bieberbach group of dimension four with point group $C_2 \times C_2$ is computed and is shown to be nonabelian. Based on the computation, the presentation of the nonabelian tensor square

of this group is constructed. The finding of this research can be used for further research in computing other homological functors of this group.

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The relative non-nil $(n - 1)$ bipartite graph of a finite group

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Abstract. Suppose G is not a nilpotent group of class at most $n - 1$ (a non-nil $(n - 1)$ group). Consider two subgroups H and K of G . Let $A = H \setminus C_H^{(n)}(K)$ and $B = K \setminus C_K^{(n)}(H)$ where $C_H^{(n)}(K) = \{h \in H : [h, k] \in Z_{n-1}(G), \forall k \in K\}$ and $C_K^{(n)}(H) = \{k \in K : [h, k] \in Z_{n-1}(G), \forall h \in H\}$. Thus, the relative non-nil $(n - 1)$ bipartite graph, $\Gamma_{H,K}^{(n)}(G)$ is a graph with vertex set, $V(\Gamma_{H,K}^{(n)}(G)) = A \cup B$. Two vertices of $\Gamma_{H,K}^{(n)}(G)$ are joined by an edge if $[x, y] \notin Z_{n-1}(G)$ where $x \in A$, $y \in B$ and $Z_n(G)$ is the n -th central series of G . In this paper, the probability which shows how close a group is to being a nil- $(n - 1)$ group is stated. Besides, some graph properties such as diameter and girth are found.

1 Introduction

Throughout this paper, $\Gamma_{H,K}^{(n)}(G)$ denotes a bipartite graph and G denotes a group which is not a nil- $(n - 1)$ group. A bipartite graph is a graph whose vertices can be divided into two independent sets, U and V such that every edge (u, v) either connects a vertex from U to V or a vertex from V to U . In other words, for every edge (u, v) , either u belongs

2010 Mathematical Subject Classification. Primary:05C25.

Keywords. Nilpotent, Bipartite graph, Diameter, Girth.

* Speaker

to U and v to V , or u belongs to V and v to U . We can also say that there is no edge that connects vertices of the same set.

A graph Γ is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively [2]. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, Γ is called undirected. Two vertices are adjacent if they are linked by an edge.

A non-commuting graph is a graph whose vertices are non central elements of G and two vertices v_1 and v_2 are adjacent whenever $v_1 v_2 \neq v_2 v_1$. This topic was the basis of many similar research about non-commuting graph of a group and many similar results about this graph have been obtained by some researchers [4, 5, 6, 7].

In this paper, some graph properties such as diameter and girth of relative non-nil $(n - 1)$ bipartite graph are found. The diameter is the maximum distance between any two vertices of Γ and $d(\Gamma)$ is used as a notation. Meanwhile, the girth of a graph, $\text{girth}(\Gamma)$ is the length of a shortest cycle contained in Γ . If Γ has no cycle then its girth is defined to be infinity.

The relative n -th nilpotency degree of two subgroups of a group, $P_{nil}(n, H, K)$ is defined as the probability that the commutator of two arbitrary elements $h \in H$ and $k \in K$ belong to $Z_{n-1}(G)$, where $Z_n(G)$ is the n -th central series of G . It was firstly investigated by Abdul Hamid *et al.* [8]. The formal definition is given in the following:

$$P_{nil}(n, H, K) = \frac{|\{(h, k) \in H \times K : [h, k] \in Z_{n-1}(G)\}|}{|H||K|}.$$

The graph related to this probability named relative non-nil $(n - 1)$ bipartite graph is found and stated in the next section.

2 Main Results

We defined the relative non-nil $(n - 1)$ bipartite graph of a finite group as in the following definition.

Definition 1. The Relative Non-Nil $(n - 1)$ Bipartite Graph of a Group

Suppose G is not a nilpotent group of class at most $n - 1$ (a non-nil $(n - 1)$ group). Consider two subgroups H and K of G . Let $A = H \setminus C_H^{(n)}(K)$ and $B = K \setminus C_K^{(n)}(H)$ where $C_H^{(n)}(K) = \{h \in H : [h, k] \in Z_{n-1}(G), \forall k \in K\}$ and $C_K^{(n)}(H) = \{k \in K : [h, k] \in Z_{n-1}(G), \forall h \in H\}$. Thus, the relative non-nil $(n - 1)$ bipartite graph, $\Gamma_{H,K}^{(n)}(G)$ is a graph with vertex set, $V(\Gamma_{H,K}^{(n)}(G)) = A \cup B$. Two vertices of $\Gamma_{H,K}^{(n)}(G)$ are joined by an edge if $[x, y] \notin Z_{n-1}(G)$ where $x \in A$, $y \in B$ and $Z_n(G)$ is the n -th central series of G . Note that we have to add the condition that $H \cap K \subseteq C_H^{(n)}(K) \cup C_K^{(n)}(H)$ to have disjoint parts A and B .

The main results of the non-nil $(n - 1)$ bipartite graph of a finite group stated as in the following theorems.

Theorem 1. The graph $\Gamma_{H,K}^{(n)}(G)$ has no isolated vertex.

Theorem 2. Assume that $h \in A$ is a vertex and $k \in B$ is a vertex and there is an edge between h and k . If $k \in aC_K^{(n)}(H)$ for some $a \in K$, then h is adjacent to any elements in coset $aC_K^{(n)}(H)$.

Theorem 3. In graph, $\Gamma_{H,K}^{(n)}(G)$, there exists a pendant vertex if and only if $|H| = 2$ or $|K| = 2$.

Theorem 4. $\text{diam}(\Gamma_{H,K}^{(n)}(G)) \leq 3$.

Theorem 5. If $|H| > 2$ and $|K| > 2$ then $\text{girth}(\Gamma_{H,K}^{(n)}(G)) = 4$.

3 Conclusion

In this paper, the graph which is the bipartite graph associated to a non-nilpotent group of class $(n - 1)$ is defined. This graph is called as relative non-nil $(n - 1)$ bipartite graph. It is proven that the graph has no isolated vertex. Also, there exists a pendant vertex if and only if $|H| = 2$ or $|K| = 2$. Besides, some graph properties such as diameter and girth are found.

Acknowledgements

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) for the financial funding through the Research University Grant (RUG) Vote No. 11J96. The first author would also like to thank Ministry of Higher Education (MOHE) Malaysia for her MyPhD scholarship.

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Some properties of CA-nilpotent groups

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Abstract. In this talk the concept of central auto-nilpotent (henceforth CA-nilpotent) groups are introduced and we discuss the relationship between such subgroups with the nilpotency of the subgroups of central automorphism of groups.

One notes that, in CA-nilpotent groups if the central automorphism runs over all automorphisms of the group, then it will generalize our work in 2013.

1 Introduction

Let G be a group then the *autocommutator* of the element $x \in G$ and the automorphism α in $\text{Aut}(G)$ is defined to be

$$[x, \alpha] = x^{-1}x^\alpha = x^{-1}\alpha(x).$$

For a given group G the central automorphism, $\text{Aut}_c(G)$, is the set of all automorphisms α in $\text{Aut}(G)$ for which $[x, \alpha] \in Z(G)$, for all $x \in G$.

In 1955, F. Haimo [6] introduced the following subgroup of a given group G , and we denote it by $L_c(G)$,

$$L_c(G) = \{x \in G : [x, \alpha] = 1, \forall \alpha \in \text{Aut}_c(G)\}.$$

2010 Mathematical Subject Classification. Primary: 20E45; 20B30; Secondary: 05A05; 05A16.

Keywords. Central autocommutator subgroup, autocentre, A-nilpotent group.

* Speaker

Note that, for every x in G the map $f_\alpha : x \mapsto x^{-1}x^\alpha$ is a homomorphism of G into $Z(G)$ and $\ker f_\alpha = \{x \in G : x^{-1}x^\alpha = [x, \alpha] = 1\}$. Hence we call $L_c(G)$ the *central kernel* of G and clearly it is a characteristic subgroup of G . For any natural number $n \geq 1$, we may define inductively the n^{th} -*central kernel* subgroup of G as follows;

$$L_{c_n}(G) = \{x \in G : [x, \alpha_1, \dots, \alpha_n] = 1, \forall \alpha_i \in \text{Aut}_c(G)\}.$$

Note that our construction does not give the one in [6]. So we work with the above definition throughout the rest of the paper.

By the above discussion, we introduce the new notion of CA-nilpotent groups as follows.

Definition 1. A group G is said to be *CA-nilpotent* of class at most n if $L_{c_n}(G) = G$, for some natural number n .

One can easily see that $L_{c_n}(G) = G$ if and only if $K_{c_n}(G) = \langle 1 \rangle$, where

$$K_{c_n}(G) = \langle [x, \alpha_1, \dots, \alpha_n] \mid \forall x \in G, \alpha_1, \dots, \alpha_n \in \text{Aut}_c(G) \rangle.$$

One notes that, if the central automorphism α runs over all automorphisms of the group, then we get the autcentre, $L(G)$, and the autocommutator subgroup, $K(G)$, respectively. It is clear that the above notion of CA-nilpotency implies A-nilpotency, which we have introduced and studied in [4].

Remark 1. The cyclic group $\mathbb{Z}_p$ of odd prime order p is not CA-nilpotent, while it is nilpotent, since $\text{Aut}(\mathbb{Z}_p) = U_{p-1}$ is a cyclic group of order $p-1$ and so it can not fix any non-identity element of $\mathbb{Z}_p$.

It will be shown that this is held for all cyclic groups of order p^n , when p is an odd prime number. On the other hand, if $p = 2$ then every automorphism of the cyclic group $\mathbb{Z}_{2^n}$ is central and hence CA-nilpotency is equivalent with A-nilpotency. So, Example 1.2(iv) from [4] gives our claim.

Here we show that some of the known results of nilpotent groups can be carried over to the case of CA-nilpotency. We begin with some elementary facts on the properties of CA-nilpotent groups.

Lemma 1. Let G be a non-trivial CA-nilpotent group, then its central kernel is non-trivial.

The following theorem gives a complete characterization of cyclic groups.

Theorem 1. The cyclic group $\mathbb{Z}_{p^n}$ of order p^n is not CA-nilpotent, for any odd prime p and $n \geq 1$.

Lemma 2. Let G be a CA-nilpotent group with a non-trivial characteristic subgroup N . Then $N \cap L_c(G)$ is non-trivial.

Corollary 1. A minimal characteristic subgroup of a CA-nilpotent group is contained in the central kernel of the group.

The central automorphism group of a direct product of finite groups has been discussed in many articles (see [1, 1, 5, 7] for more information).

In particular, the following theorem of [7] is useful for proving our final result.

Theorem 2. ([7], Theorem 3.1) If H and K are finite groups with co-prime orders, then

$$\text{Aut}_c(H \times K) \cong \text{Aut}_c(H) \times \text{Aut}_c(K).$$

Considering the above theorem, we have the following

Corollary 2. Let $G_1, G_2, \dots, G_n$ be finite CA-nilpotent groups with mutually co-prime orders. Then $G_1 \times G_2 \times \dots \times G_n$ is also CA-nilpotent.

Clearly, the above results hold for the usual nilpotent groups.

2 Main Results

Note that, subgroups and homomorphic images of CA-nilpotent groups, may not be necessarily CA-nilpotent. As there are no relations between the central automorphisms of a subgroup of a given group and the central automorphisms of the whole group, in general. For example $S_3/A_3 \cong \mathbb{Z}_2$, and so S_3/A_3 is CA-nilpotent, while S_3 is not, as $L_c(S_3) = \langle 1 \rangle$. Also the Dihedral group D_8 is CA-nilpotent, but $\mathbb{Z}_2 \times \mathbb{Z}_2$ can not be CA-nilpotent, as $L_c(\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle 1 \rangle$.

In the following, under some condition, it is shown that the property of CA-nilpotency is extension closed.

Theorem 3. For a characteristic subgroup N of a given group G , if N and G/N are both CA-nilpotent, then so is G .

Theorem 4. Let N be a proper characteristic subgroup of a given group G with G/N is CA-nilpotent of class r and $N \cap K_{c_r}(G) = \langle 1 \rangle$. Then G is CA-nilpotent.

Using the definition of $L_c(G)$, we may define a new subgroup of $\text{Aut}_c(G)$ as follows:

$$\mathcal{A}(G) = \{\alpha \in \text{Aut}_c(G) \mid x^{-1}\alpha(x) \in L_c(G) \text{ or } x\alpha(x) \in L_c(G), \forall x \in G\}.$$

One can check that $\mathcal{A}(G)$ is a characteristic subgroup of $\text{Aut}(G)$.

The following result shows that the property of CA-nilpotency of the group G gives the nilpotency property of $\mathcal{A}(G)$.

Theorem 5. Let G be a CA-nilpotent group of class $n \geq 2$, then $\mathcal{A}(G)$ is nilpotent of class $n - 1$.

By the above theorem, we obtain the following

Corollary 3. If G is a CA-nilpotent of class $n \geq 2$, then the centre of the group G is torsion of exponent dividing 2^{n-1} .

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On Laplacian eigenvalues of non-commuting graph of dihedral groups

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Abstract. Let Γ be an undirected simple graph with n vertices and m edges, the Laplacian eigenvalues of Γ are the eigenvalues of its Laplacian matrix. In this paper, by using the Laplacian matrices of non-commuting graph of dihedral groups, we present general formulas for the Laplacian eigenvalues of the non-commuting graph of dihedral groups D_{2n} .

1 Introduction

Let G be a finite group with center $Z(G)$. The non-commuting graph of a group G , denoted by Γ_G , is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices x and y are adjacent if and only if $xy \neq yx$ [1]. Let Γ be a graph with vertex set $V(\Gamma) = \{1, 2, \dots, n\}$ and edge set $E(\Gamma) = \{e_1, e_2, \dots, e_m\}$. The Laplacian matrix of Γ , denoted by $L(\Gamma)$, is an $n \times n$ matrix defined as follows: The rows and the column are indexed by $V(\Gamma)$. If $i \neq j$, then the a_{ij} entry of $L(\Gamma)$ is 0 if vertex i and j are not adjacent, and it is -1 if they adjacent. The a_{ii} entry of $L(\Gamma)$ is d_i , the degree of vertex i , $i=1, 2, \dots, n$ [2]. By the Laplacian eigenvalues of the graph we mean the eigenvalues of its Laplacian matrix. Recently there are many researches in constructing a graph by a group, for instance one can refer to the work by Abdollahi [1]. This paper consists of three parts. The first section is the introduction of the non-commuting graph

2010 Mathematical Subject Classification. Primary:20B05; Secondary: 20C30.

Keywords. Non-commuting graph, Dihedral group and Laplacian Eigenvalues.

* Speaker

which is constructed by dihedral group, Laplacian matrix and eigenvalues of the graph, followed by some fundamental concepts and definitions which are used in this work. The second section consists of some previous results. Our main results are presented in the third section, in which we compute the Laplacian matrices and the Laplacian eigenvalues of the non-commuting graphs of dihedral groups, and lastly the general formulas for the Laplacian eigenvalues of non-commuting graph of dihedral groups of order $2n$, are found.

We recall the dihedral group D_{2n} , is the group of all symmetries of a regular polygon. The group is of order $2n$ where n is an integer[3], and it has a presentation as in the following:

$$D_{2n} \cong \langle a, b : a^n = b^2 = 1, bab = a^{-1} \rangle.$$

The dihedral group D_{2n} has $2n$ elements which are listed as below:

$$D_{2n} = \{1, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}.$$

2 Preliminaries

In this section, some previous results on the Laplacian matrix and eigenvalues of graph are presented which are used in this paper.

Proposition 2.1[2]

Let $D(\Gamma)$ be the diagonal matrix of vertex degrees of the graph Γ . If $A(\Gamma)$ is the adjacency matrix of Γ , then note that: $L(\Gamma) = D(\Gamma) - A(\Gamma)$.

Proposition 2.2 [4]

Let Γ be a graph with n vertices and m edges. The Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_n$, of the graph Γ obey the following well-known relations:

1. $\sum_{i=1}^n \mu_i = 2m$.
2. $\sum_{i=1}^n \mu_i^2 = 2m + \sum_{i=1}^n d_i^2$.

3 Main Results

In this section we present our main results, namely the Laplacian eigenvalues of the non-commuting graph of dihedral groups.

Definition 3.1 The Laplacian spectrum of non-commuting graph Γ_G will denoted by $\text{Lspec}(\Gamma_G)$ and define as the set

$\{\mu_1^{m_1}, \mu_2^{m_2}, \dots, \mu_t^{m_t}\}$, where $\mu_1^{m_1}, \mu_2^{m_2}, \dots, \mu_t^{m_t}$ are distinct Laplacian eigenvalues of the Laplacian matrix of Γ_G with multiplicity $m_1, m_2, \dots, m_t$, respectively.

Using the above definition the following results are obtained.

Proposition 3.2

Let $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order 8, then:

$$\text{Lspec}(\Gamma_{D_8}) = \{(0)^1, (6)^2, (4)^3\}.$$

The following theorems gives results on the Laplacian spectrum of non-commuting graph of dihedral groups of the remaining cases.

Theorem 3.3

Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order $2n$, where $n > 2$. For n even integer and $n > 4$, then:

$$\text{Lspec}(\Gamma_{D_{2n}}) = \{(0)^1, (2n-2)^{\frac{n}{2}}, (n)^{n-3}, (2n-4)^{\frac{n}{2}}\}.$$

Theorem 3.4

Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order $2n$, where $n > 2$. For n odd integer, then:

$$\text{Lspec}(\Gamma_{D_{2n}}) = \{(0)^1, (n)^{n-2}, (2n-1)^n\}.$$

4 Conclusion

In this paper, the general formulas for the Laplacian eigenvalues of non-commuting graph of dihedral groups are found.

For $n=4$, we have $\text{Lspec}(\Gamma_{D_8}) = \{(0)^1, (6)^2, (4)^3\}$. When n is even and $n > 4$, then

$$\text{Lspec}(\Gamma_{D_{2n}}) = \{(0)^1, (2n-2)^{\frac{n}{2}}, (n)^{n-3}, (2n-4)^{\frac{n}{2}}\}.$$

For n is odd, we also have $\text{Lspec}(\Gamma_{D_{2n}}) = \{(0)^1, (n)^{n-2}, (2n-1)^n\}$.

Acknowledgment: The authors would like to acknowledge UTM for the Research University Fund (GUP) for vote no. 13H79 and the first author would like to appreciate UTM for partial financial support through International Doctorate Fellowship (IDF).

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Classification of 2-dimensional evolution algebras, their groups of automorphisms and derivation algebras

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Abstract. In the paper we give a complete classification of 2-dimensional evolution algebras over algebraically closed fields, describe their groups of automorphisms and derivation algebras.

1 Introduction

The classification problem of finite dimensional algebras and description of their invariants is one of the important problems of algebra. One of the interesting class of algebras is the class of evolution algebras. In the present paper we give a complete classifications of 2-dimensional evolution algebras over any algebraically closed field, describe their groups of automorphisms and algebras of derivations. For further information, related to similar problems, the reader is refereed to [1, 2].

2010 Mathematical Subject Classification. Primary:15A21, 15A63, 17A35; Secondary: 37A35.

Keywords. classification, structure constant, group automorphism, derivation.

* Speaker

2 The main results

Further $\mathbb{F}$ will stand for any algebraically closed field. Let $\mathbb{A}$ be a 2-dimensional algebra over $\mathbb{F}$ with multiplication $\cdot$ given by a bilinear map $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$. If $e = (e^1, e^2)$ is a basis of $\mathbb{A}$ then the bilinear map above is represented by a matrix

$$A = \begin{pmatrix} A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1 \\ A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2 \end{pmatrix} \in \text{Mat}(2 \times 4; \mathbb{F})$$

such that

$$\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v)$$

for $\mathbf{u} = eu, \mathbf{v} = ev$, where $u = (u_1, u_2)$, and $v = (v_1, v_2)$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, $(u \otimes v) = (u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $e^i \cdot e^j = A_{i,j}^1 e^1 + A_{i,j}^2 e^2$, $i, j = 1, 2$. The matrix $A \in \text{Mat}(2 \times 4; \mathbb{F})$ is said to be the matrix of structure constant (MSC) of $\mathbb{A}$ with respect to the basis e . Further, we do not differentiate $\mathbb{A}$ and its MSC A .

It is known that under change of the basis $e = (e^1, e^2)$ MSC A changes according to $B = gA(g^{-1})^{\otimes 2}$, where for $g^{-1} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix}$ one has

$$(g^{-1})^{\otimes 2} = g^{-1} \otimes g^{-1} = \begin{pmatrix} \xi_1^2 & \xi_1\eta_1 & \xi_1\eta_1 & \eta_1^2 \\ \xi_1\xi_2 & \xi_1\eta_2 & \xi_2\eta_1 & \eta_1\eta_2 \\ \xi_1\xi_2 & \xi_2\eta_1 & \xi_1\eta_2 & \eta_1\eta_2 \\ \xi_2^2 & \xi_2\eta_2 & \xi_2\eta_2 & \eta_2^2 \end{pmatrix}.$$

Definition 1. 2-dimensional algebras $\mathbb{A}, \mathbb{B}$, given by their matrices of structural constants A, B , are said to be isomorphic if $B = gA(g^{-1})^{\otimes 2}$ for some $g \in GL(2, \mathbb{F})$.

Definition 2. An n -dimensional algebra $\mathbb{E}$ is said to be an evolution algebra if it admits a basis $\{e^1, e^2, \dots, e^n\}$ such that $e^i \cdot e^j = 0$, whenever $i \neq j$ and $i, j = 1, 2, \dots, n$.

The main results of the paper are given as follows.

Theorem 1. Over an algebraically closed field $\mathbb{F}$ every nontrivial 2-dimensional evolution algebra is isomorphic to only one the algebras represented below by their matrices of structure constants:

$$\begin{aligned} E_1(\mathbf{c}) &= \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix}, \text{ where } \mathbf{c} = (b, c) \in \mathbb{F}^2, \\ E_2(\mathbf{c}) &= \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = b \in \mathbb{F}, \\ E_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Let $i \in \mathbb{F}$ stand for an element with $i^2 = -1$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $\mathbb{E}$ is an algebra given by MSC E then its group of automorphisms $Aut(E)$ is presented as follows

$$Aut(E) = \{g \in GL(2, \mathbb{F}) : gE - E(g \otimes g) = 0\}.$$

Theorem 2. Over an algebraically closed field $\mathbb{F}$, characteristic not 2, the automorphism groups of 2-dimensional evolution algebras are represented as follows.

$$\begin{aligned} Aut(E_1(\mathbf{c})) &= \{I\}, \text{ if } bc \neq 1, \text{ or } bc = 1, b \in \{-1 + 2i, -1 - 2i\} \\ Aut(E_1(\mathbf{c})) &= \left\{ I, \begin{pmatrix} -\frac{1-b}{1+b} & \frac{2b}{1+b} \\ \frac{2}{1+b} & \frac{1-b}{1+b} \end{pmatrix} \right\}, \text{ if } bc = 1, b \notin \{-1, -1 + 2i, -1 - 2i\}, \\ Aut(E_1(\mathbf{c})) &= \left\{ \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} : t \neq \frac{1}{2} \right\}, \text{ if } b = c = -1 \\ Aut(E_2(\mathbf{c})) &= \{I\}, \text{ if } b \neq 0, \quad Aut(E_2(\mathbf{c})) = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1-t \end{pmatrix} : t \neq 1 \right\}, \text{ if } b = 0, \\ Aut(E_3) &= \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix} \right\}, \\ \text{where } t &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad Aut(E_4) = \left\{ I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \\ Aut(E_5) &= \left\{ \begin{pmatrix} t^2 & s \\ 0 & t \end{pmatrix} : t \neq 0, s \in \mathbb{F} \right\}. \end{aligned}$$

In the cases of characteristic 2 the corresponding result is given as follows.

Theorem 3. Over an algebraically closed field $\mathbb{F}$ of characteristic 2 the automorphism groups of 2-dimensional evolution algebras are represented as follows.

$$\begin{aligned} Aut(E_1(\mathbf{c})) &= \{I\}, \text{ if } b \neq -1 \text{ and } c \neq -1, \\ Aut(E_1(\mathbf{c})) &= \left\{ \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix} : t \in F \right\}, \text{ if } b = c = -1, \\ Aut(E_2(\mathbf{c})) &= \{I\}, \text{ if } b \neq 0, \quad Aut(E_2) = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1-t \end{pmatrix} : t \neq 1 \right\}, \text{ if } b = 0, \\ Aut(E_3) &= \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix} \right\}, \\ \text{where } t^2 + t + 1 &= 0, \quad Aut(E_4) = \{I\}, \quad Aut(E_5) = \left\{ \begin{pmatrix} t^2 & s \\ 0 & t \end{pmatrix} : t \neq 0, s \in \mathbb{F} \right\}. \end{aligned}$$

If $\mathbb{E}$ is an algebra given by MSC E then its algebra of derivations $Der(E)$ is presented as follows

$$Der(E) = \{D \in M(2, \mathbb{F}) : E(D \otimes I + I \otimes D) - DE = 0\}.$$

Theorem 4. Over an algebraically closed field $\mathbb{F}$, characteristic not 2, 3, the derivation algebras of 2-dimensional evolution algebras are presented as follows

$$\begin{aligned} Der(E_1(\mathbf{c})) &= \{0\}, \text{ if } b \neq -1, c \neq -1, \\ Der(E_1(\mathbf{c})) &= \left\{ \begin{pmatrix} -t & t \\ t & -t \end{pmatrix} : t \in \mathbb{F} \right\}, \text{ if } b = c = -1, \\ Der(E_2(\mathbf{c})) &= \{0\}, \text{ if } b \neq 0, \quad Der(E_2) = \left\{ \begin{pmatrix} 0 & 0 \\ t & -t \end{pmatrix} : t \in \mathbb{F} \right\}, \text{ if } b = 0, \\ Der(E_3) = Der(E_4) &= \{0\}, \quad Der(E_5) = \left\{ \begin{pmatrix} 2t & s \\ 0 & t \end{pmatrix} : t, s \in \mathbb{F} \right\}. \end{aligned}$$

The corresponding results in characteristic 2 and 3 cases also are obtained.

3 Conclusion

The results of the paper is an implementation of a general approach of the authors to the problems. The similar approach can be applied in higher dimensional cases for other classes of algebras as well.

The first and third authors acknowledge MOHE for a support by grant 01-02-14-1591FR, the second author's research was supported by FRGS14-153-0394, MOHE.

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$\mathbb{R} \times \mathbb{S}$ –additive cyclic codes

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Abstract. Motivated by the applications of cyclic codes over Z_2Z_4 , in this paper we study $\mathbb{R} \times \mathbb{S}$ –additive cyclic codes for any rings R and S . $\mathbb{R} \times \mathbb{S}$ –additive codes will be defined as subgroups of the group $\mathbb{R}^\alpha \mathbb{S}^\beta$. $\mathbb{R} \times \mathbb{S}$ –additive cyclic codes will be defined as subgroups of the group $\mathbb{R}^\alpha \mathbb{S}^\beta$ that are closed under shifting. We will show that if the ring $\mathbb{R} = \mathbb{S}$ then the dual of an $\mathbb{R} \times \mathbb{S}$ –additive cyclic code is also an $\mathbb{R} \times \mathbb{S}$ –additive cyclic code.

1 Introduction

Let $\mathbb{R}$ and $\mathbb{S}$ be any finite commutative ring. Linear cyclic codes of length n are usually defined as ideals in the ring $R[x]/\langle x^n - 1 \rangle$. Additive cyclic codes of length n are defined as subgroups of $R[x]/\langle x^n - 1 \rangle$ that are closed under shifting. In this paper we are interested to study $\mathbb{R} \times \mathbb{S}$ –additive cyclic codes. These codes are generalization of additive cyclic codes over the rings $\mathbb{R}$ and $\mathbb{S}$. These codes have several applications such as the construction of optimal binary linear codes with a large Hamming distance [1] and the construction of optimal covering codes [2].

2 Main Results

Definition 1. A non-empty subset C of $\mathbb{R}^\alpha \times \mathbb{S}^\beta$ is called an $\mathbb{RS}$ –additive code if C is a subgroup of $\mathbb{R}^\alpha \times \mathbb{S}^\beta$, i.e., C is isomorphic to $\mathbb{R}^\gamma \times \mathbb{S}^\delta$, for some positive integers γ and δ .

2010 Mathematical Subject Classification. Primary: 11T71; Secondary: 94B15.

Keywords. additive codes, cyclic additive codes, dual codes.

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Definition 2. A subset C of $\mathbb{R}^\alpha \times S^\beta$ is called an $\mathbb{R} \times \mathbb{S}$ -additive cyclic code if

1. C is an additive code, and
2. For any codeword $u = (a_0 a_1 \dots a_{\alpha-1}, b_0 b_1 \dots b_{\beta-1}) \in C$, its cyclic shift

$$T(u) = (a_{\alpha-1} a_0 \dots a_{\alpha-2}, b_{\beta-1} b_0 \dots b_{\beta-2})$$

is also in C .

Suppose that the ring $\mathbb{R} = \mathbb{S}$ and define the inner product for two elements $(a_0 a_1 \dots a_{\alpha-1}, b_0 b_1 \dots b_{\beta-1}), v = (e_0 e_1 \dots e_{\alpha-1}, d_0 d_1 \dots d_{\beta-1}) \in \mathbb{R}^\alpha \times S^\beta$ to be

$$\langle u, v \rangle = \left(\sum_{i=0}^{\alpha-1} a_i e_i + \sum_{j=0}^{\beta-1} b_j d_j \right) \in \mathbb{R}.$$

Definition 3. Let C be an $\mathbb{R} \times \mathbb{R}$ -additive cyclic code. Define the dual of C to be the code

$$C^\perp = \{v \in \mathbb{R}^\alpha \times S^\beta \mid u \cdot v = 0, \forall u \in C\}.$$

Theorem 1. Let C be an $\mathbb{R} \times \mathbb{R}$ -additive cyclic code. Then the dual code $C^\perp$ of C is also an $\mathbb{R} \times \mathbb{R}$ -additive cyclic code.

3 Conclusion

In this paper we studied $\mathbb{R} \times \mathbb{S}$ -additive cyclic code. We showed that if the ring $\mathbb{R} = \mathbb{S}$, then the dual of any $\mathbb{R} \times \mathbb{S}$ -additive cyclic code is also an additive cyclic code. Using a mapping from the ring $R \times S$ to the ring R or the ring S , these codes can be used to construct good codes over the ring $\mathbb{R}$ or the ring $\mathbb{S}$.

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On the tensor isoclinism of groups

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Abstract. In 1940, P. Hall introduced the concept of isoclinism of groups. In the present paper, we introduce a new notion of tensor isoclinism between two groups. Among other results it is shown that the tensor degree of a given group G , depends only on tensor isoclinism classes of groups.

1 Introduction

The concept of non-abelian tensor product of the groups G and H , denoted by $G \otimes H$, was introduced by R. Brown et. al. [2, 1, 2], and it is generated by the symbols $g \otimes h$. The groups G and H must act on each other and by conjugations on themselves on the left and satisfy the following relations:

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),$$

all these actions must be compatible in the sense that

$$({}^g h)g' = {}^g ({}^h (g^{-1} g')), \quad ({}^h g)h' = {}^h ({}^g (h^{-1} h')),$$

2010 Mathematical Subject Classification. Primary: 47B47; Secondary: 15A78.

Keywords. Non-abelian tensor products, isoclinism groups, tensor isoclinism.

* Speaker

for all $g, g' \in G$ and $h, h' \in H$. In particular, $G \otimes G$ is the tensor square of the group G .

Furthermore, the commutator map $G \times G \rightarrow G'$ induces a homomorphism $\kappa : G \otimes G \rightarrow G'$, sending $g \otimes g'$ to the commutator element $[g, g'] = gg'g^{-1}g'^{-1}$. Moreover, the notion of tensor centre of a group G introduced by G. Ellis [3] as follows:

$$Z^{\otimes}(G) = \{g \in G \mid g \otimes x = 1_{\otimes}, \forall x \in G\},$$

in which $1_{\otimes}$ is the identity element of $G \otimes G$.

For given groups G and H , if $\alpha : \frac{G}{Z^{\otimes}(G)} \rightarrow \frac{H}{Z^{\otimes}(H)}$ is an isomorphism, then the central extension $1 \rightarrow Z^{\otimes}(G) \rightarrow G \rightarrow \frac{G}{Z^{\otimes}(G)} \rightarrow 1$ implies the following exact sequence;

$$1 \rightarrow G \otimes G \rightarrow \frac{G}{Z^{\otimes}(G)} \otimes \frac{G}{Z^{\otimes}(G)} \rightarrow 1.$$

Clearly, we have the same exact sequence for the group H and hence α induces an isomorphism $\beta : G \otimes G \rightarrow H \otimes H$.

Now, considering the above discussion, we have the following definition which is vital in our further investigations.

Definition 1. The groups G and H are said to be *tensor isoclinic* and denoted by $G \stackrel{\otimes}{\sim} H$, when there exists an isomorphism $\alpha : \frac{G}{Z^{\otimes}(G)} \rightarrow \frac{H}{Z^{\otimes}(H)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \frac{G}{Z^{\otimes}(G)} \times \frac{G}{Z^{\otimes}(G)} & \xrightarrow{\alpha \times \alpha} & \frac{H}{Z^{\otimes}(H)} \times \frac{H}{Z^{\otimes}(H)} \\ f_G \downarrow & & \downarrow f_H \\ G \otimes G & \xrightarrow{\beta} & H \otimes H, \end{array}$$

where $f_G(g_1 Z^{\otimes}(G), g_2 Z^{\otimes}(G)) = g_1 \otimes g_2$ and $f_H(h_1 Z^{\otimes}(H), h_2 Z^{\otimes}(H)) = h_1 \otimes h_2$, for each $h_i \in \alpha(g_i Z^{\otimes}(G))$, $i = 1, 2$, and β is the isomorphism induced by α .

One can easily check that the concept of tensor isoclinism is an equivalence relation. Clearly if the groups G and H are isomorphic, they are tensor isoclinic. One notes that, the converse does not hold in general. Also, if G and H are isoclinic, then they may not be tensor isoclinic. Moreover, if G and H are tensor isoclinic then they can not be isoclinic.

The following example shows our claim.

Example 1. Let $A_4 = \langle a, b \mid a^3 = b^2 = (ab)^3 = 1 \rangle$ and $\hat{A}_4 = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$. Using GAP, one can easily check that $Z^{\otimes}(A_4) = 1$ and $Z^{\otimes}(\hat{A}_4) = \mathbb{Z}_2$. Thus $A_4 \stackrel{\otimes}{\sim} \hat{A}_4$, while $A_4 \not\sim \hat{A}_4$ as $|A_4| = 12$ and $|\hat{A}_4| = 24$. On the other hand, $|A'_4| = 4$ and $|\hat{A}'_4| = 8$ and so they are not isoclinic, i. e. $A_4 \not\sim \hat{A}_4$.

Now, consider the Dihedral and Quaternion groups as follows;

$$D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle,$$

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, (ab)^2 = 1 \rangle.$$

Clearly, $Z(D_8) = Z(Q_8) = \langle a^2 \rangle$ and $D'_8 = Q'_8 = \langle a^2 \rangle$. Hence the maps $\alpha : \frac{D_8}{Z(D_8)} \rightarrow \frac{Q_8}{Z(Q_8)}$ and $\beta : D'_8 \rightarrow Q'_8$ are both isomorphisms, and so $D_8 \sim Q_8$. Now by applying GAP, we obtain $Z^{\otimes}(D_8) = Z^{\otimes}(Q_8) = \langle 1 \rangle$. So there is no isomorphism between $D_8/Z^{\otimes}(D_8)$ and $Q_8/Z^{\otimes}(Q_8)$, and hence the groups D_8 and Q_8 can not be tensor isoclinic.

2 Main Results

In this section we study and examine some properties of tensor isoclinism of groups.

Proposition 1. Let G and H be groups with coprime exponents and $H = Z^\otimes(H)$. Then $G \stackrel{\otimes}{\sim} G \times H$.

Lemma 1. Let G and H be any groups, $G \stackrel{\otimes}{\sim} H$ if and only if there exist $A \trianglelefteq G$ and $B \trianglelefteq H$ with $A \leq Z^\otimes(G)$, $B \leq Z^\otimes(H)$ and $\alpha : \frac{G}{A} \longrightarrow \frac{H}{B}$ be isomorphism.

In 1976, Bioch [1] proved that under some conditions subgroups of two isoclinic groups are isoclinic. The following example illustrates, the property of tensor isoclinic of groups can not be carried over to their subgroups even if the conditions of Lemma 2.2 are satisfied..

Example 2. Let $G = A_4$ and $H = \hat{A}_4$ as in Example 1.2. We observe that $A_4 \stackrel{\otimes}{\sim} \hat{A}_4$, $Z^\otimes(\hat{A}_4) = \mathbb{Z}_2$ and $Z^\otimes(A_4) = 1$. Now, assume that $G_1 = \langle 1 \rangle$ and $H_1 = \mathbb{Z}_2$ be the normal subgroups of G and H , respectively. One can calculate that $G_1 \leq Z^\otimes(G)$, $H_1 \leq Z^\otimes(H)$ and $G_1 \not\stackrel{\otimes}{\sim} H_1$.

One notes that, if H is a subgroup of a given group G , then H may not contain the tensor centre subgroup $Z^\otimes(G)$. In the following result we show that under some conditions a given group G is tensor isoclinic with its own subgroup.

Proposition 2. Let G be a group with a subgroup H . Then $H \stackrel{\otimes}{\sim} HZ^\otimes(G)$. In particular, if $G = HZ^\otimes(G)$ then $G \stackrel{\otimes}{\sim} H$. Conversely, if the factor group $G/Z^\otimes(G)$ is finite and $G \stackrel{\otimes}{\sim} H$, then $G = HZ^\otimes(G)$.

In the following, we give a criterion for a group to be tensor isoclinic with its factor group.

Proposition 3. Let N be a normal subgroup of a given group G . Then $N \leq Z^\otimes(G)$ if and only if $G \stackrel{\otimes}{\sim} G/N$.

The above proposition gives the following

Corollary 1. Let H be a subgroup, and N a normal subgroup of a group G .

- (i) If $N \leq Z^\otimes(G)$, then $G \stackrel{\otimes}{\sim} \frac{G}{(M \cap N)}$, for any normal subgroup M of G .
- (ii) If $G = HZ^\otimes(G)$, then $G \stackrel{\otimes}{\sim} \langle H, K \rangle$, for any subgroup K of G .
- (iii) If $N \leq Z^\otimes(H)$, then $\frac{HN}{N} \stackrel{\otimes}{\sim} H$.
- (iv) If $N \leq Z^\otimes(H)$ and $G = HZ^\otimes(G)$, then $\frac{HN}{N} \stackrel{\otimes}{\sim} \frac{G}{N}$.
- (v) Let $f : G \rightarrow H$ be an epimorphism of groups, then $\text{Ker } f \leq Z^\otimes(G)$ if and only if $G \stackrel{\otimes}{\sim} H$.

Theorem 1. Let G be any group such that the tensor central factor group satisfies descending chain condition on its normal subgroups. Then for any normal subgroup H of G the following statements are equivalent:

- (i) $G = HZ^\otimes(G)$;
- (ii) $G \stackrel{\otimes}{\sim} H$;
- (iii) $\frac{G}{Z^\otimes(G)} \cong \frac{H}{Z^\otimes(H)}$.

We remark that in 2009, Moghaddam, Niroomand and Jafari [6] proved that, for a finite group G one has always $Z^\otimes(G) \leq G'$. So in our case there is no point to search for the notion of stem groups.

In the following result, we investigate which properties of groups preserve under the notion of tensor isoclinic.

Theorem 2. Let the groups G and H be tensor isoclinic.

- (i) G is soluble if and only if H is soluble;
- (ii) The property of supersolubility transfers from G to H , when $Z^\otimes(H)$ is finite.

Theorem 3. If G is a nilpotent group of class n and tensor isoclinic with a group H , then H is also nilpotent of class at most $n + 1$, for any natural number n .

Theorem 4. Let G and H be finite groups and $G \approx^\otimes H$. Then G is π -separable if and only if H is π -separable.

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On the characterization of bi Γ –ideals of the type $(\in, \in \vee q_k)$ in ordered Γ –semigroups

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Abstract. Finite semigroups played a vital role in the study of linguistics, models of computation, reasoning and many finite automaton. The inception of fuzzy group theory have given birth and avenue for new researches. In this paper, we give some classifications of new form of fuzzy bi Γ –ideals of the type $(\in, \in \vee q_k)$ in ordered Γ –semigroup. Weakly regular and intra-regular ordered Γ –semigroups in terms of $(\in, \in \vee q_k)$ –fuzzy bi Γ –ideals, particularly we examine that an ordered Γ –semigroup G is left, right and regular simple if and only if every $(\in, \in \vee q_k)$ –fuzzy bi Γ –ideals of G is a constant function.

1 Introduction

Since the inception of the notion of a fuzzy set in 1965 by Zadeh [1], which laid the foundations of fuzzy set theory, the literature on fuzzy set theory and its applications has been growing rapidly amounting by now to several studies. These are widely scattered over many disciplines such as artificial intelligence, computer science, control engineering, expert

2010 Mathematical Subject Classification. Primary: 20N25; Secondary: 20F60

Keywords. $(\in, \in \vee q_k)$ –fuzzy bi Γ –ideals and Regular Ordered Γ –semigroup

* Speaker

systems, management science, operations research, pattern recognition, robotics, and others. Rosenfeld [2] was the first to give birth the idea of fuzzy subgroup using the concept of fuzzy set. Thereafter, amazing and fundamental concepts of researches are being produced with regards to properties of group theory to expand the scope of fuzzy groups. As a generalization of a semigroup as well as ternary semigroup, Sen [3], introduced the notion of Γ -semigroup in 1981 and developed some theory on Γ -semigroups. Since the advent of Γ -semigroups, many mathematicians have analyzed a lot towards that concept. Recently, Pal *et al.* [4] investigate some properties of fuzzy ideals, fuzzy bi-ideals and fuzzy $(1, 2)$ -ideals and characterize a po- Γ -semigroup which is left(right) simple, regular, intra-regular, regular in terms of fuzzy ideals and fuzzy bi-ideals and also give a pointwise characterisation of fuzzy regular subsemigroups in a po- Γ -semigroup. It is a widely known fact that the notion of a one-sided ideal of rings and semigroups is a generalization of the notion of an ideal of rings and semigroups.

Many classical notions and results of the theory of semigroups have been extended and generalized. Khan *et al.* [5] prove that an ordered Γ -semigroup is fuzzy simple if and only if every fuzzy interior Γ -ideal is a constant function, and they characterize intra-regular ordered Γ -semigroups in terms of interior (resp. fuzzy interior) Γ -ideals. The notion of fuzzy ideals of a Γ -semigroup has been introduced by Sardar *et al.* [6] and some of their properties have been investigated. Characterization of a regular Γ -semigroup in terms of fuzzy ideals has been obtained. Operator semigroups of a Γ -semigroup has been made to work by obtaining various relationships between fuzzy ideals of a Γ -semigroup and that of its operator semigroups by the same author. As the motivation from the cited references, in this paper we give some classifications of new form of fuzzy bi- Γ -ideals of the type $(\in, \in \vee q_k)$ in ordered Γ -semigroup. Some classification of left, right simple and regular ordered Γ -semigroups in terms of $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals are also provided.

2 Preliminaries

In this part, some previous results on bi Γ -ideals which are used in this paper are provided.

Definition 2.1 [7] Fuzzy Bi-ideals of Ordered Semigroup

A fuzzy subset f of an ordered semigroup G is called a fuzzy bi-ideal of G if the following statements hold for all $a, b, c \in G$:

- (i) $f(a) \geq f(b)$ for every $a \leq b$,
- (ii) $f(abc) \geq \min \{f(a), f(c)\}$.

3 Main Results

In this section we present our main results on the $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideal and some characterisations. A new generalization of bi Γ -ideal of the form $(\in, \in \vee q_k)$ is introduced. Throughout this section G is an ordered Γ - semigroup unless otherwise stated.

Definition 3.1 Given a fuzzy subset I of G , then λ is called $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideal of G for all $a, b, c \in G$, $\alpha, \beta \in \Gamma$ and $t_1, t_2 \in (0, 1]$ if the following three conditions are satisfied:

- (i) $b_t \in I \rightarrow a_t \in \vee q_k I$ such that $a \leq b$,
- (ii) $a_{t_1} \in I, b_{t_2} \in I \rightarrow (a\alpha b)_{\min\{t_1, t_2\}} \in \vee q_k I$,
- (iii) $a_{t_1} \in I, c_{t_2} \in I \rightarrow (a\alpha b\beta c)_{\min\{t_1, t_2\}} \in \vee q_k I$.

The proposition below gives a link between ordinary bi Γ -ideal and fuzzy bi Γ -ideal of the form $(\in, \in \vee q_k)$.

Proposition 3.2 Given a nonzero $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals of G , then $\lambda_0 = \{a \in G : \lambda(a) > 0\}$ is a bi Γ -ideal of G .

The lower part of the characteristic function and fuzzy bi Γ -ideal are linked in the following proposition.

Proposition 3.3 Given the lower part of characteristic function $\bar{\chi}_A^{-k}, \bar{\chi}_A^{-k}$ of A is a $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals of G if and only if A is a bi Γ -ideals of G .

The following proposition gives a relationship between a $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideal of G and a fuzzy bi Γ -ideal of G .

Proposition 3.4 A fuzzy subset λ of G is a $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals of G if and only if $\bar{\lambda}^{-k}$ is a fuzzy bi Γ -ideals of G .

Based on the characterization of regular, left and right simple and regular of G of the form $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideal, the following theorem is presented.

Theorem 3.5 An ordered Γ -semigroup G is left, right simple and regular if and only if for all λ a $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals of G , $\bar{\lambda}^{-k}$ is a constant function.

4 Conclusion

The classical notions and results of the theory of semigroups have been extended and generalized. The notion of Γ -semigroup is the generalization of semigroup likewise ordered Γ -semigroup is the generalization of ordered semigroup. In this paper, we give some classifications of new form of fuzzy bi- Γ -ideals of the type $(\in, \in \vee q_k)$ in ordered Γ -semigroup and similarly it's proved that in ordered Γ -semigroups of the form $(\in, \in \vee q_k)$ -fuzzy bi Γ -ideals is a constant function if and only if it is a left, right simple and regular ordered Γ -semigroups.

Acknowledgment: The authors would like to acknowledge the Ministry of Higher Education Malaysia (MOHE) for

the Fundamental Research Grant Scheme with Vote No. 4F898 and the first author would also like to express his appreciation for the partial financial support from International Doctorate Fellowship (IDF) UTM.

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Complete classifications of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras

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Abstract. To describe groups of automorphisms of 2-dimensional real algebras one needs a classification of such algebras up to isomorphism. In this paper a complete classifications of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras are given. In the case of real evolution algebras we describe their groups of automorphisms and derivation algebras as well.

1 Introduction

The classification problem of finite dimensional algebras is important in algebra. In this extended abstract we consider such problem for two-dimensional algebras over the field of real numbers $\mathbb{R}$. We provide for each class of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras the corresponding lists of algebras, by their matrices of structure constants, such that any nontrivial 2-dimensional real algebra from these algebras is isomorphic to only one algebra from the corresponding list of algebras. Similar results are stated in [1, 3]. In [3] case the authors state the existence only whereas the uniqueness can not be guaranteed. In [1] authors consider the problem over algebraically closed fields. Our approach is similar to of [1] but different than of [3]. For further information related to such problems one can see [2, 4, 5].

2010 Mathematical Subject Classification. Primary:15A72; Secondary: 17A60, 15A50.

Keywords. structure constants, division, evolution, Jordan algebras.

* Speaker

2 Main Results

Let $\mathbb{A}$ be any 2- dimensional algebra over $\mathbb{R}$ with multiplication $\cdot$ given by a bilinear map $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$ whenever $\mathbf{u}, \mathbf{v} \in \mathbb{A}$. If $e = (e^1, e^2)$ is a basis for $\mathbb{A}$ as a vector space over $\mathbb{R}$ then one can represent this bilinear map by a matrix

$$A = \begin{pmatrix} A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1 \\ A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2 \end{pmatrix} \in Mat(2 \times 4; \mathbb{R})$$

such that

$$\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v)$$

for any $\mathbf{u} = eu, \mathbf{v} = ev$, where $u = (u_1, u_2)$, and $v = (v_1, v_2)$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively, $(u \otimes v) = (u_1v_1, u_1v_2, u_2v_1, u_2v_2)$, $e^i \cdot e^j = A_{i,j}^1 e^1 + A_{i,j}^2 e^2$ whenever $i, j = 1, 2$. So the algebra $\mathbb{A}$ is presented by the matrix $A \in Mat(2 \times 4; \mathbb{R})$ (called the matrix of structure constants (MSC) of $\mathbb{A}$ with respect to the basis e).

Definition 1. Two-dimensional algebras $\mathbb{A}, \mathbb{B}$, given by their matrices of structural constants A, B , are said to be isomorphic if $B = gA(g^{-1})^{\otimes 2}$ holds true for some $g \in GL(2, \mathbb{R})$.

Theorem 1. Any non-trivial 2-dimensional real algebra is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:

$$A_{1,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 + 1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1 + 1 & -\alpha_2 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{R}^4,$$

$$A_{2,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3,$$

$$A_{3,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3,$$

$$A_{4,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}, \text{ where } \mathbf{c} = (\beta_1, \beta_2) \in \mathbb{R}^2,$$

$$A_{5,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{R}^2,$$

$$A_{6,r}(c) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1 - \alpha_1 & 0 \end{pmatrix}, \text{ where } c = \alpha_1 \in \mathbb{R},$$

$$A_{7,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1 - \alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2,$$

$$A_{8,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1 - \alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \text{ where } \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2,$$

$$A_{9,r}(c) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}, \text{ where } c = \beta_1 \in \mathbb{R},$$

$$\begin{aligned}
A_{10,r}(c) &= \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 - \alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \text{ where } c = \alpha_1 \in \mathbb{R}, \\
A_{11,r} &= \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \\
A_{12,r} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad A_{13,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \\
A_{14,r} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Definition 2. $\mathbb{A}$ is said to be a commutative Jordan algebra if it is commutative and $(\mathbf{u}\mathbf{v})\mathbf{u}^2 = \mathbf{u}(\mathbf{v}\mathbf{u}^2)$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{A}$.

Theorem 2. Any non-trivial 2-dimensional real commutative Jordan algebra is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:

$$\begin{aligned}
&A_{2,r}(1/2, 0, 1/2), \quad A_{3,r}(1/2, 0, 1/2), \quad A_{5,r}(2/3, 1/3), \\
&A_{5,r}(1/2, 1/2), \quad A_{5,r}(1, 0), \quad A_{15,r}.
\end{aligned}$$

Definition 3. A finite dimensional algebra $\mathbb{A}$ is said to be division algebra if $\mathbf{u} \cdot \mathbf{v} = 0$ is valid if and only if at least one of $\mathbf{u}, \mathbf{v} = 0$ is zero.

Theorem 3. Any nontrivial two-dimensional real division algebra is isomorphic to only one algebra from the following listed, by their matrices of structure constants, division algebras: $A_{1,r}(\mathbf{c})$, for which $\Delta_m^2 - 4\Delta_l\Delta_r < 0$, $A_{2,r}(\mathbf{c})$, for which $\beta_1^2 + 4\alpha_1(1 - \alpha_1)\beta_2 < 0$, $A_{3,r}(\mathbf{c})$, for which $\beta_1^2 - 4\alpha_1(1 - \alpha_1)\beta_2 < 0$, $A_{4,r}(\mathbf{c})$, for which $(1 - \beta_2)^2 - 4\beta_1 < 0$, $A_{7,r}(\mathbf{c})$, for which $\beta_1^2 - 4\alpha_1^2(1 - \alpha_1) < 0$, $A_{8,r}(\mathbf{c})$, for which $\beta_1^2 + 4\alpha_1^2(1 - \alpha_1) < 0$, $A_{9,r}(\mathbf{c})$, for which $1 - 4\beta_1 < 0$ and $A_{12,r}$.

Definition 4. An n -dimensional algebra $\mathbb{E}$ is said to be an evolution algebra if it admits a basis $\{e^1, e^2, \dots, e^n\}$ such that $e^i e^j = 0$ whenever $i \neq j$, $i, j = 1, 2, \dots, n$.

Theorem 4. Any nontrivial 2-dimensional real evolution algebra is isomorphic to only one algebra from the following listed, by their matrices of structure constants, algebras:

$$\begin{aligned}
E_{1,r}(c, b) &\simeq E_{1,r}(b, c) = \begin{pmatrix} 1 & 0 & 0 & b \\ c & 0 & 0 & 1 \end{pmatrix}, \text{ where } bc \neq 1, (b, c) \in \mathbb{R}^2, \\
E_{2,r}(b) &= \begin{pmatrix} 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ where } b \in \mathbb{R}, \quad E_{3,r} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
E_{4,r} &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_{5,r} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$E_{6,r} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad E_{7,r} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the talk we provide a description of their groups of automorphisms and derivation algebras as well.

3 Conclusion

The general classification Theorem 2.1 can be used for classification of different other classes of 2-dimensional real, for example, associative, noncommutative Jordan, alternative and etcetera, algebras. It can be used to describe the groups of automorphisms, algebra of derivatives of 2-dimensional real algebras as well. In general the result is helpful in any problem related to 2-dimensional real algebras.

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On the annihilator graph of a commutative semigroup

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This is a joint work with Mojgan Afkhami and Seyed Mohammad Sakhdari

Abstract. In this talk, we define and study the annihilator graph associated to a commutative semigroup with zero. Also, we characterize all annihilator graphs with three or four vertices.

1 Introduction

Let S be a commutative semigroup with zero whose operation is written multiplicatively and $S^* = S \setminus \{0\}$. The set of all zero-divisors of S is denoted by $Z(S)$ and $Z(S)^* = Z(S) \setminus \{0\}$.

The zero-divisor graph of S is a simple undirected graph $\Gamma(S)$ whose vertices are $Z(S)^*$, and x is adjacent to y in $\Gamma(S)$ if and only if $xy = 0$, for each two distinct elements x and y in $Z(S)^*$ (see [3]). In [2], A. Badawi introduced the concept of the annihilator graph for a commutative ring R , denoted by $AG(R)$, with vertices $Z(R)^*$ and $x \sim y$ is an edge in $AG(R)$ if and only if $\text{ann}_R(xy) \neq \text{ann}_R(x) \cup \text{ann}_R(y)$, where $\text{ann}_R(x) = \{r \in R \mid xr = 0\}$. Badawi shows that $AG(R)$ is a subgraph of $\Gamma(R)$ and $AG(R)$ is connected with $\text{diam}(AG(R)) \leq 2$ and $\text{gr}(AG(R)) \leq 4$.

In this talk, we introduce the annihilator graph for a commutative semigroup S and denote it by $AG(S)$. The graph $AG(S)$ is an undirected graph with vertex set $Z(S)^*$ and two distinct vertices x and y are adjacent if and only if $\text{ann}_S(xy) \neq \text{ann}_S(x) \cup \text{ann}_S(y)$, where $\text{ann}_S(x) = \{s \in S \mid xs = 0\}$. In this talk, we investigate some basic properties of $AG(S)$ by means of $\Gamma(S)$. Also we show that if $Z(S) \neq S$, then $AG(S)$ is a subgraph of $\Gamma(S)$ and if $Z(S) = S$, then we express a necessary and sufficient condition for the connectivity of $AG(S)$. Finally, we characterize all semigroups S such that $AG(S)$ has three or four vertices (see [1] and [4]). Note that if S is a group, then every elements of S have inverse, and so $AG(S)$ is the null graph.

2010 Mathematical Subject Classification. Primary: 20M14; Secondary: 05C99

Keywords. Zero-divisor graph, Annihilator graph, Reduced semigroup.

* Speaker

2 Some Basic Properties of Annihilator Graph

We assume that $|Z(S)^*| \geq 3$. The case where $|Z(S)^*| \leq 2$ is easy. Indeed, if $|Z(S)^*| = 1$, then $AG(S) \cong \Gamma(S) \cong K_1$. Let $|Z(S)^*| = 2$. Then $\Gamma(S) \cong K_2$. Now if $Z(S) = S$, then clearly $AG(S) \cong 2K_1$ and if $Z(S) \neq S$, then $AG(S) \cong \Gamma(S) \cong K_2$. We prove the following theorems

Theorem 1. If $Z(S) \neq S$, then we have $\Gamma(S) \leq AG(S)$.

Theorem 2. Let x and y be distinct adjacent vertices in $AG(S)$ such that $xy \neq 0$, $x^2y \neq 0$ and $y^2x \neq 0$. Then there exists $w \in Z(S)^*$ such that $x \sim w \sim y$ is a path in $AG(S)$ that is not a path in $\Gamma(S)$. Also in this case $\text{gr}(AG(S)) = 3$.

Theorem 3. Let S be reduced. Also suppose that $\Gamma(S) \leq AG(S)$ and $\Gamma(S) \neq AG(S)$. Then $\text{gr}(AG(S)) = 3$. Also there exists a cycle c of length three in $AG(S)$ such that each edge of c is not an edge in $\Gamma(S)$. Moreover if S is reduced, $Z(S) \neq S$ and $\Gamma(S) \neq AG(S)$, then $\text{gr}(AG(S)) = 3$.

Theorem 4. (i) Assume that S is reduced and $AG(S)$ is complete. Then $\Gamma(S)$ is complete.

(ii) Let S be reduced, and let $Z(S) = S$. Then $AG(S)$ is not a complete graph.

3 Annihilator Graphs With Three Vertices

In this section we characterize all annihilator graphs with three vertices. Throughout this section, we assume that $Z(S) = \{0, x, y, z\}$. Since $\Gamma(S)$ is connected, there exists a vertex z such that z is adjacent to both vertices x and y in $\Gamma(S)$. Therefore, without loss of generality, we may assume that $zx = zy = 0$. If $Z(S) \neq S$, then $\Gamma(S) \leq AG(S)$, and so $AG(S)$ is connected, which implies that $AG(S)$ is isomorphic to non of the graphs $K_1 \cup K_2$ and $3K_1$. Therefore $AG(S)$ is isomorphic to $K_{1,2}$ or K_3 .

Theorem 5. Let $Z(S) \neq S$. Then $AG(S) \cong K_{1,2}$ (with center z) if and only if one of the following statements holds:

- (1) $xy = x$ or $xy = y$.
- (2) $xy = z$ and $x^2 = y^2 = z^2 = 0$.

Theorem 6. Let $Z(S) \neq S$. Then $AG(S) \cong K_3$ if and only if one of the following statements holds:

- (1) $xy = 0$.
- (2) $xy = z$ and we either have $x^2 \neq 0$ or $y^2 \neq 0$.

Theorem 7. Let $Z(S) = S$. Then $AG(S) \cong K_2 \cup K_1$ (with x is adjacent to y , and z is an isolated vertex) if and only if $xy = z$ and we either have $x^2 \neq 0$ or $y^2 \neq 0$. Otherwise $AG(S) \cong 3K_1$.

4 Annihilator Graphs With Four Vertices

Throughout this section we assume that $Z(S) = \{0, x, y, z, w\}$. We use $K_3 + \{wx\}$ to denote a complete graph K_3 with vertex set $\{x, y, z\}$ together with the end vertex w , where w is adjacent to x . Also we use $K_4 \setminus \{wy\}$ to denote a

complete graph K_4 , such that an edge between w and y is deleted. As usual P_n and C_n will denote the path of length n and a cycle with n vertices, respectively.

Let $\Gamma(S)$ be connected with vertex set $\{x, y, z, w\}$. If $\Gamma(S)$ is not isomorphic to $K_{1,3}$, then $\Gamma(S)$ contains at least a path of length three and we denote this path by $w \sim x \sim y \sim z$. If $\Gamma(S) \cong K_{1,3}$, then we denote the center of $\Gamma(S)$ by x . Thus, without loss of generality, we may assume that $wx = xy = yz = 0$ or $wx = xy = xz = 0$.

Theorem 8. Let $S = \{0, 1, x, y, z, w\}$ be a commutative semigroup, with non-zero identity 1, and $Z(S) = \{0, x, y, z, w\}$. Then the following statements hold.

- (1) $AG(S)$ is connected and it is not isomorphic to any of the graphs $4K_1$, $2K_2$, $K_2 \cup 2K_1$, $P_2 \cup K_1$ or $K_3 \cup K_1$.
- (2) There is no semigroup S such that $AG(S) \cong P_3$.
- (3) $AG(S) \cong C_4$ if and only if $\Gamma(S) \cong P_3$ or $\Gamma(S) \cong C_4$. Moreover there are exactly sixty six semigroups S such that their annihilator graphs are isomorphic to C_4 .
- (4) If $AG(S) \cong K_4 \setminus \{wy\}$, then $\Gamma(S) \cong K_4 \setminus \{wy\}$ or $\Gamma(S) \cong K_3 + \{wx\}$. Moreover there exist fifty eight semigroups S such that their annihilator graphs are isomorphic to $K_4 \setminus \{wy\}$.
- (5) If $AG(S) \cong K_4$, then $\Gamma(S)$ is isomorphic to one of the graphs K_4 , $K_4 \setminus \{wy\}$, $K_3 + \{wx\}$ or $K_{1,3}$. Moreover there are exactly one hundred sixty six semigroups S such that their annihilator graphs are isomorphic to K_4 .
- (6) If $AG(S) \cong K_3 + \{wx\}$, then $\Gamma(S) \cong K_3 + \{wx\}$. Moreover there are exactly seventeen semigroups S such that $AG(S) \cong K_3 + \{wx\}$.
- (7) If $AG(S) \cong K_{1,3}$, then $\Gamma(S) \cong K_{1,3}$. Moreover there are exactly one hundred sixty two semigroups S such that $AG(S) \cong K_{1,3}$.

Theorem 9. Let $S = Z(S) = \{0, x, y, z, w\}$ be a commutative semigroup. Then the following statements hold.

- (1) There are only two semigroups S such that $AG(S)$ is connected and in this case $AG(S) \cong P_3$.
- (2) $AG(S)$ is not isomorphic to $2K_2$.
- (3) If $AG(S) \cong 4K_1$, then $\Gamma(S)$ is isomorphic to one of the graphs C_4 , K_4 , $K_4 \setminus \{wy\}$ or $K_{1,3}$. Moreover, there are exactly three hundred seventy five semigroups S such that annihilator graphs are isomorphic to $4K_1$.
- (4) If $AG(S) \cong K_2 \cup 2K_1$, then $\Gamma(S)$ is isomorphic to one of the graphs $K_4 \setminus \{wy\}$ or $K_3 + \{wx\}$. Moreover, there are exactly fifty nine semigroups S such that their annihilator graphs are isomorphic to $K_2 \cup 2K_1$.
- (5) If $AG(S) \cong K_{1,2} \cup K_1$, then $\Gamma(S)$ is isomorphic to $K_3 + \{wx\}$. Moreover, there are exactly four semigroups S such that their annihilator graphs are isomorphic to $K_{1,2} \cup K_1$.
- (6) If $AG(S) \cong K_3 \cup K_1$, then $\Gamma(S)$ is isomorphic to one of the graphs $K_3 + \{wx\}$ or $K_{1,3}$. Moreover, there are exactly twenty two semigroups S such that their annihilator graphs are isomorphic to $K_3 \cup K_1$.

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Generating some finite groups using sequential insertion systems

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Abstract. The operation of insertion in theoretical computer science has spurred much interest among researchers across many disciplines, where initially, it was introduced to generalize the concatenation operation. Previously, it was shown that a new variant of insertion, called sequential insertion systems with interactions, could generate all finite cyclic and dihedral groups. In this paper, we show that sequential insertion systems with interactions are also able to generate the quaternion group and all symmetric groups. We also determine the relation between sequential insertion systems with interactions and bonded sequential insertion systems.

1 Introduction

Numerous studies in theoretical computer science on the operation of insertion have been conducted, where most aim to determine its computational properties. The many variants of insertion serve as different mechanisms to generate languages by means of concatenation at arbitrary positions in an axiom or an iterated/generated word. Some of the variants of insertion include sequential, parallel, controlled, permuted, and scattered [1]. Recently, a multidisciplinary interest in theoretical computer science and biochemistry led to the introduction of bonded variants of insertion systems

2010 Mathematical Subject Classification. Primary: 20F99; Secondary: 20B99.

Keywords. Sequential insertion systems with interactions, Groups, Quaternion group, Symmetric group, Formal Languages.

* Speaker

[2, 3], which include bonded sequential insertion systems (shortly, bSINS-system s). Furthermore, inspired by the application of formal languages in group theory as seen in [4, 5, 6, 7], Fong *et. al* introduced sequential insertion systems with interactions, which enabled the generation of all finite cyclic and dihedral groups [8].

Sequential insertion systems with interactions, introduced in [8], is formally defined as follows: A *sequential insertion system with interaction* (*SINS-system for short) is a quadruple $\zeta = (\Sigma, A, I, *)$, where Σ is an alphabet, $A \subseteq \Sigma^*$ is a finite set of axioms, $I \subseteq \Sigma^*$ is a finite set of insertion rules and $*$ is a binary operation, such that for all $\beta \in \Sigma^*$, $\beta = \alpha_1 \alpha_2 = \alpha_1 * \alpha_2$, where $\alpha_1, \alpha_2 \in \Sigma^*$.

The derivation relation $\Rightarrow_\zeta$ is defined as follows: for $\alpha, \beta \in \Sigma^*$, $\alpha \Rightarrow_\zeta \beta$ if and only if there exists an $\alpha' \in I$ such that $\alpha * \alpha' = \beta$.

The reflexive and transitive closure of $\Rightarrow_\zeta$ is denoted by $\Rightarrow_\zeta^*$. Should there be no danger of confusion, we write $\Rightarrow$ and $\Rightarrow^*$ instead of $\Rightarrow_\zeta$ and $\Rightarrow_\zeta^*$, respectively.

The language generated by a *SINS-system $\zeta = (\Sigma, A, I, *)$ is defined as

$$L(\zeta) = \{\beta \mid \text{there exists an axiom } \alpha \in A \text{ such that } \alpha \Rightarrow_\zeta^* \beta\}.$$

From there, it was shown that the *SINS-system s are able to generate all finite cyclic and dihedral groups. In this paper, we extend this idea and show that the *SINS-system s are able to generate the quaternion group $\mathcal{Q}$ and also all symmetric groups S_n of order $n!$. Not only that, we compare the generative power of *SINS-system s against bSINS-system s and determine their relation.

2 Main Results

A trivial yet overlooked result in [8] regarding the sequential insertion systems with interactions is that it is in fact a generalization of sequential insertion systems. By selecting concatenation as the interaction of the system, we immediately obtain the usual sequential insertion system. Hence, we obtain the following theorem.

Theorem 1. For every sequential insertion system $\text{SINS} = (\Sigma, A, I)$, there exists a *SINS-system $\zeta = (\Sigma, A, I, *)$ such that $L(\zeta) = L(\text{SINS})$, where the interaction $*$ is concatenation.

Furthermore, we show that sequential insertion systems with interactions are able to generate some special types of finite groups.

Theorem 2. For the quaternion group $\mathcal{Q}$ of order 8, there exists a *SINS-system $\zeta = (\Sigma, A, I, *)$ such that $L(\zeta) = \mathcal{Q}$.

Theorem 3. For every symmetric group S_n of order $n!$, there exists a *SINS-system $\zeta = (\Sigma, A, I, *)$ such that $L(\zeta) = S_n$.

Next, we determine the relation between sequential insertion systems with interactions and bonded sequential insertion systems.

Theorem 4. For every bSINS-system $\gamma = (\Sigma, A, I)$, there exists a *SINS-system $\zeta = (\Sigma, A, I, *)$ such that $L(\zeta) = L(\gamma)$.

By showing that there exists a language that belongs to $\mathcal{L}(*\text{SINS})$ but not $\mathcal{L}(\text{bSINS})$, we prove that the relation $\mathcal{L}(\text{bSINS}) \subset \mathcal{L}(*\text{SINS})$ holds.

3 Conclusion

We have shown that sequential insertion systems with interactions are able to generate the quaternion group of order 8 and all symmetric groups. We have also determined the relation between sequential insertion systems with interactions and bonded sequential insertion systems, where the relation $\mathcal{L}(\text{bSINS}) \subset \mathcal{L}(*\text{SINS})$ holds. The next step is to compare *SINS-systems with bPINS-systems to construct the language hierarchy.

Acknowledgements

The first author would like to thank the Ministry of Higher Education Malaysia (MOHE) for his MyBrainSC scholarship. The second and third author would like to acknowledge MOHE and Research Management Centre (RMC) of Universiti Teknologi Malaysia (UTM) for the financial funding through Fundamental Research Grant Scheme (FRGS) Vote No. 4F590 and Research University Grant Vote No. 13H18. The fourth author would like to express his gratitude to the International Islamic University Malaysia through MOHE for his financial funding of Vote No. FRGS13-066-0307.

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The Schur multiplier and capability of pairs of groups of order p^4

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Abstract. In this research, the Schur multiplier and capability of pairs of all abelian groups of order p^4 are determined.

1 Introduction

The Schur multiplier of a group G , denoted by $M(G)$, was first introduced by Schur [1] in 1904. In 1998, Ellis [2] extended the notion of the Schur multiplier of groups to the Schur multiplier of a pair of groups.

The study of capability of groups was initiated by Baer [3] in 1938 who determined all capable abelian groups. Following Hall and Senior [4], a group G is capable if there exists a group H such that $G \cong H/Z(H)$. In 1996, Ellis [5] extended the theory of the capability of groups to a theory for a pair of groups. He defined the capability of a pair of groups in terms of Loday's notion of a relative central extension.

Groups, Algorithms and Programming (GAP) software [6] is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. In this paper, we use GAP software to compute the capable pairs of abelian groups of order p^4 .

In this research, we consider the classification of abelian groups of order p^4 in Theorem 1 to determine the Schur multiplier and capability of pairs of abelian groups of order p^4 .

2010 Mathematical Subject Classification. Primary:20F05; Secondary: 20J05.

Keywords. Schur multiplier, capability, pairs of groups.

* Speaker

Theorem 1. [7] Let G be an abelian group of order p^4 where p is an odd prime. Then exactly one of the following holds:

$$G \cong \mathbb{Z}_{p^4}; \quad (1)$$

$$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p; \quad (2)$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}; \quad (3)$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p; \quad (4)$$

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \quad (5)$$

2 Main Results

In this section, the Schur multiplier and capability of pairs of abelian groups order p^4 are presented. The following theorem gives the Schur multiplier of pairs of pairs of abelian groups order p^4 is stated.

Theorem 2. Let G be an abelian group of order p^4 where p is an odd prime and (G, N) be an arbitrary pair of finite groups where N is a normal subgroup of G . Then exactly one of the following holds:

$$M(G, N) \cong \begin{cases} 1 & ; \text{ if } G \text{ is of Type (1) or} \\ & G \text{ is of Type (2) to Type (5) when } N \cong 1, \\ \mathbb{Z}_p & ; \text{ if } G \text{ is of Type (2) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_{p^3}, \mathbb{Z}_p \text{ or } G, \\ \mathbb{Z}_{p^2} & ; \text{ if } G \text{ is of Type (3) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \text{ or } G, \\ (\mathbb{Z}_p)^2 & ; \text{ if } G \text{ is of Type (2) when } N \cong \mathbb{Z}_{p^2}, \\ & G \text{ is of Type (3) when } N \cong \mathbb{Z}_p, \\ & G \text{ is of Type (4) when } N \cong \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_p, \\ (\mathbb{Z}_p)^3 & ; \text{ if } G \text{ is of Type (2) when } N \cong (\mathbb{Z}_p)^2, \\ & G \text{ is of Type (4) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, (\mathbb{Z}_p)^3, (\mathbb{Z}_p)^2 \text{ or } G, \\ (\mathbb{Z}_p)^5 & ; \text{ if } G \text{ is of Type (3) when } N \cong (\mathbb{Z}_p)^2, \\ & G \text{ is of Type (5) when } N \cong (\mathbb{Z}_p)^2, \\ (\mathbb{Z}_p)^6 & ; \text{ if } G \text{ is of Type (5) when } N \cong (\mathbb{Z}_p)^3 \text{ or } G. \end{cases}$$

Next, the capability of pairs of abelian groups of order p^4 is stated in the following conjectures.

Conjecture 3. Let G be a cyclic group of order p^4 where p is an odd prime. Suppose that (G, N) is a pair of G where N is a normal subgroup of G . Then (G, N) is capable if $N = 1, \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.

Conjecture 4. Let G be an abelian group of order p^4 where p is an odd prime. Suppose that (G, N) is a pair of G where N is a normal subgroup of G . Then (G, N) is capable if

1. $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ for $N = 1, \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

2. $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ for all N of G .
3. $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ for $N = 1, \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p$ or $(\mathbb{Z}_p)^3$.
4. $G \cong (\mathbb{Z}_p)^4$ for all N of G .

3 Conclusion

The Schur multiplier and capability of pairs of abelian groups of p^4 are presented in this paper. The results are shown in Theorem 2, Conjecture 3 and Conjecture 4.

Acknowledgments. The authors would like to thank Ministry of Higher Education (MOHE) Malaysia and Research Management Centre, Universiti Teknologi Malaysia (RMC UTM), for the financial support through the Fundamental Research Grant Scheme (FRGS) Vote No. 4F898 The first author would also like to thank Ministry of Higher Education (MOHE) Malaysia for her MyPhD Scholarship.

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On elliptic curves as cryptographic pairing groups

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Abstract. Elliptic curve is a set of two variable points on polynomials of degree 3 over a field acted by an addition operation forms a group structure. The motivation of this note is that the mathematics behind elliptic curve to the applicability within a cryptosystem. Nowadays, pairings, bilinear maps, on elliptic curve are popular to construct cryptographic protocols. Pairings help to transform a discrete logarithm problem on an elliptic curve to the discrete logarithm problem in finite fields. The purpose of this paper is to introduce elliptic curve, bilinear pairings on elliptic curves as based on pairing cryptography. Also this paper serve as a basis in guiding anyone interested to understand one of the applications group theory in cryptosystem.

1 Introduction

Cryptography is an evolving field that studies discrete mathematical equations which are representable by computer algorithm to provide message confidentiality. Modern mathematical cryptography involves many areas of mathematics, including especially number theory, abstract algebra (groups, rings, fields), probability and statistics. In this paper the researcher focuses on elliptic curves groups, that are used extensively as a based cryptography. In brief, elliptic curve is a study of points on two-variable polynomials of degree 3 over a field. With curve defined over a finite field, this set of points acted by an addition operation forms a finite group structure.

The discrete logarithm problem (DLP) in an additively written group $G = \langle P \rangle$ of order n is the problem, given P and Q , of finding the integer $x \in [0, n - 1]$ such that $Q = xP$. The closely related Diffie-Hellman problem (DHP) is the

2010 Mathematical Subject Classification. Primary:11-01; Secondary: 11Gxx, 11G05.

Keywords. elliptic curve, bilinear map, pairing-based cryptography.

* Speaker

problem, given P , aP , and bP , of finding abP . The protocol can easily be extended to three parties, as illustrated by the two-round protocol depicted in the following that Joux[3] devised.

Let $(G_1, +)$ and $(G_2, \cdot)$ be cyclic groups of prime order n , $G = \langle P \rangle$ and $\hat{e} : G_1 \times G_1 \rightarrow G_2$ be a bilinear map. consider three parties A, B, C with secret keys $a, b, c \in \mathbb{Z}_n$.

- . A broadcasts aP to both B,C
- . B broadcasts bP to both A,C
- . C broadcasts cP to both A,B
- . A computes $\hat{e}(bP, cP)^a$
- . B computes $\hat{e}(aP, cP)^b$
- . C computes $\hat{e}(aP, bP)^c$
- . Common agreed key is $\hat{e}(P, P)^{abc}$.

The protocol is secure against eavesdroppers if the problem of computing $\hat{e}(P, P)^{abc}$ given P, aP, bP, cP and pairing $\hat{e}$ is intractable. This problem is presumably no easier than DHP. Pairings have been accepted as an indispensable tool for protocol designers. There has also been a tremendous amount of work on the realization and efficient implementation of bilinear pairings using elliptic curves[5].

The purpose of this paper is to provide an introduction to elliptic curve groups and pairing-based cryptography on elliptic curves. Elliptic curves are explained in section 2. In section 3 the researcher describes the bilinear map on elliptic curves that can be used to construct bilinear pairings.

2 Elliptic Curves

An elliptic curve E over a field K is defined by a non-singular Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad (1)$$

where $a_1, a_2, a_3, \dots, a_6 \in K$. The set $E(K)$ consists of the points $(x, y) \in K \times K$ that satisfy (1) and the point at infinity, which is denoted by ∞ .

The chord-and-tangent rule for adding two points in $E(K)$ endows $E(K)$ with the structure of an abelian groups. The point at infinity ∞ serves the identity element. The negative of a point $p = (x_1, y_1)$ is $-P = (x_1, y_2)$ where y_1, y_2 are the two roots of the defining equation for E with $x = x_1$. If $P, Q \in E(K) - \{\infty\}$ with $p \neq \pm Q$, then $P + Q$ is defined to be R where $-R$ is the third point of intersection of line through P and Q with the curve. If $P = Q$, then the tangent line through P that intersect curve is defined $-R$.

In Cryptography the researcher, only ever instantiate elliptic curves, defined over finite fields. Suppose now that K is a finite field F_q of order q and characteristic p . Hasse's theorem [5] gives tight bounds for the cardinality of $E(F_q)$: $(\sqrt{q} - 1)^2 \leq |E(F_q)| \leq (\sqrt{q} + 1)^2$.

If $p > 3$, then a linear change of variables transforms equation (1) into the simpler form:

$$y^2 = x^3 + ax + b$$

where $a, b \in K$ and $4a^3 + 27b^2 \neq 0$. We will always be working over large prime fields where the short Weierstrass equation covers all possible isomorphism classes of elliptic curves.

The rank of $E(F_q)$, is at most two. More precisely, we have $E(F_q) \cong Z_{n_1} \times Z_{n_2}$ where $n_2 | n_1$ and $n_2 | q - 1$. Now let E be an elliptic curve over the finite field F_q and let P and Q be points in $E(F_q)$. The elliptic curve discrete logarithm problem (ECDLP) is the problem of finding an integer n such that $Q = nP$. The best generic algorithm known for solving the ECDLP is Pollard's rho method. However, there may be other discrete log solvers that are faster for certain families of elliptic curves. In particular, it was shown in [2, 4] that Weil and Tate pairings can be used to transfer the ECDLP instance to an instance of the discrete logarithm problem in an extension field F_{q^k} , where the embedding degree k is defined as follows.

Let E be an elliptic curve defined over F_q , and $P \in E(F_q)$ be a point of prime order r . suppose that $\gcd(r, q) = 1$. Then the embedding degree of $\langle P \rangle$ is the smallest positive integer k such that $r | q^k - 1$. Elliptic curves with small embedding degree and large prime order subgroup are key ingredients for implementing pairing-based cryptographic systems.

3 Pairing-based Cryptography

Let G_1 be a cyclic additive group generated by P , whose order is a prime p , and G_2 be a cyclic multiplicative group of order p . A bilinear pairing is a map $\hat{e} : G_1 \times G_1 \rightarrow G_2$ with the following properties:

- 1) Bilinearity: $\hat{e}(aP, bP) = \hat{e}(P, P)^{ab}$ for all $a, b \in \mathbb{Z}_p^*$,
- 2) non-degeneracy: $\hat{e}(P, P) \neq 1$,
- 3) computability: There is an efficient algorithm to compute $\hat{e}(P_1, P_2)$ for all $P_1, P_2 \in G_1$.

For instance, choosing $G_1 = E(F_q)$ and $G_2 \subset F_{q^k}^*$ with k an embedding degree, defines bilinear pairings. In practice, elliptic curves are the only groups used to implement pairings. The Weil and Tate pairings for elliptic curves that can be used for design bilinear pairings, see [5]. The security of many pairing-based protocols is dependent on the intractability of the following problem.

Let $\hat{e}$ be a bilinear pairing on (G_1, G_2) . The bilinear Diffie-Hellman problem (BDHP) is to compute the value of bilinear pairing $\hat{e}(P, P)^{abc}$, whenever P, aP, bP, cP are given. Hardness of the BDHP implies the hardness of DHP in both G_1 and G_2 . First, if the DHP in G_1 can be efficiently solved, then one could solve an instance of BDHP by computing abP and then $\hat{e}(abP, cP) = \hat{e}(P, P)^{abc}$. Also, if the DHP in G_2 can be efficiently solved, then the BDHP instance could be solved by computing $g = \hat{e}(P, P)$, $g^{ab} = \hat{e}(aP, bP)$, $g^c = \hat{e}(P, cP)$ and then g^{abc} . Nothing else is known about the intractability of the BDHP, and the problem is generally assumed to be just as hard as the DHP in G_1 and G_2 .

4 Conclusion

In this paper, a broad view of elliptic curve has been discussed. Elliptic curve is an abelian group that is a suitable candidate for public key cryptosystems. Pairings are being used to design elegant solutions to protocol problems, some

of which have been open for many years. Many techniques have been developed for generating suitable elliptic curves in pairing, see [1] for a comprehensive survey. Implementing ECC with applying the combination of software and hardware is advantageous as it provides flexibility and favorable performance. Its disadvantage is its lack of maturity, as mathematicians believe that not enough research has been done in ECDLP.

5 Acknowledgement

I would like to thank Islamic Azad university of Shahr-e-Qods for supporting this work.

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The sub-multiplicative degree for noncyclic subgroups of some nonabelian metabelian groups

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Abstract. A metabelian group is a group G that have at least an abelian normal subgroup N such that the quotient group G/N is also abelian. The concept of commutativity degree plays as an important role in determining the abelianness of the group. This concept has been extended to the relative commutativity degree of a subgroup H of a group G which is defined as the probability that an element of H commutes with an element of G . Further extension to it, the multiplicative degree of a group G is discovered. It is defined as the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup of H . In this paper, the sub-multiplicative degree for noncyclic subgroups of some nonabelian metabelian groups are determined.

1 Introduction

Probability theory is the mathematical study of uncertainty. Everyday we face some situations where the result is uncertain, and perhaps without realizing it, we guess about the likelihood of one outcome or another.

In 2007, Erfanian *et al.* [1] proposed the concept of relative commutativity degree of a subgroup H of a group G , denoted as $P(H, G)$. The relative commutativity degree of a subgroup H which is the probability for an element of H to commute with an element of G , is the extension of the concept of commutativity degree.

2010 Mathematical Subject Classification. Primary:20P05; Secondary: 11R20.

Keywords. Metabelian group, Multiplicative degree, Noncyclic subgroup.

* Speaker

In early 2016, Abd Rhani *et al.* [2] introduced the multiplicative degree of a group G denoted as $P_{mul}(G)$ is the extension from the concept of relative commutativity degree. Where it is defined as, the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup of G .

A metabelian group is a group G that has at least a normal subgroup N such that both N and G/N are abelian. Metabelian group can be considered as a group close to being abelian, in the sense that every abelian group is metabelian, but not all metabelian groups are abelian. From the definition, any dihedral group is metabelian since it has a cyclic normal subgroup and the quotient group is cyclic of order two. Abdul Rahman [3] has found all the metabelian groups of order at most 24. She listed all the 59 groups of order less than 24 with their presentation including abelian and nonabelian groups. From the results, only 25 out of 59 groups are nonabelian metabelian groups of order less than 24. Narrowing down there are eight dihedral groups in those 25 groups.

The commutativity degree of nonabelian metabelian groups of order at most 24 has been determined by Che Mohd [4] in 2011 while the relative commutativity degree of a cyclic subgroup of nonabelian metabelian group of order at most 24 has been determined by Hassan in [5]. In 2013, Abdul Hamid *et al.* [6] presented some results on the relative commutativity degree for cyclic subgroups of dihedral groups of order at most 26. Then in 2016, Abd Rhani *et al.* [2] determined the multiplicative degree of a cyclic subgroup of dihedral groups. The formal definition of $P_{mul}(G)$ is stated as follows.

Definition 1. [2] The Multiplicative Degree of a Group

Let G be a group and H be a subgroup of G . For any $x, y \in G$ then the multiplicative degree of a group G , denoted as $P_{mul}(G)$, is defined as:

$$P_{mul}(G) = \frac{|\{(x, y) \in G \times G : xy \in H\}|}{|G|^2}.$$

Based on Definition 1, there are four cases that can be considered which are $x, y \in H$, $x \in G \setminus H$ but $y \in H$, $x \in H$ but $y \in G \setminus H$ and lastly $x, y \in G \setminus H$. For the case $x, y \in H$, then we have $P_{mul}(G) = \frac{|H|^2}{|G|^2}$. For the case $x, y \in G \setminus H$, we defined it as sub-multiplicative degree as in Definition 2.

The sub-multiplicative degree can be defined as the probability that the product of a pair of elements chosen randomly from a group G but not in H is in the given subgroup of H . In the early 2016, Mustafa [7] and Jaafar [8] have determined the sub-multiplicative degree for cyclic subgroups of nonabelian metabelian groups of order less than 24. Hence, in this paper, the sub-multiplicative degree for noncyclic subgroups of some nonabelian metabelian groups are determined.

Definition 2. The Sub-Multiplicative Degree of a Group

Let G be a group and H be a subgroup of G . For any $x, y \in G \setminus H$ then the sub-multiplicative degree of a group G , denoted as $P_{subm}(G)$, is defined as:

$$P_{subm}(G) = \frac{|\{(x, y) \in G \setminus H \times G \setminus H : xy \in H\}|}{|G|^2}$$

2 Main Results

In this section, the results on the sub-multiplicative degree for noncyclic subgroups of some nonabelian metabelian groups are determined. This paper focussed only on nonabelian metabelian groups of order 8, 12, 14, 16 and all finite dihedral groups.

Theorem 1. Let G be a nonabelian metabelian group of order 12.

Suppose H is a noncyclic subgroup of G and $x, y \in G \setminus H$ such that $xy \in H$, then

$$P_{subm}(G) = \begin{cases} \frac{2}{9} & \text{if } |H| = 4, \\ \frac{1}{4} & \text{if } |H| = 6. \end{cases}$$

Theorem 2. Let G be a nonabelian metabelian group of order 16.

Suppose H is a noncyclic subgroup of G and $x, y \in G \setminus H$ such that $xy \in H$, then

$$P_{subm}(G) = \begin{cases} \frac{3}{16} & \text{if } |H| = 4, \\ \frac{1}{4} & \text{if } |H| = 8. \end{cases}$$

Theorem 3. Let G be a finite dihedral group, D_n where $n \geq 3$.

Suppose H is a noncyclic subgroup of D_n and $x, y \in D_n \setminus H$ such that $xy \in H$, then

$$P_{subm}(D_n) = \begin{cases} \frac{1}{4} & \text{if } |H| = n, \\ \frac{2(n-2)}{n^2} & \text{if } |H| \neq n. \end{cases}$$

3 Conclusion

In this paper, the multiplicative degree for noncyclic subgroups of nonabelian metabelian of order 8,12,14,16 and all finite dihedral groups are determined.

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g -noncommuting graph of finite groups

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Abstract. Let $Z(G)$ be the center of a non-abelian group G and g be a fixed element of G . The g -noncommuting graph of G , denoted by Δ_G^g , is an undirected graph with vertex set $G \setminus Z(G)$ and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1} . In this talk, we survey some graph theoretical properties like connectivity and planarity.

1 Introduction

Let $Z(G)$ be the center of a group G . Associate a graph Γ_G to G as follows: Take $G \setminus Z(G)$ as vertex set of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. This graph is called the non-commuting graph of G . Now, we are going to consider the new generalization of non-commuting graph called g -noncommuting graph which is associated to a fixed element g of a group G given by Tolue et al. in [4] as the following.

Definition 1. For any non-abelian group G and fixed element g in G , the g -noncommuting graph of G is the graph with vertex set G and two distinct vertices x and y join by an edge if $[x, y] \neq g$ and g^{-1} .

In this article, we consider an induced subgraph of Γ_G^g whose vertices are all non-central elements of G . It is called the g -noncommuting graph of G and is denoted by Δ_G^g .

2010 Mathematical Subject Classification. Primary:05C25; Secondary: 20P05.

Keywords. g -noncommuting graph, Non-commuting graph, Connected graph.

* Speaker

It is clear that Δ_G^e coincides with the non-commuting graph.

Let $K(G) = \{[x, y] | x, y \in G\}$ be the set of all commutators of G and set $G' = \langle K(G) \rangle$, where G' is the commutator subgroup of G . The g -noncommuting graph Δ_G^g is a complete graph if $g \notin K(G)$. Therefore, we always assume that g is a non-identity element, $g \in K(G)$ and G is a finite group.

2 Main Results

In this section, we may investigate some graph theoretical properties of Δ_G^g . Let us start with to mention some relations between the new graph Δ_G^g and commuting graph.

Lemma 1. The commuting graph of group G is a spanning subgraph of Δ_G^g .

Lemma 2. If $K(G) = \{e, g\}$ or $\{e, g, g^{-1}\}$ then Δ_G^g is equal to commuting graph.

In the following theorems, we determine connectivity and diameter of the g -noncommuting graph.

Theorem 1. $\Delta_{D_{2n}}^g$ is connected if and only if $n \neq 3, 4$ and 6 .

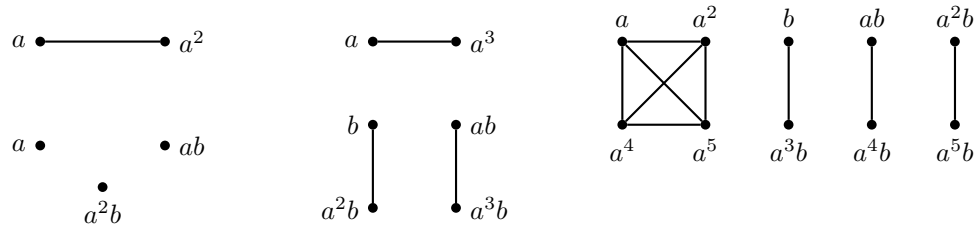


Figure 1: $\Delta_{D_6}^{a^2}$, $\Delta_{D_8}^{a^2}$ and $\Delta_{D_{12}}^{a^2}$

Theorem 2. Let g be a non-central element of G . Then

1. If $|g| \neq 3$, then $\text{diam}(\Delta_G^g) = 2$.
2. If $|g| = 3$, then the following cases occur:
 - (a) Let $[C(g) : Z(G)] = 3$. If there exists a vertex x such that $d(x, g) > 2$, then $\frac{G}{Z(G)} \cong S_3$ and $\Delta_G^g = K_{2|Z(G)|} \cup 3K_{|Z(G)|}$, otherwise $\text{diam}(\Delta_G^g) \leq 4$.
 - (b) Let $[C(g) : Z(G)] > 3$. Then $\text{diam}(\Delta_G^g) \leq 4$.

Theorem 3. Let g be a central element of G . If there are two vertices such that their distance is greater than 5, then Δ_G^g is disconnected and the following cases occur:

- (i) If $|g| \geq 3$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Delta_G^g = 4K_{2|Z(G)|}$,
- (ii) If $|g| = 2$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Delta_G^g = 3K_{|Z(G)|}$.

Otherwise Δ_G^g is the connected graph and $\text{diam}(\Delta_G^g) = 2$.

Corollary 1. If G is a group of odd order and Δ_G^g is connected, then $\text{diam}(\Delta_G^g) = 2$.

Theorem 4. The girth of g -noncommuting graph is 3, unless $G \cong S_3, D_8$ or Q_8 . Moreover, g -noncommuting graph of groups S_3, D_8 and Q_8 are forest.

In the following theorem, we classify all groups that their g -noncommuting graph is planar.

Theorem 5. Let G be a finite non-abelian group. Then Δ_G^g is planar if and only if G is isomorphic to one of the following groups:

- (1) $S_3, D_8, Q_8, D_{10}, D_{12}, D_8 \times Z_2, Q_8 \times Z_2,$
- (2) $\langle a, b : a^3 = b^4 = e, ab = a^{-1} \rangle \cong Z_3 \rtimes Z_4,$
- (3) $\langle a, b : a^4 = b^4 = e, ab = a^{-1} \rangle \cong Z_4 \rtimes Z_4,$
- (4) $\langle a, b : a^8 = b^2 = e, ab = a^{-3} \rangle \cong Z_8 \rtimes Z_2,$
- (5) $\langle a, b : a^4 = b^2 = (ab)^4 = [a^2, b] = e \rangle \cong (Z_4 \times Z_2) \rtimes Z_2,$
- (6) $\langle a, b, c : a^2 = b^2 = c^4 = [a, c] = [b, c] = e, [a, b] = c^2 \rangle \cong (Z_4 \times Z_2) \rtimes Z_2,$

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Primary ideal in quotient semirings

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Abstract. Semirings $(R, +, \times)$ have a relationship with group. As opposed to groups, where all elements are required to have an inverse, in semirings, there is no requirement that all elements have an inverse w.r.t. $+$. In this paper, we prove three results. First, we characterize primary ideal in semiring $\mathbb{Z}_0^+$. Then, we characterize prime and primary ideals in semiring $\mathbb{Z}_0^+$. Finally, we show the relationship between primary ideal over semirings and primary ideal over quotient semirings.

1 Introduction

Semirings are a mathematical structure that arise naturally in many areas, such as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and noncommutative ring theory, optimization theory, discrete event dynamical systems, automata theory, formal language theory and the mathematical modeling of quantum physics and parallel computation systems [4]. There are many works that have been done on characterizing some types of ideals in semirings. In the sequel, we mention some of them. Allen [3] defines the notion of Q ideals. Chaudhari et al. [2] extend the primary decomposition theorem for commutative Noetherian rings to non-commutative right Noetherian, right k -semirings. Gupta et al. [6] characterize prime ideals in semirings. Gupta et al. [7] show the relationship between partitioning and subtractive ideals over strongly Euclidean semiring. Chaudhari

2010 Mathematical Subject Classification. Primary: 20M12; Secondary: 16Y60.

Keywords. Primary ideal, Prime ideal, Quotient Semirings.

* Speaker

et al. [8] introduce the notion of subtractive extension of ideals and generalize the results in [1]. Chaudhari et al. [8] characterize the subtractive extension of ideals in semiring $\mathbb{Z}_0^+$.

This paper extends the work in [8, 6]. Theorem 2.1 in [6] characterizes prime ideal in semiring $\mathbb{Z}_0^+$. In this work, we characterize primary ideal in semiring $\mathbb{Z}_0^+$. Lemma 2.2 in [8] characterizes Q ideal and subtractive ideal in semiring $\mathbb{Z}_0^+$. Here, we characterize prime and primary ideals in semiring $\mathbb{Z}_0^+$. Theorem 2.12 in [8] shows the relationship between prime ideal over semirings and prime ideal over quotient semirings. In this paper, we show the relationship between primary ideal over semirings and primary ideal over quotient semirings.

2 Main Results

Theorem 1. A proper ideal $I = \langle x \rangle$ for $x \in \mathbb{Z}_0^+$ over semiring $(\mathbb{Z}_0^+, +, \cdot)$ is primary if and only if $x = p^n$ where p is a prime number and $n \in \mathbb{Z}_0^+ - \{0\}$.

Theorem 2. Let $I = J_1 \times J_2$ be an ideal in the semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then

- I is a prime ideal if and only if $I = J_1 \times \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+ \times J_2$ where J_1, J_2 are prime ideals in semiring $(\mathbb{Z}_0^+, +, \cdot)$,
- I is a primary ideal if and only if $I = J_1 \times \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+ \times J_2$ where J_1, J_2 are primary ideals in semiring $(\mathbb{Z}_0^+, +, \cdot)$.

Theorem 3. Let R be a semiring, I be a Q ideal of R and P be a subtractive extension of I . Then P is a primary ideal of R if and only if $P/I_{(P \cap Q)}$ is a primary ideal of $R/I_{(Q)}$.

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On the order commutativity degree of some finite groups

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Abstract. The commutativity degree is the probability that a pair of elements chosen randomly from a group G commute. This concept was widely generalized by many authors in many directions. In this paper, the concept of order commutativity degree is introduced. It is a method to compute the number of commutative elements of the same order out of all elements of the same order in a non-abelian group. This paper focuses on the dihedral groups of degree n .

1 Introduction

The theory of commutativity degree has played a very important role in determining the abelianness of a group G . This concept was firstly introduced by Miller in 1944 [1]. The definition of commutativity degree is given as follows:

Definition 1. [1] Let G be a group. The commutativity degree is the probability that two random elements (x, y) in G commute, defined as follows:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

The above probability is less than or equal to $5/8$ for finite non-abelian groups [2, 3]. Gustafson [2] and MacHale [3] showed that this probability can be computed using conjugacy classes. Numerous researches have been done on

2010 Mathematical Subject Classification. Primary:20P99; Secondary: 20N99.

Keywords. Commutativity degree, non-abelian groups, dihedral groups.

* Speaker

the commutativity degree and many results have been achieved. In this research, a new generalization is introduced, namely the order commutativity degree of a finite nonabelian group.

The scope in this research is on the dihedral groups of degree n . They are defined as follows:

Definition 2. [4] For $n \geq 3$ dihedral groups, D_n is denoted as the set of symmetries of a regular n -gon. Furthermore, the order of D_n is $2n$, or equivalently $|D_n| = 2n$. D_n can be represented in a form of generators and relations given as in the following.

$$D_n = \langle x, y | x^n = y^2 = 1, yx = x^{-1}y \rangle.$$

2 Main Results

In this paper, we introduce a new concept of commutativity degree, namely the order commutativity degree, defined in the following:

Definition 3. Let G be a finite non-abelian group. Suppose that x and y are two random elements of G of the same order. The order commutativity degree is the probability that two random elements (x, y) of the same order in G commute, out of all elements of the same order in G , defined as follows:

$$P_o(G) = \frac{|\{(x, y) \in G \times G : |x| = |y|, xy = yx, x \neq y\}|}{|\{(x, y) \in G \times G : |x| = |y|, x \neq y\}|}$$

Notice that if $xy = yx$, then the ordered pair $(x, y) = (y, x)$ will be counted once.

The order commutativity degree for the dihedral groups of degree n when n is odd and when n is an even number are stated in the following theorems.

Theorem 1. Let G be a Dihedral Group of order $2n$ where n is an odd integer. If n is a prime number, then the order commutativity degree in this case is:

$$P_o(G) = \frac{\binom{n-1}{2}}{\binom{n}{2} + \binom{n-1}{2}}$$

If n is not a prime number, then the order commutativity degree in this case is:

$$P_o(G) = \frac{\sum_{d_i} \binom{\phi(d_i)}{2}}{\binom{n}{2} + \sum_{d_i} \binom{\phi(d_i)}{2}}$$

Theorem 2. Let G be a Dihedral Group of order $2n$ where n is an even integer, then the order commutativity degree of G is given as follows:

$$P_o(G) = \frac{\frac{3n}{2} + \sum_{d_i \neq 2} \binom{\phi(d_i)}{2}}{\binom{n+1}{2} + \sum_{d_i \neq 2} \binom{\phi(d_i)}{2}}$$

3 Conclusion

In this paper, the concept of order commutativity degree is introduced. Moreover, the order commutativity degree for diherdral groups of degree n is computed.

4 Acknowledgment

The first author would like to express her appreciation for Dr. Mustafa Anis El-Sanfaz for his assistant during the preparation of this paper. The second author would also like to acknowledge the Ministry of Higher Education (MOHE) Malaysia and Research Management Centre (RMC) Universiti Teknologi Malaysia (UTM) for the financial funding through the Research University Grant (GUP), Vote No: 13H79 and 11J96.

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On the generalized conjugacy class graph of metabelian groups

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Abstract. The concepts of graph theory have been applied in many fields such as chemistry, physics, computer science as well as group theory. In this research, the generalized conjugacy class graph is found for some metabelian groups. The method of constructing the graph is by obtaining the probability that an element of the metabelian groups fixes a set. The set considered in this research is the set of pairs of commuting elements where the greatest common divisor of the order of the elements is two. The orbit of the set is computed under conjugation action. The results are then applied to the generalized conjugacy class graph and some of the graph properties are determined.

1 Introduction

A graph is used to model relations between objects. In 2012, Omer *et al.* found the generalized conjugacy class graph for finite metacyclic 2-groups [1]. In this paper, the generalized conjugacy class graph is constructed to show the relations between the orbits and the order of the elements. The extension of the commutativity degree which is the probability of an element of a group fixes a set is used as the method to obtain the generalized conjugacy class graph. The groups in the scope of this research are three metabelian groups of order 12, namely the alternating group A_4 , the dihedral group D_6 and the semidirect product T . The set Ω under this study is the set of all pairs in the form of (a, b) , $a, b \in G$, where a and b commute, $a \neq b$ and $\gcd(|a|, |b|) = 2$. From the results, the generalized conjugacy class graph

2010 Mathematical Subject Classification. Primary: 20E45; Secondary: 20N99.

Keywords. element fixes a set, Orbit, Metabelian groups.

* Speaker

is constructed and some of its properties namely the chromatic number, independent number and dominating number are found.

2 Preliminaries

This section includes some definitions and result that are used in this research.

Definition 1. [2]Orbit
Suppose G is a finite group that acts on a set Ω and $\omega \in \Omega$. The orbit of ω , denoted by $O(\omega)$, is the subset $O(\omega) = \{g\omega | g \in G, \omega \in \Omega\}$. In this research the groups act on themselves by conjugation, so the orbit is $O(\omega) = \{g\omega g^{-1} | g \in G, \omega \in \Omega\}$. The number of orbit is denoted by $K(\Omega)$.

Definition 2. [3]Probability that an Element of a Group Fixes a Set
The probability that an element of a group fixes a set is given as:

$$P_G(\Omega) = \frac{|\{(g, \omega) \in G \times \Omega | g\omega = \omega \text{ for } g \in G, \omega \in \Omega\}|}{|G||\Omega|}.$$

Proposition 2.1. [4] Let G be a finite group and $K(G)$ be the number of orbit in G . Then the probability that an element of a group fixes a set, $P_G(\Omega)$ is given as $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$.

Definition 3. [5] Generalized conjugacy class graph
The generalized conjugacy class graph, Γ_G^Ω is defined as a graph whose vertices are non-central orbits under group action of a set. Two vertices in Γ_G^Ω are adjacent if their cardinalities are not coprime.

From the generalized conjugacy class graph, some properties of graph are obtained. The chromatic number is the maximum number of colors in a proper coloring in Γ , denoted by $\chi(\Gamma)$. The independent number, $\alpha(\Gamma)$ is the number of vertices in maximum independent set. The dominating number is the minimum size of the dominating set and it is denoted by $\gamma(\Gamma)$ [6].

3 Main Results

In this section the main results of the probability of an element of the metabelian groups fixes the set Ω and its generalized conjugacy class graph are formulated. In addition, some properties of the graph for one of the groups mentioned in the Introduction, namely the dihedral group of order 12 are also stated.

Proposition 3.1. Let D_6 be the dihedral group of order 12 and $\Omega = \{(a, b) \in D_6 \times D_6 | a, b \in D_6, a \neq b, ab = ba, \gcd(|a|, |b|) = 2\}$. D_6 acts on Ω by conjugation. Then the number of orbits of Ω is $K(\Omega)=8$ and the probability that an element of D_6 fixes the set Ω is $P_{D_6}(\Omega) = \frac{4}{11}$.

Theorem 1. The generalized conjugacy class graph of D_6 , $\Gamma_{D_6}^\Omega$ is the union of complete graphs K_2 and K_6 .

Proposition 3.2. The chromatic number for the generalized conjugacy class graph of D_6 , $\chi(\Gamma_{D_6}^\Omega) = 6$.

Proposition 3.3. The independent number for the generalized conjugacy class graph of D_6 , $\alpha(\Gamma_{D_6}^\Omega) = 2$.

Proposition 3.4. The dominating number for the generalized conjugacy class graph of D_6 , $\gamma(\Gamma_{D_6}^\Omega) = 2$.

4 Conclusion

In this research, the probability that an element fixes a set Ω is found for three metabelian groups of order 12 namely the dihedral group D_6 , the alternating group A_4 and the semidirect product T . In order to obtain the probability, the number of orbits are first calculated. It is found that the probability that an element of the group D_6 fixes the set Ω is $P_{D_6}(\Omega) = \frac{4}{11}$. Next, the generalized conjugacy class graph of D_6 is found to be the union of complete graphs K_2 and K_6 .

5 Acknowledgement

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education Malaysia (MOHE) for financial funding through Research University Grant (GUP) Vote No. 13H79.

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The commutativity degree in terms of centralizers of metacyclic 2-groups of negative type of nilpotency class two and their centralizer graphs

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Abstract. Let G be a metacyclic 2-group of negative type of nilpotency class two. The probability that two random elements commute in G is called the commutativity degree. The commutativity degree can be computed using two different techniques, one of them is using the conjugacy classes and another technique is by using Cayley table. However, in this paper, the centralizers are used as another way to obtain the commutativity degree. The commutativity degree in terms of centralizers of metacyclic 2-groups of negative type of nilpotency class two are found. The obtained sizes of the centralizers are then applied to the graph theory, more precisely to the centralizer graphs.

2010 Mathematical Subject Classification. Primary:20P05; Secondary: 20B40.

Keywords. The commutativity degree, graph theory.

* Speaker

1 Introduction

Throughout this paper, G denotes a non-abelian metacyclic 2-group and Prcom_ψ is a notation used for the commutativity degree in terms of centralizers. In addition, $C(a)$ represents the centralizer of a in G . The conjugacy class is denoted by $\text{cl}(G)$.

The probability that a pair of elements x and y selected randomly from a group G commute, is called the commutativity degree and it was firstly introduced in 1944 [1]. The definition of commutativity degree is stated formally in the following.

Definition 1. [1] Let G be a group. The commutativity degree is the probability that two random elements (x, y) in G commute, defined as follows:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

The above probability is less than or equal to $\frac{5}{8}$ for finite non-abelian groups ([2] and [3]). Numerous researches have been done on the commutativity degree and many results have been achieved.

Much later in 2005, the classification of non-Abelian metacyclic p -groups of class two and class at least three were done by Beuerle [4]. Based on Beuerle's classification, the metacyclic p -groups of nilpotency class two are partitioned into two families of non-isomorphic p -groups stated as follows:

- (1) $G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$,
- (2) $G \cong \langle a, b : a^4 = 1, b^2 = [b, a] = a^{-2} \rangle$, namely a Quaternion of order 8, Q_8 .

Throughout this paper, we refer to these classifications as groups of type (1) and (2).

In 2012, Moradipour *et al.* [5] computed the conjugacy classes which are considered as a way to find the commutativity degree. They computed the exact value of the commutativity degree of some metacyclic p -groups. The following result is used in this paper.

Theorem 1. [5] Let G be a metacyclic p -group where p is any prime number, $G \cong \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$. Then $K(G) = p^{\alpha+\beta} \left(\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}} \right)$.

As mentioned, the concept of the commutativity degree has been generalized and extended by several authors. One of these extensions is the commutativity degree in terms of centralizers. This concept was firstly introduced by Omer *et al.* [6]. From Definition 1, the commutativity degree is $P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}$. Using the set of commuting elements $\{(x, y) \in G \times G | xy = yx\}$ and multiplying the right side by x^{-1} yields $\{(x, y) \in G \times G | xyx^{-1} = y\}$ which is the set of centralizers in G . Thus Definition 1 can be modified and is given in the following definition.

Definition 2. [7] Let G be a finite non-Abelian group. If G acts on itself by conjugation, then the commutativity degree in terms of centralizers is

$$\text{Prcom}_\psi(G) = \frac{|\{(x, y) \in G \times G | xyx^{-1} = y\}|}{|G|^2},$$

where ψ is a function from G to itself

The commutativity degree in terms of centralizers was firstly computed for dihedral groups as below.

Proposition 1. [6] Let G be a dihedral group. If G acts on itself by conjugation, then the commutativity degree in terms of centralizers of dihedral groups is given as follows:

$$\text{Prcom}_{\psi}(G) = \begin{cases} \frac{|C(G)|+1}{|G|}, & \text{if } n \text{ is even,} \\ \frac{|C(G)|+|Z(G)|}{2|G|}, & \text{if } n \text{ is odd.} \end{cases}$$

In 2015, Omer and Sarmin [8] introduced a new graph called the centralizer graph. The vertices of this graph are proper centralizers in which two vertices are adjacent if their cardinalities are identical. The centralizer graph was determined for dihedral groups and dicyclic groups. The following is the definition of the centralizer graph.

Definition 3. [8] Let G be a finite non-Abelian group. The centralizers graph, denoted by Γ_{cent} , is a graph whose vertices are proper centralizers, $|V(\Gamma_{\text{cent}})| = |C(G)| - |A|$, where $C(G)$ is the number of proper centralizers in G and A is the number of improper centralizers in G . Two vertices of Γ_{cent} are joined by an edge if their cardinalities are identical.

2 Main Results

This section consists of two parts. First part focuses on the commutativity degree in terms of centralizers, while the second part focuses on the centralizer graphs.

2.1 The Commutativity Degree in Terms of Centralizers of Metacyclic 2-Groups of Negative Type of Nilpotency Class Two

In this section, the main results are introduced. Starting with the commutativity degree in terms of centralizers of metacyclic 2-group of type (1).

Theorem 2. Let G be a metacyclic 2-group of type (1), $G \cong \langle a, b : a^{p^{\alpha}} = 1, b^{p^{\beta}} = 1, [a, b] = a^{p^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$. If G acts on itself by conjugation, then

$$\text{Prcom}_{\psi}(G) = \begin{cases} \frac{(|C(G)|+1)2^{\alpha-2}}{|G|}, & \text{if } \beta = \gamma = 1, \\ \frac{2^{\alpha-1} \cdot |C(G)|}{|G|}, & \text{if } \beta = 2, \gamma = 1. \end{cases}$$

Next, the commutativity degree in terms of centralizers of metacyclic 2-group of type (2) is found.

Theorem 3. Let G be a metacyclic 2-group of type (2), $G \cong \langle a, b : a^4 = 1, b^2 = [b, a] = a^{-2} \rangle, Q_8$. If G acts on itself by conjugation, then $\text{Prcom}_{\psi}(G) = \frac{5}{8}$.

2.2 The Centralizer Graphs of Metacyclic 2-Groups of Negative Type of Nilpotency Class Two

In this section, the centralizers' sizes that were obtained in Section 2.1 are applied to the centralizer graph. We begin with the centralizer graph of metacyclic 2-group of type (1).

Theorem 4. Let G be a metacyclic 2-group of type (1), $G \cong \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = 1, [a, b] = a^{2^{\alpha-\gamma}} \rangle$, where $\alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma$ and $\beta \geq \gamma \geq 1$. If G acts on itself by conjugation, then $\Gamma_{\text{cent}} = K_2$.

Next, the centralizer graph of metacyclic 2-group of type (2).

Theorem 5. Let G be a metacyclic 2-group of type (2), $G \cong \langle a, b : a^4 = 1, b^2 = [b, a] = a^{-2} \rangle$. If G acts on itself by conjugation, then $\Gamma_{\text{cent}} = K_3$.

Remark 1. The centralizer graph of Q_8 is connected since there is only one complete component of K_3 . In addition, the chromatic number $\chi(\Gamma_{\text{cent}})$ and the clique number $\omega(\Gamma_{\text{cent}})$ are identical and equal to three.

3 Conclusion

In this paper, the commutativity degree in terms of centralizers of metacyclic 2-groups of negative type of nilpotency class two is computed. In addition, the centralizers graphs of metacyclic 2-groups of negative type of nilpotency class two are determined. It can be concluded that the centralizer graphs of metacyclic 2-groups of types (1) and (2) have one complete component along with isolated vertices.

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The relative co-prime graph of a group

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Abstract. Let G be a finite group and H any subgroup of G . The co-prime graph of a group is a graph having the set of all elements of G as its vertex set and two distinct vertices are adjacent if and only if they have co-prime order. In this paper, the concept of co-prime graph of a group is further extended to the relative co-prime graph of a group G with respect to a subgroup H , which is defined as a graph whose vertices are all elements of G and two distinct vertices x and y are adjacent if and only if the order of x and y are co-prime and at least one of elements x or y is in H . Some properties of graph on diameter, girth, chromatic number and clique number are found. We show that, for the relative co-prime graph of a group, the clique number and the chromatic number are equal. Besides, some types of graph such as complete, complete r -partite, star and planar graphs are classified.

1 Introduction

In this paper, G is assumed to be a finite group. A graph Γ is a mathematical structure consisting of two sets, namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively and the maximum distance between any pairs of $V(\Gamma)$ is called the diameter of Γ , denoted by $diam(\Gamma)$ [1]. The first investigation of the diameter of Cayley graphs for general groups is started by Erdos and Renyi [2].

In [3], a graph is connected if every pair of points are joined by a path. In 2006, Lai and Yao [4] investigated the group connectivity number for a 2-edge-connected graph G with diameter at most two. Some other properties of graph are girth and chromatic number. The girth of a graph, $girth(\Gamma)$ is the length of a shortest cycle contained in Γ . The girth is defined to be infinity if the graph does not contain any cycles. Meanwhile, the chromatic number of a graph is the minimum number of colors needed to fully color the graph and is denoted by $\chi(\Gamma)$.

A complete graph is a graph where each ordered pair of distinct vertices is adjacent, denoted by K_n , where n is the number of connected vertices [5]. A bipartite graph means that $V(\Gamma)$ can be partitioned into two subsets U and W , called partite sets, such that every edge of Γ joins a vertex of U and a vertex of W . If every vertex of U is adjacent to every vertex of W , Γ is called a complete bipartite graph, where U and W are independent. In [3], a complete bipartite graph of the form $K_{1,n-1}$ is called a star graph with n -vertices.

For any integer $n > 1$, $\pi(n)$ denotes the set of all prime divisors of n . If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$.

Ma *et al.* [6] introduced a new graph called the co-prime graph of a group. This graph is stated as follows.

Definition 1. [6] Let G be a group. The co-prime graph, denoted as Γ_G is a graph whose vertices are element of G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$.

In [7], Sattanathan and Kala defined the order prime graph of a group G , which is a graph having the set of all elements of G as its vertices and two distinct vertices are adjacent if and only if the orders of the corresponding subgroups are co-prime. They determined some properties of this graph. This graph was further investigated by Ma *et al.* [6] and Dorbidi [8]. They called the order prime graph of a group as the co-prime graph of a group.

The relation of co-primeness of the orders of the elements of a group plays a significant role in the determination of the structural properties of that group. In this research, we define the relative co-prime graph of a group G with respect to a subgroup H , denoted by $\Gamma_{copr}(H, G)$. It is a graph having all the elements of G as its vertices and two distinct vertices x and y are adjacent if and only if $|x|$ and $|y|$ are co-prime and the element of x or y is in H .

In this paper, some graph properties such as diameter, girth, chromatic number and clique number are determined. Besides, some types of graph such as complete r -partite, star and planar graphs are classified.

2010 Mathematical Subject Classification. Primary:05C25.

Keywords. Coprime graph, Complete graph, Star graph, Planar graph.

* Speaker

2 Main Results

The definition of the relative co-prime graph of G with respect to H and the main results of this research are stated in this section.

Definition 2. The Relative Co-prime Graph

Let G be a finite group and H a subgroup of G . The relative co-prime graph of G with respect to H , denoted as $\Gamma_{copr}(H, G)$ is a graph whose vertices are element of G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$ and x or y is in H .

The following theorem shows some properties of the relative co-prime graph of a group with respect to H .

Theorem 1. $\Gamma_{copr}(H, G)$ is connected and $diam(\Gamma_{copr}(H, G)) \leq 2$. If $|H| \neq 1$ and H is not a p -group, then $girth(\Gamma_{copr}(H, G)) = 3$.

The next theorem shows that for the relative co-prime graph of a group, the clique and chromatic numbers are equal.

Theorem 2. Let G be a group and H a subgroup of G . Then $\chi(\Gamma_{copr}(H, G)) = \omega(\Gamma_{copr}(H, G)) = \pi(H) + 1$ or $\pi(H) + 2$, whenever $\pi(H) = \pi(G)$ or $\pi(H) \not\subseteq \pi(G)$.

The following theorem provides complete graph of the relative co-prime graph of a group.

Theorem 3. $\Gamma_{copr}(H, G)$ is a complete graph if and only if $|G| = 2$ and $|H| = 1$.

The next theorem shows a star graph of the relative co-prime graph of a group.

Theorem 4. $\Gamma_{copr}(H, G)$ is a star graph if and only if $|H| = 1$ or G is a p -group.

The next theorem gives complete r -partite subgraph of the relative co-prime graph of a group.

Theorem 5. Let G be a group and H a subgroup of G . If $\pi(H) = \pi(G)$, then $\Gamma_{copr}(H, G)$ has a complete r -partite subgraph.

The last theorem gives planar graph of the relative co-prime graph of a group when G is a p -group where p is prime.

Theorem 6. $\Gamma_{copr}(H, G)$ is planar graph if and only if $|H| = 1$ or $|H| = 2$ or G is a p -group, where p is prime.

3 Conclusion

In this paper, we generalized the co-prime graph of a group by defining the relative co-prime graph of a group G with respect to a subgroup H . Furthermore, some properties of the relative co-prime graph of a group such as diameter,

girth, chromatic number and clique number are found. It is found that the clique number and the chromatic number are equal. Lastly, some types of graph such as complete r -partite, star and planar graphs are classified.

Acknowledgements

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) for the financial funding through the Research University Grant (RUG) Vote No. 15H79. The first author would also like to thank Universiti Teknologi MARA (UiTM) and Ministry of Higher Education (MOHE) Malaysia for the PhD fellowship scheme.

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Energy of commuting and non-commuting graphs for some metabelian groups

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Abstract. The energy of a graph G is the sum of all absolute values of the eigenvalues of its adjacency matrix. Meanwhile, an adjacency matrix is a square matrix where the rows and columns consist of 0 or 1-entry depending on the adjacency of the vertices. The main objectives of this study is to determine the energy of commuting and non-commuting graphs for some nonabelian metabelian groups of order 24. The energy computations are assisted by Groups, Algorithms and Programming (GAP) and Maple2016 softwares.

1 Introduction

In this paper, ten metabelian groups of order 24 determined by Abd Rahman and Sarmin [1] are used. The groups are $S_3 \times \mathbb{Z}_4$, $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $D_4 \times \mathbb{Z}_3$, $Q \times \mathbb{Z}_3$, $A_4 \times \mathbb{Z}_2$, Q_{12} , D_{12} , $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ and $\mathbb{Z}_3 \rtimes Q$.

The followings are some definitions that are used in this work.

Definition 1. [2] **Metabelian** A group G is metabelian if there exists a normal subgroup A in G such that both A and the factor group G/A are abelian.

2010 Mathematical Subject Classification. Primary:20B05; Secondary: 20C30.

Keywords. Energy of graph, Commuting graph, Non-commuting graph, Metabelian.

* Speaker

Definition 2. [3] **Center of a Group** The center, $Z(G)$, of a group G is the subset of elements in G that commute with every element of G . In symbols,

$$Z(G) = \{a \in G \mid ax = xa \text{ for all } x \text{ in } G\}.$$

Definition 3. [4] **Commuting Graph** Let G be a finite group. The commuting graph of G denoted $\Delta(G)$ is the graph whose vertex set is $G - Z(G)$ and whose edges are pairs $\{h, g\} \subseteq G - Z(G)$, such that $h \neq g$ and $[h, g] \in Z(G)$.

Definition 4. [5] **Non-commuting Graph** For any non-abelian group G , the non-commuting graph of G is defined as the graph with vertex set $G - Z(G)$, and two distinct vertices x and y are joined by an edge whenever $xy \neq yx$. It is denoted by $\Delta(G)$.

Definition 5. [6] **Energy of Graph** For any graph G , the energy of the graph is defined as $\varepsilon(G) = \sum_{i=1}^n |\lambda_i|$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix of G .

2 Main Results

In this section, some of the main results are presented.

Lemma 1. Let $G = S_3 \times \mathbb{Z}_4$. Then the adjacency matrix for the commuting graph of G is given as the following :

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Its characteristic polynomial is

$$f(\lambda) = \lambda^{20} - 46\lambda^{18} - 136\lambda^{17} + 393\lambda^{16} + 2752\lambda^{15} + 3640\lambda^{14} - 10848\lambda^{13} - 48958\lambda^{12} - 69056\lambda^{11} + 25740\lambda^{10} + 288080\lambda^9 + 614042\lambda^8 + 796992\lambda^7 + 724984\lambda^6 + 480928\lambda^5 + 233565\lambda^4 + 81344\lambda^3 + 19314\lambda^2 + 2808\lambda + 189.$$

Lemma 2. Let $G = S_3 \times \mathbb{Z}_4$. Then the adjacency matrix for the non-commuting graph of G is given as the following :

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Its characteristic polynomial is

$$f(\lambda) = \lambda^{20} - 144\lambda^{18} - 896\lambda^{17} - 1536\lambda^{16}.$$

Theorem 1. Let $G = S_3 \times \mathbb{Z}_4$. Then, the energy of the commuting graph of G , $\varepsilon(\Delta(G))$ is 32.

Theorem 2. Let $G = S_3 \times \mathbb{Z}_4$. Then, the energy of the non-commuting graph of G , $\varepsilon(\Delta(G))$ is $8 + 8\sqrt{7}$.

3 Conclusion

In this paper, the energy of commuting and non-commuting graphs for 10 nonabelian metabelian groups of order 24 are determined.

4 Acknowledgement

The authors would like to express their appreciation for the support of sponsor; Ministry of Higher Education (MOHE) Malaysia and Research Management Centre (RMC), Universiti Teknologi Malaysia (UTM) Johor Bahru for the financial funding through the Research University Grant (GUP) Vote No. 13H79. The first author is also indebted to Ministry of Higher Education (MOHE) Malaysia for her MyBrain15 Scholarship.

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Invariants and contractions of low-dimensional Leibniz algebras

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Abstract. In this paper we study some invariants of Leibniz algebras derived from a given concept of generalized derivations and apply the values of the invariants in finding the contractions in low-dimensional cases.

1 Introduction

Different kind generalizations of derivation of Lie algebras were introduced and studied in [2, 3, 7]. For results on generalized derivations of some other classes of algebras we refer to [1, 4, 5, 6] etc. In this paper we study generalized derivations of Leibniz algebras.

Definition 1. A Leibniz algebra L is a vector space over a field $\mathbb{K}$ equipped with a bilinear mapping $[\cdot, \cdot] : L \times L \rightarrow L$ satisfying the following identity;

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \text{ for all } x, y \in L.$$

Note that if in Leibniz algebra the identity $[x, x] = 0$ holds then L becomes Lie algebra, hence a Leibniz algebra is a generalization of Lie algebra.

The generalized derivation of Leibniz algebras is defined as follows.

Definition 2. Let $(L, [\cdot, \cdot])$ be a Leibniz algebra over a field $\mathbb{K}$, and α, β, γ be elements of $\mathbb{K}$. A linear transformation $d : L \rightarrow L$ is said to be (α, β, γ) -derivation of L if

$$\alpha d([x, y]) = \beta[d(x), y] + \gamma[x, d(y)], \text{ for all } x, y \in L. \quad (1)$$

2010 Mathematical Subject Classification. Primary: 20XX; Secondary: 20XX

Keywords. Leibniz algebra, Invariants, Contractions

* Speaker

The set of all (α, β, γ) -derivations of L we denote by $Der_{(\alpha, \beta, \gamma)}L$. In the study of (α, β, γ) -derivations of Leibniz algebras it is observed that some of $Der_{(\alpha, \beta, \gamma)}L$ have parametric form, in these cases we define functions depending on these parameters. We found such two subspaces of $EndL$, they are $Der_{(\alpha, 1, 1)}L$ and $Der_{(\alpha, 1, 0)}L$. We define two functions from $\mathbb{K}$ to $\{0, 1, 2, 3, \dots, (dimL)^2\}$ as follows: $\xi_L(\alpha) = dim_{\mathbb{K}} Der_{(\alpha, 1, 1)}L$ and $\xi'_L(\alpha) = dim_{\mathbb{K}} Der_{(\alpha, 1, 0)}L$. These functions are isomorphism invariants of Leibniz algebras over any field $\mathbb{K}$ ($char \mathbb{K} = 0$). We refer to the functions as the basic independent functions and use the functions to construct a contraction criterion. Using the criterion we compute the contractions of two- and three-dimensional complex Leibniz algebras.

Proposition 1.1. Let L be a Leibniz algebra over a field $\mathbb{K}$ with $char \mathbb{K} = 0$ and $\alpha, \beta, \gamma, \in \mathbb{K}$, then for $Der_{(\alpha, \beta, \gamma)}L$ the values of α, β, γ are given as follows

- $Der_{(1, 1, 1)}L = DerL$;
- $Der_{(1, 1, 0)}L = \{d \in EndL \mid [d(xy)] = [d(x), y]\}$;
- $Der_{(1, 0, 1)}L = \{d \in EndL \mid [d(xy)] = [x, d(y)]\}$;
- $Der_{(1, 0, 0)}L = \{d \in EndL \mid [d(xy)] = 0\}$;
- $Der_{(0, 1, 1)}L = \{d \in EndL \mid [d(x), y] = [x, d(y)]\}$;
- $Der_{(0, 0, 1)}L = \{d \in EndL \mid [x, d(y)] = 0\}$;
- $Der_{(0, 1, 0)}L = \{d \in EndL \mid [d(x), y] = 0\}$;
- $Der_{(0, 1, \delta)}L = \{d \in EndL \mid [d(x), y] = \delta[x, d(y)]\}$.

Proposition 1.2. Any two-dimensional Leibniz algebra L is isomorphic to one of the following non-isomorphic Leibniz algebras of

1. L is Abelian
2. $L_2^1 : [e_1, e_1] = e_2$
3. $L_2^2 : [e_1, e_2] = -[e_2, e_1] = e_2$
4. $L_2^3 : [e_1, e_2] = e_1, [e_2, e_2] = e_1$ where e_1, e_2 are basis

Proposition 1.3. Up to isomorphism there exist nine non-Lie complex Leibniz algebras on dimension three:

1. $L_3^1 : [e_1, e_3] = -2e_2, [e_2, e_2] = e_1, [e_3, e_2] = e_2, [e_2, e_3] = -e_2$
2. $L_3^2 : [e_1, e_3] = \alpha e_1, [e_3, e_3] = e_1, [e_3, e_2] = e_2, [e_2, e_3] = -e_2$
3. $L_3^3 : [e_2, e_2] = e_1, [e_3, e_3] = \alpha e_1, [e_2, e_3] = e_1$.
4. $L_3^4 : [e_2, e_2] = e_1, [e_3, e_3] = e_1, [e_2, e_3] = e_1$.
5. $L_3^5 : [e_1, e_3] = e_2, [e_2, e_3] = e_1, [e_2, e_3] = e_1$.

$$6. L_3^6 : [e_1, e_3] = e_2, [e_2, e_3] = \alpha e_1 + e_2, [e_2, e_3] = e_1.$$

$$7. L_3^7 : [e_1, e_3] = e_1, [e_2, e_3] = e_2.$$

$$8. L_3^8 : [e_3, e_3] = e_1, [e_1, e_3] = e_2.$$

$$9. L_3^9 : [e_3, e_3] = e_1, [e_1, e_3] = e_1 + e_2.$$

2 Main Results

In this section we give a criterion for a Leibniz algebra to be a contraction of another Leibniz algebra and construct all possible contractions of two- and three-dimensional complex Leibniz algebras.

Definition 3. Let $L_1 = (V, [\cdot, \cdot]_1)$ be a complex Leibniz algebra and $f_t : (0, 1] \rightarrow GL(V)$ a continuous mapping where $f_t(\epsilon) \in GL(V)$ for $0 < \epsilon \leq 1$. If the limit $[x, y]_2 = \lim_{\epsilon \rightarrow 0+} f_t(\epsilon)^{-1} [f_t(\epsilon)x, f_t(\epsilon)y]_1$ exists $\forall x, y \in V$, then algebra $L_2 = (V, [\cdot, \cdot]_2)$ is called a one-parametric continuous contraction (or in short contraction) of $L_1 = (V, [\cdot, \cdot]_1)$ written as $L_1 \rightarrow L_2$.

Proposition 2.1. Let L_1 and L_2 be complex Leibniz algebras and L_2 be a proper contraction of L_1 . Then

1. $\xi_{L_1}(\alpha) \leq \xi_{L_2}(\alpha)$, for $\alpha \in \mathbb{C} \setminus \{1\}$;
2. $\xi_{L_1}(1) < \xi_{L_2}(1)$.

Proposition 2.2. Among all three-dimensional complex Leibniz algebras only the following one-parametric continuous contraction exist:

1. L_3^1 is a contraction of $L_3^2, L_3^1, L_3^4, L_3^5, L_3^8, L_3^9$;
2. L_3^2 is a contraction of L_3^3 ;
3. L_3^3 is a contraction of L_3^9 ;
4. L_3^4 is a contraction of L_3^9 ;
5. L_3^5 is a contraction of L_3^8 and L_3^9 ;
6. L_3^6 is a contraction of L_3^8 and L_3^9 ;
7. L_3^7 is a contraction of $L_3^6, L_3^1, L_3^2, L_3^3, L_3^4, L_3^5, L_3^8$ and L_3^9 .

3 Conclusion

In this work contractions of 2 and 3 dimensional complex Leibniz algebras are found. Some suggestions for future work are given as follows; consider more invariant functions to study the contraction of Leibniz algebras of higher dimensions and over finite fields.

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Isomorphism classes of ultra-groups and normal subgroups

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Abstract. The ultra-group ${}_H M$ is an algebraic new structure whose underlying set depends on the group G and its subgroup H . At first, we present a framework to define the notation ultra-groups. We say two right ultra-groups are isomorphic if they are isomorphic with respect to the induced right ultra-group structures. The aim of this paper is to construct the homomorphism of ultra-groups and calculate the number of isomorphism classes of all finite ultra-groups. Finally, we focus on the subgroup H is non normal if the number of isomorphism classes of right ultra-groups of the subgroup H over the group G is three.

1 Introduction

Let H be a subgroup in group G . A subset M of group G is called a (right) transversal to H in G if $G = \bigcup_{m \in M} Hm$. Therefore the pair (H, M) is a (right) transversal if and only if the subset M of the group G obtained by selecting one and only one member from each right coset of G modulo subgroup H of G . Throughout the paper, a right (left) transversal will be assumed to contain the identity of the group. For the group G which satisfies the above conditions, we have $MH \subseteq G = HM$, and subset M of G is called (right unitary) transversal set with respect to subgroup H . This notion was introduced by Kurosh in [1] which is the base of the concept of an ultra-group. Therefore, for every element $mh \in MH$ there exists unique $h' \in H$ and $m' \in M$ such that $mh = h'm'$. We denote h' and m' by ${}^m h$ and m^h , respectively. Similarly for any elements $m_1, m_2 \in M$ there exists unique elements $[m_1, m_2] \in M$ and

2010 Mathematical Subject Classification. 20A05; 05D15, 20K30.

Keywords. Ultra-group, Transversal, Homomorphism.

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$(m_1, m_2)h \in H$ such that $m_1 m_2 = (m_1, m_2)h[m_1, m_2]$. Similarly for every element $a \in M$, there exists a^{-1} belonging to G . As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Now we are ready to define an ultra-group.

Definition 1. A (right) ultra-group ${}_H M$ is a transversal set of H over group G with a binary operation $\alpha : {}_H M \times {}_H M \rightarrow {}_H M$ and unary operation $\beta_h : {}_H M \rightarrow {}_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.

It is easy to see that every group is an ultra-group. The following example is an ultra-group which is not a group.

Example 1. Let $G = S_3$. Then $H = \{(1), (13)\}$ is a subgroup of G . By $H \setminus G = \{(1), (13)\}, \{(23), (123)\}, \{(12), (132)\}$ we have four possibility for set M . For example let $M = \{(1), (12), (23)\}$, then M is an ultra-group with the following maps :

α	(1)	(23)	(12)	β	(1)	(13)
(1)	(1)	(23)	(12)	(1)	(1)	(1)
(23)	(23)	(1)	(23)	(23)	(23)	(12)
(12)	(12)	(12)	(1)	(12)	(12)	(23)

A (left) ultra-group M_H is defined similarly via (left unitary) transversal set. From now on, unless specified otherwise, ultra-group means right ultra group. A right ultra-group with left cancellation is a quasi-group. A subset $S \subseteq {}_H M$ which contains e , is called a *subultra-group* of H over G , if S is closed under the operations α and β_h in the Definition 1. A subultra-group N of ${}_H M$ is called *normal* if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in {}_H M$. In addition $[N, S]$ is a subultra-group of ${}_H M$, where S is a subultra-group of ${}_H M$. Moreover, $[N, S]$ is a normal subultra-group of ${}_H M$ if S is also normal subultra-group of ${}_H M$ (see[2, 3, 4]).

Definition 2. Suppose ${}_{H_i} M_i$ is an ultra-group of H_i over group G_i , $i = 1, 2$, and φ is a group homomorphism between two subgroups H_1 and H_2 . A function $f : {}_{H_1} M_1 \rightarrow {}_{H_2} M_2$ is an ultra-group homomorphism if for all $m, m_1, m_2 \in {}_{H_1} M_1$ and $h \in H_1$ hold:

- (i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,
- (ii) $(f(m))^{\varphi(h)} = f(m^h)$.

If f is a surjective and injective ultra-group homomorphism, we call it isomorphism and denote it by ${}_{H_1} M_1 \cong {}_{H_2} M_2$. All homomorphism between the two ultra-groups preserves the identity and left inverse elements. If S is a subultra-group of ${}_{H_1} M_1$, then $f(S)$ is a subultra-group of ${}_{H_2} M_2$ if and only if φ is a surjective homomorphism between two subgroups H_1 and H_2 . Isomorphism theorems which are valid for any algebra can be translated for ultra-groups.

2 Main Results

Being isomorphic is an equivalence relation on the set of ultra-groups ${}_H M$ of the subgroup H over a group G . the equivalence classes under this relation is called isomorphism classes of ultra-groups. Due to following lemma, we can restrict ourself to right ultra-groups for determining isomorphism classes of ultra-groups.

Theorem 1. The number of isomorphism classes of right ultra-groups is equal to the number of isomorphism classes of left ultra-groups.

In this paper, we denote the number of isomorphism classes of right ultra-groups of the subgroup H over the group G by $|H_G|$. Now we will calculate the isomorphism class of right ultra-groups of subgroup over a group.

Theorem 2. Let H be a subgroup of the group G . Then H is normal if and only if $|H_G| = 1$.

In addition, if $[G : H] = 2$, then $|H_G| = 1$.

Theorem 3. Let H be a subgroup of the group G . Then $[G : H] = 3$ and H be non normal subgroup if and only if $|H_G| = 3$.

Theorem 4. If G is a finite group and H is its subgroup. Then $|H_G| \neq 4$.

Also, ultra-groups M and M' of the subgroup H over a group G are isomorphism if there exists $\theta \in S_{n-1}$ such that $\theta M = M'\theta$, where S_{n-1} is the set of permutations fixing 1 in S_n .

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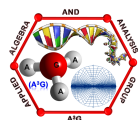
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With Special Thanks to the following Universities, Institutes, Center of Excellences and Organizations for partially scientific and financial support of 4rd Biennial International Group Theory Conference (4BIGTC2017).



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