

FUKAYA CATEGORIES OF SURFACES WITH DG COEFFICIENTS

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ABSTRACT. Given a compact surface S admitting a Liouville form, a finite set $Q \subset S$, a line field on S , and a local system of dg-categories on $S \setminus Q$ together with a section of the monodromy functor around each interior $q \in Q$, we construct a Fukaya category. We give examples where such Fukaya categories are quasi-equivalent to derived categories of projective completions of line bundles and of iterated blow-ups of the affine plane; and to wrapped Fukaya categories of cotangent bundles of spheres.

We define two versions of the Fukaya category, a topological, which is constructed combinatorially from a full arc system, and a geometric, which includes all Lagrangian branes supported on immersed curves. We prove that the topological category embeds into the geometric category in an essentially unique way. To define Lagrangian branes we develop a generalisation of the Chekanov–Eliashberg algebra incorporating dg-coefficients and wrapping along the boundary.

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INTRODUCTION

Let S be a compact oriented surface admitting a Liouville form (i.e. each component of S has non-empty boundary) and $Q \subset S$ a finite set, called the *marking*. The *directed relative path 2-category* $\mathcal{P}_2^+(S, Q)$ is the (strict) 2-category whose objects are $S \setminus Q$, 1-morphisms are lazy paths in $S \setminus Q$ up to thin homotopy, and 2-morphisms are isotopy classes of homotopies of lazy paths in S that intersect Q positively, see Definition 1.2 for details.

Let k be a commutative ring and $\mathrm{dgc}at_k$ the 2-category whose objects are small dg-categories over k , 1-morphisms are dg-functors, and 2-morphisms are closed degree zero morphisms of dg-functors. A *weighted system of dg-categories* on (S, Q) is a strict 2-functor $\mathcal{C}: \mathcal{P}_2^+(S, Q) \rightarrow \mathrm{dgc}at_k$. Informally one can think of $\mathcal{C}$ as a local system of dg-categories on $S \setminus Q$ (it is flattened by the coherent isomorphisms of dg-functors given by homotopies in $S \setminus Q$) together with a morphism of dg-functors from id to the monodromy around q for each interior $q \in Q$. The points of $Q \cap \partial S$ are stops, i.e. boundary paths are not allowed to pass through them.

Given a line field η on S and a weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(S, Q) \rightarrow \mathrm{dgc}at_k$ we call $(\mathcal{S} = (S, Q, \eta); \mathcal{C})$ a *weighted surface*.

In this paper we define and study Fukaya categories of weighted surfaces.

There is a related approach that associates the ∞ -category of global sections to a perverse schober on a ribbon graph, see Kapranov–Schechtman [28] and Christ–Haiden–Qiu [14]. We expect that [14, Proposition 7.7] can be extended to associate a perverse schober $\mathcal{G}$ on the skeleton of any weighted surface $(\mathcal{S}; \mathcal{C})$ in such a way that the global sections of $\mathcal{G}$ are equivalent to the ∞ -categorical Ind-completion of the Fukaya category of $(\mathcal{S}; \mathcal{C})$. We will address this elsewhere.

Topological Fukaya category. Let $(\mathcal{S} = (S, Q, \eta); \mathcal{C})$ be a weighted surface. An *arc brane* in $(\mathcal{S}; \mathcal{C})$ is a properly embedded segment $\gamma: [0, 1] \rightarrow S \setminus Q$ (an *arc*) together with a grading (i.e. a homotopy class of paths from $\gamma^*\eta$ to the tangent line $\dot{\gamma}$) and an object M in $\mathcal{C}_{\gamma(0)}$. A set of arc branes Γ is called an *arc brane system* if the underlying curves form a no more than countable set Γ and arcs in Γ are pairwise disjoint. An arc brane system Γ is called *full* if Γ cuts S into topological discs such that each disc has at most one point from Q and for each arc $\gamma \in \Gamma$ the objects in $\mathcal{C}_{\gamma(0)}$ obtained from arc branes supported on γ generate $\mathcal{C}_{\gamma(0)}$ (i.e. the smallest full subcategory of $\mathrm{Tw} \mathcal{C}_{\gamma(0)}$ containing these objects and closed under taking shifts, cones, and homotopy equivalent objects contains $\mathcal{C}_{\gamma(0)}$).

Generalising the construction of Haiden–Katzarkov–Kontsevich [25, §3] to our setting we associate an A_∞ -category $\mathcal{V}_\Gamma$ to any arc brane system Γ . More concretely, $\mathrm{Ob} \mathcal{V}_\Gamma = \Gamma$, morphism spaces come from boundary paths in $\partial S \setminus Q$ between the ends of arcs in Γ , and A_∞ -operations come from the differential and composition in fibres of $\mathcal{C}$, concatenation of boundary paths, and $\mathcal{C}$ on 2-morphisms obtained from immersed polygons in S bounded by arcs from Γ , see §1.2.2 for details. Our first result is the following.

Theorem 1. *Let $(\mathcal{S} = (S, Q, \eta); \mathcal{C})$ be a weighted surface. Then*

- (1) $\mathcal{V}_\Gamma$ is a strictly unital $\mathbb{Z}$ -graded k -linear A_∞ -category for any arc brane system Γ in $(\mathcal{S}; \mathcal{C})$.
- (2) For a full arc brane system Γ the triangulated envelope $\mathrm{Tw} \mathcal{V}_\Gamma$ does not depend on the choice of Γ up to homotopy equivalence of strictly unital A_∞ -categories.

By *triangulated* in Theorem 1 and everywhere in the paper we mean closed under shifts and cones, not necessarily idempotent complete. We define the *topological Fukaya category* $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C})$ of $\mathcal{S}$ with coefficients in $\mathcal{C}$ as (the homotopy equivalence class of) $\mathrm{Tw} \mathcal{V}_\Gamma$ for a full arc brane system Γ in $(\mathcal{S}; \mathcal{C})$. This definition generalises topological Fukaya categories by Haiden–Katzarkov–Kontsevich [25] and relative wrapped Fukaya categories of surfaces by Evans–Lekili [22]. Theorem 1 follows from Theorem 5 below.

Homological mirror symmetry. We call any geometrically motivated quasi-equivalence between a Fukaya category (A-side) and a derived category of algebro-geometric origin (B-side) or representation-theoretic origin (D-side) homological mirror symmetry (HMS). The subject starts

with Kontsevich's ICM address [31], by now it is a vast area of research and we will not attempt to give an overview.

In the topological Fukaya category $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ we have a geometric input given by $\mathcal{S}$ and an algebraic input given by fibres of $\mathcal{C}$ (with the rest of the structure of $\mathcal{C}$ thought of as relating the two), and so we can consider it on the A- or D-side. The theorem below is intended to illustrate the kind of HMS statements involving $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ one can obtain and is not at all exhaustive. $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ is on the A-side in (1)&(2) and on the D-side in (3).

Theorem 2. *Each of the following is quasi-equivalent to $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ for the indicated $\mathcal{S}$ and a suitable $\mathcal{C}$.*

- (1) $D_{\text{perf}}^{\text{dg}}(\mathcal{O}_X \oplus \mathcal{L})$ where X is a separated Noetherian scheme and $\mathcal{L}$ a line bundle on X . The graded marked surface in this case is $\mathcal{S} = (S = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}, Q = \{1, -2\}, \eta \text{ radial})$. The fibres of $\mathcal{C}$ are $D_{\text{perf}}^{\text{dg}} X$.
- (2) $D_{\text{perf}}^{\text{dg}} X$ where X is a bamboo forest blow-up of the affine plane $\mathbb{A}_k^2$ over an infinite field k . The graded marked surface in this case is $\mathcal{S} = (S = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}, Q, \eta \text{ radial})$ with $Q \cap \partial S = \{1\}$. The fibres of $\mathcal{C}$ are $k[t]$ viewed as a dg-category with one object.
- (3) The wrapped Fukaya category $\mathcal{W}(T^*S^n; k)$ of T^*S^n with coefficients in a commutative ring k for any $n \geq 2$. The graded marked surface in this case is $\mathcal{S} = (S = \{z \in \mathbb{C} : |z| \leq 2\}, Q = \{1, -1\}, \eta \text{ vertical})$. For $n \geq 3$ the fibres of $\mathcal{C}$ are quasi-equivalent to $\mathcal{W}(T^*S^{n-1}; k)$ and for $n = 2$ to $\text{Tw } k[z]$.

$D_{\text{perf}}^{\text{dg}}$ above is a dg-enhancement of the derived category of perfect complexes, it is known to be unique up to quasi-equivalence in the setting of Theorem 2(1)&(2) by Canonaco–Neeman–Stellari [11].

Taking $X = \text{Spec } k$ in Theorem 2(1) we recover the classical homological mirror symmetry for $\mathbb{P}^1$ going back to Kontsevich [32, Pictures]. Our result is therefore a relative version of this over X . It can probably be extended to any rank two vector bundle $\mathcal{V}$ on X in place of $\mathcal{O}_X \oplus \mathcal{L}$ by representing $\mathcal{V}$ as a twisted complex of line bundles. We prove Theorem 2(1) in §2.1.

The *bamboo forest blow-up* in Theorem 2(2) is a successive blow-up of $\mathbb{A}_k^2$ at k -points such that the exceptional locus is a disjoint union of chains of smooth rational curves with self-intersections $-2, \dots, -2, -1$. In the proof of Theorem 2(2) we use a projection $X \rightarrow \mathbb{A}^1$ with regular fibre $\mathbb{A}^1$ and singular fibres chains $\mathbb{A}^1 - \mathbb{P}^1 - \dots - \mathbb{P}^1$. Our result is therefore a relative version of the mirror symmetry, proven by Lekili–Polishchuk [37], between $\mathcal{S}$ as in Theorem 2(2) and the chain $\mathbb{A}^1 - \mathbb{P}^1 - \dots - \mathbb{P}^1$ with $|Q \cap \text{Int } S|$ projective lines. A similar relative statement for versal deformations of the chain $\mathbb{A}^1 - \mathbb{P}^1 - \dots - \mathbb{P}^1 - \mathbb{A}^1$ is proven by Evans–Lekili [22]. We prove Theorem 2(2) in §2.2.

Theorem 2(3) for $n = 1$ holds trivially, but with S an annulus. For $n = 2$ it is proven by Lekili–Segal [39] as the starting point for their conjectures on algebraic torus fibrations. For $n \geq 3$ our construction is different from their framework as the regular fibre of the fibration we use is T^*S^{n-1} , not a torus $(\mathbb{C}^\times)^{n-1}$. More precisely, the standard Lefschetz fibration $T^*S^n \rightarrow \mathbb{C}$ is singular over $Q = \{1, -1\}$ and has T^*S^{n-1} as regular fibre. Moreover, $(\mathcal{S}; \mathcal{C})$ can be viewed as a singular symplectic reduction of (T^*S^n, k) with respect to a Hamiltonian action of $\text{SO}(n)$. We prove Theorem 2(3) in §2.3.

Geometric Fukaya A_∞ -pre-category. We give a second, geometric construction of a Fukaya category associated to a graded marked surface (S, Q, η) with coefficients in $\mathcal{C} : \mathcal{P}_2^+(S, Q) \rightarrow \text{dgc}_{\text{at}, k}$. To this end we choose a Liouville form λ on S compatible with Q (see Definition 3.1). We call $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C})$ a *weighted Liouville surface* and define the *geometric Fukaya A_∞ -pre-category* $\mathcal{W}_{\text{pre}}^{\text{geom}} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ of $\mathcal{S}$ with coefficients in $\mathcal{C}$. The reason we introduce $\mathcal{W}_{\text{pre}}^{\text{geom}}$ is to have arbitrary Lagrangian branes supported on immersed curves in $S \setminus Q$ as objects. On a more technical level, we use the geometric category to prove Theorem 1 about the topological category.

A_∞ -pre-categories were introduced by Kontsevich–Soibelman [33]. An A_∞ -pre-category $\mathcal{W}$ is a set of objects $\text{Ob } \mathcal{W}$, a collection of transverse sequences in $(\text{Ob } \mathcal{W})^n$ for each $n \geq 1$ such that all sequences of length one are transverse and subsequences of transverse sequences are transverse,

a graded module of morphisms $\mathcal{W}(X_0, X_1)$ for each transverse sequence (X_0, X_1) , and an A_∞ -operation

$$\mu^n: \mathcal{W}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{W}(X_0, X_1) \rightarrow \mathcal{W}(X_0, X_n)$$

of degree $2 - n$ for each $n \geq 1$ and transverse sequence $(X_0, \dots, X_n)$ such that the A_∞ -relations are satisfied, see §5.1.1 for details.

The objects of $\mathcal{W}_{\text{pre}}^{\text{geom}}$ are Lagrangian branes, which are certain exact immersed curves γ in $S \setminus Q$ with additional brane data on them. It takes some work to define what a Lagrangian brane is. Namely, a *Lagrangian pre-brane* $\mathcal{L}$ in $(S; \mathcal{C})$ is a compact proper generic immersion $\gamma: (I, \partial I) \rightarrow (S \setminus Q, \partial S \setminus Q)$ of a possibly disconnected 1-dimensional manifold I together with a generic *potential* $P: I \rightarrow \mathbb{R}$ (i.e. $\gamma^* \lambda + dP = 0$), a grading, and a local system of objects over $\gamma^* \mathcal{C}$, see Definition 4.1. We define a curved A_∞ -algebra $\mathcal{A}_{\mathcal{L}}$, the *obstruction algebra* of the Lagrangian pre-brane $\mathcal{L}$. Note that the pair (γ, P) defines an embedded Legendrian (the *Legendrian lift*) in the contactisation $((S \setminus Q) \times \mathbb{R}_z, \lambda + dz)$. The bar construction on our $\mathcal{A}_{\mathcal{L}}$ is a generalisation of the Chekanov–Eliashberg algebra of a Legendrian link introduced by Chekanov [13] and Eliashberg–Givental–Hofer [20]. The generalisation goes in two directions, one is that (γ, P) can go to the boundary, cf. Sivek [55], and another is that the coefficients are in $\mathcal{C}$ instead of a commutative ring.

The elements of $\mathcal{A}_{\mathcal{L}}$ are given by double points of γ and Reeb chords of positive action in $\partial S \setminus Q$ between the ends of γ (each such point or chord gives a graded k -module of maps between respective fibres of the local systems). The A_∞ -operations μ^n , $n \geq 0$ in $\mathcal{A}_{\mathcal{L}}$ come from the differential and composition in fibres of $\mathcal{C}$, concatenation of Reeb chords, and $\mathcal{C}$ on 2-morphisms obtained from immersed polygons in S bounded by γ , see §4.1.3 for details.

A *Lagrangian brane* is a pair $\mathbb{L} = (\mathcal{L}, \delta)$ where $\mathcal{L}$ is a Lagrangian pre-brane and $\delta \in \mathcal{A}_{\mathcal{L}}$ a bounding cochain (i.e. a solution to the Maurer–Cartan equation $\mu^0(1_k) + \mu^1(\delta) + \mu^2(\delta, \delta) + \dots = 0$, the sum is finite since $\mathcal{A}_{\mathcal{L}}$ is nilpotent). The objects of $\mathcal{W}_{\text{pre}}^{\text{geom}}$ are the Lagrangian branes. A sequence of Lagrangian branes is transverse if the underlying curves are in general position, see §6.1.1 for details. The morphism space $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathbb{L}_0, \mathbb{L}_1)$ is given by the intersection points between γ_0 and γ_1 and the Reeb chords in $\partial S \setminus Q$ from an end of γ_0 to an end of γ_1 . The A_∞ -structure μ is defined similarly to that in the obstruction algebra, with bounding cochains deforming μ so that $\mu^0 = 0$, see §6.1.2 for details.

The geometric Fukaya A_∞ -pre-category defined above enjoys the following properties.

Theorem 3. *Let $(S = (S, Q, \eta, \lambda); \mathcal{C})$ be a weighted Liouville surface. Then*

- (1) *The geometric Fukaya A_∞ -pre-category $\mathcal{W}_{\text{pre}}^{\text{geom}} = \mathcal{W}_{\text{pre}}^{\text{geom}}(S; \mathcal{C})$ is a small strongly triangulated $\mathbb{Z}$ -graded k -linear A_∞ -pre-category with enough homotopy equivalences.*
- (2) *If $\mathbb{L}_0 = (L_0 = (I, \gamma_0, P_0), \text{gr}_0, \mathcal{M}_0, \delta_0)$ is a Lagrangian brane in $(S; \mathcal{C})$ and $L_1 = (I, \gamma_1, P_1)$ is isotopic to L_0 , then there is a Lagrangian brane $\mathbb{L}_1$ supported on L_1 such that $\mathbb{L}_0$ and $\mathbb{L}_1$ are homotopy equivalent in $\mathcal{W}_{\text{pre}}^{\text{geom}}$.*
- (3) *The strict equivalence class of $\mathcal{W}_{\text{pre}}^{\text{geom}}$ is invariant under isotopy of η or $\mathcal{C}$ and does not depend on the choice of a compatible Liouville form λ .*

A homotopy equivalence in an A_∞ -pre-category is a closed morphism of degree zero invertible up to homotopy, see Definition 5.1, it is stronger than being a quasi-isomorphism in the sense of [33, Definition 9]. Having enough homotopy equivalences is an appropriate notion of unitality for A_∞ -pre-categories in our context. Another option would be to use the homotopy unit framework of Fukaya–Oh–Ohta–Ono [23, §3.3], but it is significantly more involved. By strongly triangulated we mean that there is a strict totalisation functor from the triangulated envelope $\text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}$ to $\mathcal{W}_{\text{pre}}^{\text{geom}}$.

In Theorem 3 (2) two objects in an A_∞ -pre-category are homotopy equivalent if they can be connected by a chain of homotopy equivalences. By ‘isotopic’ we mean connected by a path in the space of generic (I, γ, P) satisfying $\gamma^* \lambda + dP = 0$ with the C^∞ -topology. This notion is weaker than Legendrian isotopy of the lifts. We expect that the latter also induces homotopy equivalence in $\mathcal{W}_{\text{pre}}^{\text{geom}}$. We will address this elsewhere.

In Theorem 3(3) a strict equivalence $\mathcal{F}: \mathcal{W}_0 \rightarrow \mathcal{W}_1$ is a bijection on objects $\mathcal{F}^0$ that establishes a bijection on transverse sequences together with isomorphisms of graded k -modules $\mathcal{F}^1: \mathcal{W}_0(-, -) \rightarrow \mathcal{W}_1(\mathcal{F}^0-, \mathcal{F}^0-)$ on transverse pairs such that $\mathcal{F}^1 \circ \mu_0^n = \mu_1^n \circ (\mathcal{F}^1)^{\otimes n}$. An isotopy of η is any path in $C^\infty(S, \mathbb{P}TS)$. An isotopy of $\mathcal{C}$ is induced by a path starting at id_S in the space of diffeomorphisms of S fixing Q with the C^∞ -topology.

Geometric Fukaya strictly unital A_∞ -category. Kontsevich and Soibelman [33, Conjecture 4] conjectured that the quasi-equivalence classes of k -linear A_∞ -pre-categories with enough quasi-isomorphisms are in bijection with quasi-equivalence classes of strictly unital k -linear A_∞ -categories. This was proven when k is a field by Efimov [17], but remains open for arbitrary commutative rings, which is the setting we are in. We therefore resort to a hands-on way to construct an A_∞ -category spanned by a collection of objects in an A_∞ -pre-category that works well for our geometric setup.

We sketch the construction. Call a collection $\mathbf{L}^0$ of Lagrangian branes *tame* if the branes in it are supported on a no more than countable set of transverse curves. Call a tame collection $\mathbf{L}^0$ *full* if any Lagrangian brane in $(\mathcal{S}; \mathcal{C})$ is homotopy equivalent in $\mathcal{W}_{\text{pre}}^{\text{geom}}$ to a Lagrangian brane from $\mathbf{L}^0$. For any tame collection $\mathbf{L}^0 = \{\mathbb{L}_\alpha^0\}_{\alpha \in A}$ we construct a collection $\mathbf{L}^\bullet = \{\mathbb{L}_\alpha^j\}_{\alpha \in A}^{j=0,1,\dots}$ of Lagrangian branes and homotopy equivalences $e_\alpha^{j_0, j_1} \in \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathbb{L}_\alpha^{j_0}, \mathbb{L}_\alpha^{j_1})$ for each $j_0 \neq j_1$ that behave like strict units, see §5.3.1. We call such $\mathbf{L}^\bullet$ a *system of strict units* and associate to it a strictly unital A_∞ -category $\mathcal{W}_{\mathbf{L}^\bullet}$ with $\text{Ob } \mathcal{W}_{\mathbf{L}^\bullet} = \mathbf{L}^0$, see §5.3.2. This system of strict units is obtained by taking small perturbations of the underlying curve of $\mathbb{L}_\alpha^0$, see §6.4.1. The result of the construction is the following.

Theorem 4. *Let $(\mathcal{S}; \mathcal{C})$ be a weighted Liouville surface. Then*

- (1) *For any system of strict units $\mathbf{L}^\bullet$ the strictly unital A_∞ -category $\mathcal{W}_{\mathbf{L}^\bullet}$ extends the full subcategory of $\mathcal{W}_{\text{pre}}^{\text{geom}}$ on $\mathbf{L}^0$, i.e. $\text{Ob } \mathcal{W}_{\mathbf{L}^\bullet} = \mathbf{L}^0$, $\mathcal{W}_{\mathbf{L}^\bullet}(-, -) = \mathcal{W}_{\text{pre}}^{\text{geom}}(-, -)$ on transverse pairs, and $\mu_{\mathcal{W}_{\mathbf{L}^\bullet}} = \mu_{\mathcal{W}_{\text{pre}}^{\text{geom}}}$ on transverse sequences. Moreover, two objects of $\mathcal{W}_{\mathbf{L}^\bullet}$ are homotopy equivalent in $\mathcal{W}_{\mathbf{L}^\bullet}$ if and only if they are homotopy equivalent in $\mathcal{W}_{\text{pre}}^{\text{geom}}$.*
- (2) *For any two systems of strict units $\mathbf{L}_0^\bullet \subset \mathbf{L}_1^\bullet$ we have a natural strict full embedding $\mathcal{W}_{\mathbf{L}_0^\bullet} \hookrightarrow \mathcal{W}_{\mathbf{L}_1^\bullet}$.*
- (3) *For any two systems of strict units $\mathbf{L}_0^\bullet, \mathbf{L}_1^\bullet$ there exists a strict autoequivalence $\mathcal{I}$ of $\mathcal{W}_{\text{pre}}^{\text{geom}}$ such that $\mathbf{L}_0^\bullet \sqcup \mathcal{I}(\mathbf{L}_1^\bullet)$ is a system of strict units.*
- (4) *Any tame collection can be extended to a full tame collection.*
- (5) *If $\mathbf{L}^0$ is a full tame collection, then $\mathcal{W}_{\mathbf{L}^\bullet}$ is triangulated.*

A strict full embedding above is a strict A_∞ -functor which is injective on objects and an isomorphism of graded modules on morphisms. Theorem 4 implies that the homotopy equivalence class of $\mathcal{W}_{\mathbf{L}^\bullet}$ for a system of strict units $\mathbf{L}^\bullet$ on a full tame collection $\mathbf{L}^0$ does not depend on these choices. We call the resulting category the *geometric Fukaya strictly unital A_∞ -category* $\mathcal{W}_{\text{su}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ of $\mathcal{S}$ with coefficients in $\mathcal{C}$. Note that $\mathcal{W}_{\text{su}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ is a small strictly unital triangulated $\mathbb{Z}$ -graded k -linear A_∞ -category.

We now tie back to the topological Fukaya category. Any arc brane can be interpreted as a Lagrangian brane and an arc brane system Γ gives a tame collection of Lagrangian branes Γ^0 .

Theorem 5. *Let $(\mathcal{S}; \mathcal{C})$ be a weighted Liouville surface. Then*

- (1) *For any arc brane system Γ the A_∞ -category $\mathcal{V}_\Gamma$ is strictly equivalent to $\mathcal{W}_{\mathbf{L}^\bullet}$ for a system of strict units $\mathbf{L}^\bullet$ on the corresponding tame collection Γ^0 .*
- (2) *For a full tame collection $\mathbf{L}^0$ the subcategory of $\mathcal{W}_{\mathbf{L}^\bullet}$ generated by $\Gamma^0 \subset \mathbf{L}^0$ corresponding to a full arc brane system Γ does not depend on such Γ .*

The subcategory generated by $B \subset \text{Ob } \mathcal{W}$ is the smallest full subcategory of $\mathcal{W}$ containing B and closed under taking shifts, cones, and homotopy equivalent objects. In the proof of Theorem 5(1) we do not use that $\mathcal{V}_\Gamma$ satisfies the A_∞ -relations hence Theorem 5 implies Theorem 1 and moreover that $\mathcal{W}^{\text{top}}$ is homotopy equivalent to a subcategory of $\mathcal{W}_{\text{su}}^{\text{geom}}$, well-defined up to homotopy autoequivalence of $\mathcal{W}_{\text{su}}^{\text{geom}}$. We expect that $\mathcal{W}^{\text{top}}$ is in fact homotopy equivalent to $\mathcal{W}_{\text{su}}^{\text{geom}}$. We will address this elsewhere.

Structure of the paper and other results. In §1 we define weighted surfaces and their topological Fukaya categories. In Proposition 1.9 we give a sufficient condition for $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ to have a formal generator. We describe the relation of $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ to classical topological Fukaya categories in §1.2.4 and to relative wrapped Fukaya categories of surfaces in §1.2.5.

In §2 we prove Theorem 2.

In §3 we study Liouville structures on a given surface, Lagrangian structures on a given curve and deformations of such structures. We describe the moduli space of graded Liouville marked structures on a given surface in Proposition 3.2 and prove that diffeomorphisms of marked surfaces deform to symplectomorphisms in Proposition 3.4. We characterise Lagrangian structures on a given curve in terms of linear inequalities on actions and areas in Propositions 3.14 and 3.15. In Proposition 3.17 we characterise isotopies of Lagrangians topologically. We then define the relevant polygons bounded by curves and discuss the grading.

In §4 we define Lagrangian pre-branes and their obstruction algebras. In Proposition 4.8 we prove that the obstruction algebra does not change under isotopies of Lagrangians and another move that allows to exchange Reeb chords for double points. In Proposition 4.10 we prove that the obstruction algebra is a curved A_∞ -algebra.

In §5 we study A_∞ -pre-categories with enough homotopy equivalences. After establishing basic facts about homotopy equivalences in Lemmas 5.3 to 5.6, we explain how to get a strictly unital A_∞ -category from a certain A_∞ -pre-category in Proposition 5.12 and give a sufficient condition for this construction to be canonical in Proposition 5.14. We define the triangulated envelope of an A_∞ -pre-category in §5.2.1 and prove that homotopy equivalences behave well for strongly triangulated A_∞ -categories in Proposition 5.8.

In §6 we define the geometric Fukaya A_∞ -pre-category in §6.1 and prove Theorems 1 and 3 to 5.

Conventions. We adopt the following conventions throughout.

For geometric considerations we work in the smooth category unless otherwise stated. Manifold means (smooth) manifold with (possibly empty) boundary. A smooth map on a subset A of a manifold is the restriction of a smooth map on a neighbourhood of A .

Given a smooth map $\varphi: M \rightarrow N$ the pullback on n -forms is $\varphi^*: \Omega^n(N) \rightarrow \Omega^n(M)$, for any other fibre bundle E on N the pullback on sections is $\varphi^*: C^\infty(N, E) \rightarrow C^\infty(M, \varphi^*E)$. The pullback of a along an embedding is denoted by $a|_M$.

A component of a topological space T is its path-connected component, the set of all components is denoted by $\pi_0 T$.

For algebraic considerations we work over a unital commutative ring k . A linear map is a morphism of (left) k -modules. The tensor product over k is denoted by $\otimes$.

All (dg-, A_∞ -, 2-)categories except $\text{dgc}at_k$, $Ch(k)$, and $D_{\text{perf}} X$ are small. All 2-categories and 2-functors are strict.

Compositions of morphisms and concatenations of paths are written from right to left, e.g. the concatenation of a path a from p to q and a path b from q to r is the path ba from p to r .

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search, and implementing notational and editorial changes. All main ideas, constructions and results are the author's. The author has verified all AI-assisted content and takes full responsibility for the paper. The wording of this declaration was suggested by AI.

1. TOPOLOGICAL FUKAYA CATEGORY

In §1.1 we define weighted systems of dg-categories on a marked surface. In §1.2 we define the topological Fukaya category of a weighted surface, give a sufficient condition for existence of a formal generator in it, and explain how to specialise it to earlier constructions.

1.1. Weighted system of dg-categories. Let S be an oriented smooth surface and $Q \subset S$ a finite set, called the *marking*. In §1.1.1 we define the *directed relative path 2-category* $\mathcal{P}_2^+(S, Q)$ by modifying the definition of the path 2-groupoid of a smooth manifold by Schreiber–Waldorf [48, §2–3]. In §1.1.2 we define a weighted system of dg-categories on a marked surface (S, Q) as a 2-functor $\mathcal{C}: \mathcal{P}_2^+(S, Q) \rightarrow \text{dgcats}_k$ for a commutative ring k . In §1.1.3 we define the composition along a 2-morphism in $\mathcal{P}_2^+(S, Q)$ with respect to $\mathcal{C}$, which is the key construction to define all A_∞ -operations in the paper. In §1.1.4 we describe two more manageable quotients of $\mathcal{P}_2^+(S, Q)$, the directed relative fundamental and homological 2-categories $\Pi_2^+(S, Q)$ and $\mathcal{H}_2^+(S, Q)$.

1.1.1. Directed relative path 2-category. First recall general notions on lazy paths, see Schreiber–Waldorf [48] for details. Let M be a smooth manifold. A *lazy 1-path* (a.k.a. a path with sitting instants) from $p_0 \in M$ to $p_1 \in M$ is a smooth map $\gamma: [0, 1] \rightarrow M$ such that

- (1) $\gamma(s) = p_0$ for s in a neighbourhood of 0 and
- (2) $\gamma(s) = p_1$ for s in a neighbourhood of 1.

We use notation $p_0 \xrightarrow{\gamma} p_1$ to denote such a lazy 1-path.

A *lazy 2-path* from a lazy 1-path $p_0 \xrightarrow{\gamma_0} p_1$ to a lazy 1-path $p_0 \xrightarrow{\gamma_1} p_1$ is a smooth map $\sigma: [0, 1] \times [-1, 0] \rightarrow M$ such that

- (1) $\sigma(s, t) = p_i$ for fixed $i = 0, 1$ and (s, t) in a neighbourhood of $\{i\} \times [-1, 0]$ and
- (2) $\sigma(s, t) = \gamma_i(s)$ for fixed $i = 0, 1$ and (s, t) in a neighbourhood of $[0, 1] \times \{-i\}$.

We use notation $\gamma_0 \xrightarrow{\sigma} \gamma_1$ to denote such a lazy 2-path.

Remark 1.1. The choice of the domain for a lazy 2-path is made to satisfy the following conditions.

- (1) The first coordinate is horizontal and the second coordinate is vertical (so the horizontal and vertical compositions are the respective stackings of the domains).
- (2) The orientation of the domain $[0, 1] \times \{-1\}$ of the *target* 1-path coincides with its orientation as boundary of $[0, 1] \times [-1, 0]$.

A lazy 2-path σ is called a *thin homotopy* if $\text{rk } d\sigma \leq 1$ everywhere. Observe that thin homotopy is an equivalence relation on lazy 1-paths.

Specialising to $Q \subset S$ as above, we say that a lazy 2-path $\gamma_0 \xrightarrow{\sigma} \gamma_1$ in S is *positive* if

- (1) γ_0 and γ_1 are lazy 1-paths in $S \setminus Q$ and
- (2) $(\frac{\partial \sigma}{\partial s}, \frac{\partial \sigma}{\partial t})$ is a positive basis of $T_{\sigma(s, t)}S$ for each $(s, t) \in [0, 1] \times [-1, 0]$ with $\sigma(s, t) \in Q$.

Note that a positive 2-path cannot have $Q \cap \partial S$ in its image. We therefore divide Q into two parts, the *divisor* $D = Q \cap \text{Int } S$ and the *stop* $\Lambda = Q \cap \partial S$.

An *isotopy* from a positive 2-path $\gamma_0^0 \xrightarrow{\sigma^0} \gamma_1^0$ to a positive 2-path $\gamma_0^1 \xrightarrow{\sigma^1} \gamma_1^1$ is a smooth map $h: [0, 1] \times [-1, 0] \times [0, 1] \rightarrow S$ such that

- (1) $h|_{[0, 1] \times \{-i\} \times [0, 1]}$ is a thin homotopy from γ_i^0 to γ_i^1 in $S \setminus Q$ for $i = 0, 1$;
- (2) $h(s, t, r) = \sigma^i(s, t)$ for fixed $i = 0, 1$ and (s, t, r) in a neighbourhood of $[0, 1] \times [-1, 0] \times \{i\}$
- (3) $h|_{[0, 1] \times [-1, 0] \times \{r\}}$ is a positive 2-path for any $r \in [0, 1]$.

Observe that isotopy induces an equivalence relation on positive 2-paths. Note that there is no notion of thinness for isotopies since $\dim S = 2$.

Definition 1.2. Let S be an oriented surface and $Q \subset S$ a finite set. The *directed relative path 2-category* $\mathcal{P}_2^+(S, Q)$ is the following (strict) 2-category.

- (1) $\text{Ob } \mathcal{P}_2^+(S, Q) = S \setminus Q$.
- (2) For $p_0, p_1 \in S \setminus Q$ the set of 1-morphisms is

$$\mathcal{P}_2^+(S, Q)(p_0, p_1) = \frac{\{\text{lazy 1-paths from } p_0 \text{ to } p_1 \text{ in } S \setminus Q\}}{\text{thin homotopy}}.$$

- (3) For $[\gamma_0], [\gamma_1] \in \mathcal{P}_2^+(S, Q)(p_0, p_1)$ the set of 2-morphisms is

$$\mathcal{P}_2^+(S, Q)(p_0, p_1)([\gamma_0], [\gamma_1]) = \frac{\left\{ \text{positive 2-paths } \gamma'_0 \xrightarrow{\sigma} \gamma'_1 \text{ in } S \text{ with } \gamma'_i \in [\gamma_i] \text{ for } i = 0, 1 \right\}}{\text{isotopy}}.$$

The composition $\circ$ of 1-morphisms is induced by the concatenation of paths; the vertical and horizontal compositions (we denote both by $\cup$ if it does not cause confusion) of 2-morphisms are induced by vertical and horizontal stackings of the domains $[0, 1]^2$ respectively.

We refer the reader to Kelly–Street [30] for the 2-categorical notation used. That Definition 1.2 indeed gives a 2-category is a straightforward check analogous to the case of the path 2-groupoid, see [48, §3.1] for details of the latter.

Observe that any piece-wise smooth path γ in $S \setminus Q$ defines a 1-morphism $[\gamma]$ in $\mathcal{P}_2^+(S, Q)$. Moreover, if we have a smooth map $u: \Pi \rightarrow M$ where Π is a compact region in $\mathbb{R}^2$ bounded by a piecewise-smooth Jordan curve, then for any two points $v_0, v_1 \in \partial\Pi$ we get a 2-morphism $[u]: [\gamma^0] \rightarrow [\gamma^1]$ where γ^0 (resp. γ^1) is the restriction of u to the part of $\partial\Pi$ from v_0 to v_1 that has Π on the right (resp. left).

Note that any 1-morphism in $\mathcal{P}_2^+(S, Q)$ is invertible with the inverse for a lazy 1-path γ given by $\gamma^{-1}(s) = \gamma(1 - s)$. Also observe that the *degree function*

$$\deg_\sigma: D \rightarrow \mathbb{Z}_{\geq 0}, \deg_\sigma(q) = |\sigma^{-1}(q)|$$

of a positive 2-path $\gamma_0 \xrightarrow{\sigma} \gamma_1$ is constant in isotopy classes and additive with respect to vertical and horizontal composition. In particular, if $\deg_\sigma \neq 0$, then σ is not invertible. On the other hand, if $\deg_\sigma = 0$, then $\sigma(s, -1 - t)$ is the inverse of $\sigma(s, t)$ (with respect to vertical composition).

1.1.2. Weighted system of dg-categories on a graded marked surface. We define the surfaces we consider in this paper.

Definition 1.3. A *graded marked surface* is the data of

- (1) A compact smooth oriented surface S such that each component of S has non-empty boundary.
- (2) A finite set $Q \subset S$ called the *marking*.
- (3) A line field $\eta \in C^\infty(S, \mathbb{P}TS)$ called the *grading*.

Let $\mathcal{S} = (S, Q, \eta)$ be a graded marked surface. We associate to it the 2-category $\mathcal{P}_2^+(\mathcal{S})$ from Definition 1.2. We recall the naïve 2-category structure on dg-categories.

Fix a commutative ring k and recall that the category $Ch(k)$ of dg- k -modules is symmetric monoidal. A *dg-category* $\mathcal{C}$ (over k) is a category enriched over $Ch(k)$, a *dg-functor* is a functor enriched over $Ch(k)$, see Keller [29, §2] for details. $Z^0(\mathcal{C})$ and $H^0(\mathcal{C})$ are the categories with the same objects as $\mathcal{C}$ and morphisms given by

$$Z^0(\mathcal{C})(M, N) = Z^0(\mathcal{C}(M, N)) \text{ and } H^0(\mathcal{C})(M, N) = H^0(\mathcal{C}(M, N))$$

where Z^0 and $H^0: Ch(k) \rightarrow k\text{-Mod}$ are the 0-cycle and 0-cohomology functors respectively. We will only consider small dg-categories. They form a closed symmetric monoidal category with the internal hom-functor Fun_{dg} . By *morphisms* of dg-functors from $\mathcal{C}$ to $\mathcal{D}$ we mean that in the category $Z^0(\text{Fun}_{dg}(\mathcal{C}, \mathcal{D}))$. A dg-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a (*strict*) *dg-equivalence* if there exists a dg-functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G$ is isomorphic to $\text{id}_{\mathcal{D}}$ (resp. $F \circ G = \text{id}_{\mathcal{D}}$) and $G \circ F$ is isomorphic to $\text{id}_{\mathcal{C}}$ (resp. $G \circ F = \text{id}_{\mathcal{C}}$). We denote by $\text{dgc}_{\mathbf{k}}$ the 2-category of dg-categories where an object is a small dg-category over k , a 1-morphism is a dg-functor, and a 2-morphism is a morphism of dg-functors.

Definition 1.4. A *weighted system of dg-categories* on the graded marked surface $\mathcal{S}$ is the data of

- (1) a commutative ring k ;
- (2) a strict 2-functor $\mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgcat}_k$.

We call the pair $(\mathcal{S}; \mathcal{C})$ a *weighted surface*.

For $p \in \text{Ob } \mathcal{P}_2^+(\mathcal{S}) = S \setminus Q$ we denote the corresponding dg-category, called the *fibre* at p , by $\mathcal{C}_p$. For a 1-morphism in $\mathcal{P}_2^+(\mathcal{S})$ represented by a lazy 1-path γ we denote the corresponding dg-functor, called the *transition functor* along γ , by $\mathcal{C}(\gamma)$. For a 2-morphism in $\mathcal{P}_2^+(\mathcal{S})$ represented by a positive 2-path σ we denote the corresponding morphism of dg-functors, called the *weight* of σ , by $\mathcal{C}(\sigma)$.

Note that all transition functors in a weighted system of dg-categories are strict dg-equivalences and all weights of positive 2-paths that do not intersect Q are isomorphisms of dg-functors.

Remark 1.5. As Fukaya categories are A_∞ -categories, it might seem more natural to consider A_∞ -category coefficients instead. We do not do this for three reasons. First, similarly to the case of tensor product, the formulae in the case of A_∞ -categories are complicated and unenlightening, see Ornaghi [42] for an overview of the tensor product constructions. Secondly, to obtain an object of the Fukaya category, one chooses a local system of modules on a Lagrangian, and modules over an A_∞ -category form a dg-category, so we can simply consider that dg-category and local system of objects over it. Thirdly, A_∞ -categories do not form a 2-category at the chain level since the vertical composition of A_∞ -natural transformations is not strictly associative.

1.1.3. Composition along a 2-morphism. Fix a weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(\mathcal{S}, Q) \rightarrow \text{dgcat}_k$. Let $p^0, \dots, p^n \in S \setminus Q$, $\gamma^0, \dots, \gamma^n$ be lazy 1-paths in $S \setminus Q$ such that γ^0 goes from p^0 to p^n and γ^i goes from p^{i-1} to p^i for $i = 1, \dots, n$, and $\mathcal{M}(p^i)$ be an object in $\mathcal{C}_{p^i}$ for $i = 0, \dots, n$. Let γ be a lazy parametrisation of $\gamma^n \dots \gamma^1$ and consider a positive 2-path $\sigma: \gamma^0 \Rightarrow \gamma$. The *composition along σ* is the morphism

$$\begin{array}{c} \mathcal{C}_{p^n}(\mathcal{C}(\gamma^n)\mathcal{M}(p^{n-1}), \mathcal{M}(p^n)) \otimes \dots \otimes \mathcal{C}_{p^1}(\mathcal{C}(\gamma^1)\mathcal{M}(p^0), \mathcal{M}(p^1)) \\ \downarrow \mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma) \\ \mathcal{C}_{p^n}(\mathcal{C}(\gamma^0)\mathcal{M}(p^0), \mathcal{M}(p^n)) \end{array}$$

given by

$$(1.1) \quad \mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma)(\varphi^n, \dots, \varphi^1) = \varphi^{\gamma^n, \dots, \gamma^1} \cdot \mathcal{C}(\sigma)_{\mathcal{M}(p^0)}$$

where $\varphi^{\gamma^n, \dots, \gamma^1} = \varphi^{\gamma^n} \dots \varphi^{\gamma^1}(\varphi^n, \dots, \varphi^1)$ is the composition

$$(1.2) \quad \mathcal{C}(\gamma^n \dots \gamma^1)\mathcal{M}(p^0) \xrightarrow{\mathcal{C}(\gamma^n \dots \gamma^2)\varphi^1} \mathcal{C}(\gamma^n \dots \gamma^2)\mathcal{M}(p^1) \xrightarrow{\mathcal{C}(\gamma^n \dots \gamma^3)\varphi^2} \dots \xrightarrow{\varphi^n} \mathcal{M}(p^n)$$

in $\mathcal{C}_{p^n}$.

We list elementary properties of composition along a positive 2-path for future reference.

Lemma 1.6. Let $\gamma^0, \dots, \gamma^n$ and σ be as above. Then

- (1) $\mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma)$ only depends on the thin homotopy classes of $\gamma^0, \dots, \gamma^n$ and the isotopy class of σ .
- (2) $\mathcal{C}(\gamma^0; \gamma^0; \text{id}) = \text{id}$.
- (3) If $\mathcal{C}(\gamma^i; \gamma_{m^i}^i, \dots, \gamma_1^i; \sigma^i)$ is defined for $i = 1, \dots, n$, then $\mathcal{C}(\gamma^0; \gamma_{m^n}^n, \dots, \gamma_1^n, \dots, \gamma_{m^1}^1, \dots, \gamma_1^1; \sigma \cup \sigma^n \cup \dots \cup \sigma^1)$ is defined and equals $\mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma) \circ (\mathcal{C}(\gamma^n; \gamma_{m^n}^n, \dots, \gamma_1^n; \sigma^n), \dots, \mathcal{C}(\gamma^1; \gamma_{m^1}^1, \dots, \gamma_1^1; \sigma^1))$.

Proof. (1) and (2) are immediate from construction. (3) is the computation

$$\begin{aligned}
& \mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma) \circ (\mathcal{C}(\gamma^n; \gamma_{m^n}^n, \dots, \gamma_1^n; \sigma^n), \dots, \mathcal{C}(\gamma^1; \gamma_{m^1}^1, \dots, \gamma_1^1; \sigma^1))(\varphi_{m^n}^n, \dots, \varphi_1^1) = \\
& \mathcal{C}(\gamma^0; \gamma^n, \dots, \gamma^1; \sigma) \left(\varphi_{m^i}^i, \dots, \gamma_1^i \cdot \mathcal{C}(\sigma^i)_{\mathcal{M}(p^{i-1})} \right)_{i=1}^n = \\
& \left(\prod_{i=1}^n \mathcal{C}(\gamma^n \dots \gamma^{i+1}) \left(\varphi_{m^i}^i, \dots, \gamma_1^i \cdot \mathcal{C}(\sigma^i)_{\mathcal{M}(p^{i-1})} \right) \right) \cdot \mathcal{C}(\sigma)_{\mathcal{M}(p^0)} = \\
& \left(\prod_{i=1}^n \mathcal{C}(\gamma^n \dots \gamma^{i+1}) \left(\varphi_{m^i}^i, \dots, \gamma_1^i \right) \cdot \mathcal{C}(\text{id}_{\gamma^n \dots \gamma^{i+1}} \cup \sigma^i)_{\mathcal{M}(p^{i-1})} \right) \cdot \mathcal{C}(\sigma)_{\mathcal{M}(p^0)} =^* \\
& \left(\prod_{i=1}^n \mathcal{C}(\gamma_{m^n}^n \dots \gamma_1^{i+1}) \left(\varphi_{m^i}^i, \dots, \gamma_1^i \right) \right) \cdot \left(\prod_{i=1}^n \mathcal{C}(\text{id}_{\gamma^n \dots \gamma^{i+1}} \cup \sigma^i)_{\mathcal{C}(\gamma_{m^{i-1}}^{i-1} \dots \gamma_1^1)_{\mathcal{M}(p^0)}} \right) \cdot \mathcal{C}(\sigma)_{\mathcal{M}(p^0)} = \\
& \varphi_{m^n}^n, \dots, \gamma_1^1(\varphi_{m^n}^n, \dots, \varphi_1^1) \cdot \left(\prod_{i=1}^n \mathcal{C}(\text{id}_{\gamma^n \dots \gamma^{i+1}} \cup \sigma^i \cup \text{id}_{\gamma_{m^{i-1}}^{i-1} \dots \gamma_1^1})_{\mathcal{M}(p^0)} \right) \cdot \mathcal{C}(\sigma)_{\mathcal{M}(p^0)} = \\
& \mathcal{C}(\gamma^0; \gamma_{m^n}^n, \dots, \gamma_1^1; \sigma \cup \sigma^n \cup \dots \cup \sigma^1)(\varphi_{m^n}^n, \dots, \varphi_1^1)
\end{aligned}$$

where $\prod_{i=1}^n x_i = x_n \dots x_1$ and for $=^*$ we repeatedly use that

$$\mathcal{C}(\tau)_{M_1} \cdot \mathcal{C}(\delta_0)\varphi = \mathcal{C}(\delta_1)\varphi \cdot \mathcal{C}(\tau)_{M_0} \in \mathcal{C}_{q_1}(\mathcal{C}(\delta_0)M_0, \mathcal{C}(\delta_1)M_1)$$

for any $q_0, q_1 \in S \setminus Q$, lazy 1-paths δ_0, δ_1 from q_0 to q_1 , positive 2-path $\tau: \delta_0 \Rightarrow \delta_1$, $M_0, M_1 \in \mathcal{C}_{q_0}$, and $\varphi \in \mathcal{C}_{q_0}(M_0, M_1)$ $\square$

1.1.4. Fundamental and homological 2-category. The equivalence relation on paths in Definition 1.2 is the finest that allows us to define a Fukaya category. There are two coarser and more manageable 2-categories which one can associate to $Q \subset S$ as above.

Definition 1.7. Let S be an oriented surface and $Q \subset S$ a finite set. The *directed relative fundamental 2-category* $\Pi_2^+(S, Q)$ is the following 2-category.

- (1) $\text{Ob } \Pi_2^+(S, Q) = S \setminus Q$.
- (2) For $p_0, p_1 \in S \setminus Q$ the set of 1-morphisms is

$$\Pi_2^+(S, Q)(p_0, p_1) = \frac{\{\text{lazy 1-paths from } p_0 \text{ to } p_1 \text{ in } S \setminus Q\}}{\text{homotopy rel endpoints in } S \setminus Q}.$$

- (3) For $[\gamma_0], [\gamma_1] \in \Pi_2^+(S, Q)(p_0, p_1)$ the set of 2-morphisms is

$$\Pi_2^+(S, Q)(p_0, p_1)([\gamma_0], [\gamma_1]) = \frac{\left\{ \text{positive 2-paths } \gamma'_0 \xRightarrow{\sigma} \gamma'_1 \text{ in } S \text{ with } \gamma'_i \in [\gamma_i] \text{ for } i = 0, 1 \right\}}{\text{isotopy not necessarily thin on } [0, 1] \times \{-i\} \times [0, 1]}.$$

Note that the 1-category truncation of $\Pi_2^+(S, Q)$ is the fundamental groupoid $\Pi_1(S \setminus Q)$. Also observe that the identity on objects, lazy 1-paths, and positive 2-paths induce a 2-functor $\mathcal{P}_2^+(S, Q) \rightarrow \Pi_2^+(S, Q)$.

Definition 1.8. Let S be an oriented surface and $Q \subset S$ a finite set. The *directed relative homology 2-category* $\mathcal{H}_2^+(S, Q)$ is the following 2-category.

- (1) $\text{Ob } \mathcal{H}_2^+(S, Q) = S \setminus Q$.
- (2) For $p_0, p_1 \in S \setminus Q$ the set of 1-morphisms is

$$\mathcal{H}_2^+(S, Q)(p_0, p_1) = \frac{\{x \in C_1(S \setminus Q): \partial x = p_1 - p_0\}}{\partial C_2(S \setminus Q)}$$

where $(C_*(S \setminus Q), \partial)$ is the chain complex of singular smooth simplices in $S \setminus Q$ with coefficients in $\mathbb{Z}$.

(3) For $[x_0], [x_1] \in \mathcal{H}_2^+(S, Q)(p_0, p_1)$ the set of 2-morphisms is

$$\mathcal{H}_2^+(S, Q)(p_0, p_1)([x_0], [x_1]) = \left\{ v \in (\mathbb{Z}_{\geq 0})^D : x_1 - x_0 - \sum_{q \in D} v(q) \gamma_q \in \partial C_2(S \setminus Q) \right\}$$

where γ_q is the boundary of a small disc in S around $q \in D$.

The composition of 1-morphisms is induced by addition in $C_1(S \setminus Q)$ and the vertical and horizontal compositions are given by addition in $(\mathbb{Z}_{\geq 0})^D$.

Note that the set of 2-morphisms between two fixed 1-morphisms has at most one element since $\{[\gamma_q]\}_{q \in D}$ is a linearly independent set in $H_1(S \setminus Q)$. Also observe that the identity on $S \setminus Q$, sending a 1-path to the corresponding 1-simplex, and sending a positive 2-path σ to its degree $\deg_\sigma$ induce a 2-functor $\Pi_2^+(S, Q) \rightarrow \mathcal{H}_2^+(S, Q)$.

1.2. Fukaya category from arc systems. Fix a weighted surface $(S = (S, Q, \eta); \mathcal{C})$.

In §1.2.1 we define arc brane systems, and degrees of boundary paths. In §1.2.2 we construct an A_∞ -category from an arc brane system. In §1.2.3 we define topological Fukaya categories and give a sufficient condition for existence of a formal generator. In §§1.2.4 and 1.2.5 we recall the classical topological Fukaya category and the relative wrapped Fukaya category of a surface respectively, and discuss their relation to our topological Fukaya category.

1.2.1. Arc systems and boundary paths. An *arc* in S is an embedded curve $\gamma: [0, 1] \rightarrow S \setminus Q$ such that $\gamma\{0, 1\} \subset \partial S$, $\gamma(0, 1) \subset \text{Int } S$, and γ is transverse to ∂S . An *arc system* is a no more than countable set of arcs such that any two arcs from the set are disjoint. An arc system is called *full* if its arcs cut S into topological discs and each such disc has at most one point from Q . Note that full arc systems exist for any marked surface (S, Q) since we can first cut S into a single disc by an arc system, and then for each $q \in Q$ add an arc that cuts off q from the disc.

Let γ_0 and γ_1 be two (not necessarily distinct) arcs from an arc system Γ . A *boundary path* from γ_0 to γ_1 is a homotopy class of lazy 1-paths $y: [0, 1] \rightarrow \partial S \setminus \Lambda$ such that $y(0) = \gamma_0(y_+)$, $y(1) = \gamma_1(y_-)$ for $y_+, y_- \in \{0, 1\}$, and dy/dt is always non-negative with respect to the natural orientation of ∂S .

A *grading* on an arc γ is a homotopy class of paths from $\gamma^*\eta$ to the tangent line $\dot{\gamma}$, i.e. an element $\text{gr} \in \Pi_1 C^\infty([0, 1], \gamma^* \mathbb{P}TS)(\gamma^*\eta, \dot{\gamma})$. If the arcs γ_0 and γ_1 are graded by gr_0 and gr_1 respectively, then we define the *degree* $|y| \in \mathbb{Z}$ of a boundary path y from γ_0 to γ_1 as follows. First, obtain a path from $\eta(y(1))$ to $T_{y(1)}\partial S$ in $\mathbb{P}(T_{y(1)}S)$ as a concatenation of $\text{gr}_1(y_-)$ and the smallest counter-clockwise rotation from $\dot{\gamma}(y_-)$ to $T_{y(1)}\partial S$. Secondly, transport it along y to a path from $\eta(y(0))$ to $T_{y(0)}\partial S$ in $\mathbb{P}(T_{y(0)}S)$. Thirdly, compose it with the smallest clockwise rotation from $T_{y(0)}\partial S$ to $\dot{\gamma}(y_+)$ and then with the inverse of $\text{gr}_0(y_+)$ to obtain an element in $\pi_1(\mathbb{P}T_{y(0)}, \eta(y(0)))$. Finally, identify $\pi_1(\mathbb{P}T_{y(0)}, \eta(y(0)))$ with $\mathbb{Z}$ by sending the counter-clockwise rotation by π to 1 to obtain the degree of y . Note that the degree is additive with respect to concatenation of paths.

An *arc brane* is a graded arc (γ, gr) together with an object M^0 of $\mathcal{C}_{\gamma(0)}$. Note that the parallel transport along γ defines an object M^t of $\mathcal{C}_{\gamma(t)}$ for any $t \in [0, 1]$. An *arc brane system* is a set of arc branes $\mathbf{\Gamma}$ such that

$$\mathbf{\Gamma} = \{\gamma \mid (\gamma, \text{gr}, M) \in \mathbf{\Gamma}\}$$

is an arc system. For $\gamma \in \mathbf{\Gamma}$ set

$$\mathcal{M}_\gamma = \{M \mid (\gamma, \text{gr}, M) \in \mathbf{\Gamma}\} \subset \text{Ob } \mathcal{C}_{\gamma(0)}.$$

We say that $\mathcal{M} \subset \text{Ob } \mathcal{C}$ *generates* the dg-category (or A_∞ -category) $\mathcal{C}$ if the smallest full subcategory of the triangulated envelope $\text{Tw } \mathcal{C}$ containing $\mathcal{M}$ and closed under taking shifts, cones of closed degree zero morphisms, and homotopy equivalent objects, contains $\mathcal{C}$. An arc brane system $\mathbf{\Gamma}$ is called *full* if $\mathbf{\Gamma}$ is a full arc system and for any $\gamma \in \mathbf{\Gamma}$ the set $\mathcal{M}_\gamma$ generates $\mathcal{C}_{\gamma(0)}$. Our conventions for the triangulated envelope Tw are given in §5.2.1. To define Tw for dg-categories we view them as A_∞ -categories via $\mu^1(a) = (-1)^{|a|}da$ and $\mu^2(a, b) = (-1)^{|b|}a \cdot b$. This coincides with the usual definition of twisted complexes over a dg-category (see e.g. Drinfeld [16, §2.4]) when the shift by m is realised as $k[m] \otimes -$.

Note that for any full arc system Γ we can take $\mathcal{M}_\gamma = \text{Ob } \mathcal{C}_{\gamma(0)}$ for each $\gamma \in \Gamma$ and obtain a full arc brane system.

1.2.2. A_∞ -category associated to an arc system. We generalise the construction of the A_∞ -category of an arc system from [25, §3.3] to our setting.

Given an arc brane system $\mathbf{\Gamma}$ in $(\mathcal{S}; \mathcal{C})$ we define the strictly unital k -linear A_∞ -category $\mathcal{V}_\Gamma$ as follows.

Objects. $\text{Ob } \mathcal{V}_\Gamma = \Gamma$.

Morphisms. Fix two arc branes $\mathbb{L}_i = (\gamma_i, \text{gr}_i, M_i) \in \Gamma$, $i = 0, 1$.

For a boundary path y from γ_0 to γ_1 define a graded k -module

$$\mathcal{V}_\Gamma(y) = \mathcal{C}_{y(1)}(\mathcal{C}(y)(M_0^{y^+}), M_1^{y^-}).$$

The morphism space between arc branes is the graded k -module

$$(1.3) \quad \mathcal{V}_\Gamma(\mathbb{L}_0, \mathbb{L}_1) = \bigoplus_y \mathcal{V}_\Gamma(y)[-|y|] \Big/ \sim$$

where y runs through boundary paths from γ_0 to γ_1 up to reparametrisation, and $\sim$ identifies the summands for the two constant paths via the transition functor if $\gamma_0 = \gamma_1$.

A_∞ -operations. There are three sources of A_∞ -operations: operations in the fibres of $\mathcal{C}$ (i.e. the differential d and the composition $\cdot$), concatenation of boundary paths, and polygons bounded by the arcs.

If $\gamma_1, \dots, \gamma_n$ are arcs from Γ_Γ and y^i , $i = 1, \dots, n$ are boundary paths from γ_{i-1} to γ_i (the index is mod n) such that $y_+^i \neq y_-^{i-1}$, then the concatenation $\gamma_n y^n \gamma_{n-1} y^{n-1} \dots \gamma_1 y^1$ (whenever we write γ_i in a concatenation we tacitly assume that it can be travelled in either direction) is a closed curve. This closed curve can bound at most one immersed disc, and if it does, then $(y^n, \dots, y^1)$ is called a *disc sequence*.

For $n \geq 1$ and $\mathbb{L}_0 = (\gamma_0, \text{gr}_0, M_0), \dots, \mathbb{L}_n = (\gamma_n, \text{gr}_n, M_n) \in \Gamma$ the A_∞ -operation

$$\mu^n: \mathcal{V}_\Gamma(\mathbb{L}_{n-1}, \mathbb{L}_n) \otimes \dots \otimes \mathcal{V}_\Gamma(\mathbb{L}_0, \mathbb{L}_1) \rightarrow \mathcal{V}_\Gamma(\mathbb{L}_0, \mathbb{L}_n)$$

is determined by its entries

$$\mu^{y^0; y^n, \dots, y^1}: \mathcal{V}_\Gamma(y^n) \otimes \dots \otimes \mathcal{V}_\Gamma(y^1) \rightarrow \mathcal{V}_\Gamma(y^0)$$

where y^i is a boundary path from γ_{i-1} to γ_i for $i = 1, \dots, n$, and y^0 is a boundary path from γ_0 to γ_n .

For $\varphi^i \in \mathcal{C}_{y^i(1)}(\mathcal{C}(y^i)(M_{i-1}^{y_+^i}), M_i^{y_-^i})$, $i = 1, \dots, n$ the element

$$\varphi^0 = \mu^{y^0; y^n, \dots, y^1}(\varphi^n, \dots, \varphi^1) \in \mathcal{C}_{y^0(1)}(\mathcal{C}(y^0)(M_0^{y_+^0}), M_n^{y_-^0})$$

is possibly non-zero in the following cases.

(1) $n = 1$ and $y^0 = y^1$. Then

$$(1.4) \quad \varphi^0 = (-1)^{|\varphi^1| + |y^0|} d\varphi^1.$$

(2) $n = 2$ and $y^0 = y^2 y^1$. If $y_-^1 = 1$ and $y_-^0 = 0$, then

$$(1.5) \quad \varphi^0 = (-1)^{|\varphi^1| + (|y^2| - 1)} \varphi^2 \cdot \mathcal{C}(y^2)(\varphi^1).$$

(3) There exists a boundary path z from γ_{n-1} to γ_0 such that $y^n = y^0 z$ and $(z, y^{n-1}, \dots, y^1)$ is a disc sequence bounding an immersed polygon $\mathbf{u}$. If $\gamma_1, \dots, \gamma_{n-1}$ are oriented as the boundary of $\mathbf{u}$ and $y_-^0 = 0$, then

$$(1.6) \quad \varphi^0 = (-1)^{\sum_{i < j} |\varphi^i| (|y^j| - 1)} \mathcal{C}(y^0; y^n, \gamma_{n-1}, y^{n-1}, \dots, \gamma_1, y^1, \gamma_0; \sigma_{\mathbf{u}}) \\ (\varphi^n, \text{id}, \varphi^{n-1}, \dots, \text{id}, \varphi^1, \text{id})$$

where the 2-morphism $\sigma_{\mathbf{u}}: [y^0] \Rightarrow [y^n \gamma_{n-1} y^{n-1} \dots \gamma_1 y^1 \gamma_0]$ is obtained from $\mathbf{u}$ (which can be viewed as a 2-morphism $[y^0(0)] \Rightarrow [z \gamma_{n-1} y^{n-1} \dots \gamma_1 y^1 \gamma_0]$) by post-composing with $[y^0]$.

- (4) There exists a boundary path z from γ_n to γ_1 such that $y^1 = zy^0$ and $(z, y^n, \dots, y^2)$ is a disc sequence bounding an immersed polygon $\mathbf{u}$. If $\gamma_1, \dots, \gamma_n$ are oriented as the boundary of $\mathbf{u}$, then

$$(1.7) \quad \varphi^0 = (-1)^{\sum_{i < j} |\varphi^i|(|y^j| - 1)} \mathcal{C}(y^0; \gamma_n, \quad y^n, \gamma_{n-1}, \quad y^{n-1}, \dots, \quad \gamma_1, y^1; \sigma_{\mathbf{u}}) \\ (\text{id}, \quad \varphi^n, \text{id}, \quad \varphi^{n-1}, \dots, \quad \text{id}, \varphi^1)$$

where the 2-morphism $\sigma_{\mathbf{u}}: [y^0] \Rightarrow [\gamma_n y^n \gamma_{n-1} y^{n-1} \dots \gamma_1 y^1]$ is obtained from $\mathbf{u}$ (which can be viewed as a 2-morphism $[y^0(1)] \Rightarrow [\gamma_n y^n \gamma_{n-1} y^{n-1} \dots \gamma_1 z]$) by pre-composing with $[y^0]$.

- (5) In the cases (2)–(4) above the change of orientation of γ_i for $i = 1, \dots, n-1$ (resp. γ_n) multiplies the right-hand side in (1.5) to (1.7) by $(-1)^{|y^i|}$ (resp. $(-1)^{|y^n| + |y^0|}$).

It is not hard to see that the cases above are mutually exclusive except when y^0 is a constant path and $(y^n, \dots, y^1)$ is a disc sequence. The formulae in (1.6) and (1.7) are then related by the transition functor along $\gamma_0 = \gamma_n$ and hence get identified under the equivalence relation in (1.3).

The strict units in $\mathcal{V}_{\Gamma}$ are given by $\text{id} \in \mathcal{C}_{y(1)}(M^{y+}, M^{y-}) = \mathcal{V}_{\Gamma}(y) \subset \mathcal{V}_{\Gamma}(\mathbb{L}, \mathbb{L})$ for a constant path y (for which $y_+ = y_-$). That $\mathcal{V}_{\Gamma}$ is a strictly unital A_{∞} -category is the content of Theorem 1(1). It can be proven similarly to [25, Proposition 3.1], we get it as a corollary of Theorem 5.

1.2.3. Basic properties. Recall from the introduction that Theorem 1 implies that we have a well-defined up to homotopy equivalence *topological Fukaya category* $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$, which is given by $\text{Tw } \mathcal{V}_{\Gamma}$ for an arbitrary full arc brane system Γ in $(\mathcal{S}; \mathcal{C})$. We prove Theorem 1 in §6.7. We now give conditions under which $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ has a dg- or formal generator.

Proposition 1.9. *Let $(S = (S, Q, \eta); \mathcal{C})$ be a weighted surface.*

- (1) *If each component of S containing no points from $\Lambda = Q \cap \partial S$ is a disc or an annulus, then there exists a full arc brane system Γ such that $\mathcal{V}_{\Gamma}$ is a dg-category (i.e. $\mu^n = 0$ for $n \geq 3$).*
- (2) *If in addition to (1) each disc component of S has at least two points from Λ and the fibres of $\mathcal{C}$ have zero differential, then Γ can be chosen so that $\mathcal{V}_{\Gamma}$ is a graded $\mathbf{k}$ -linear category (i.e. $\mu^n = 0$ for $n \neq 2$).*

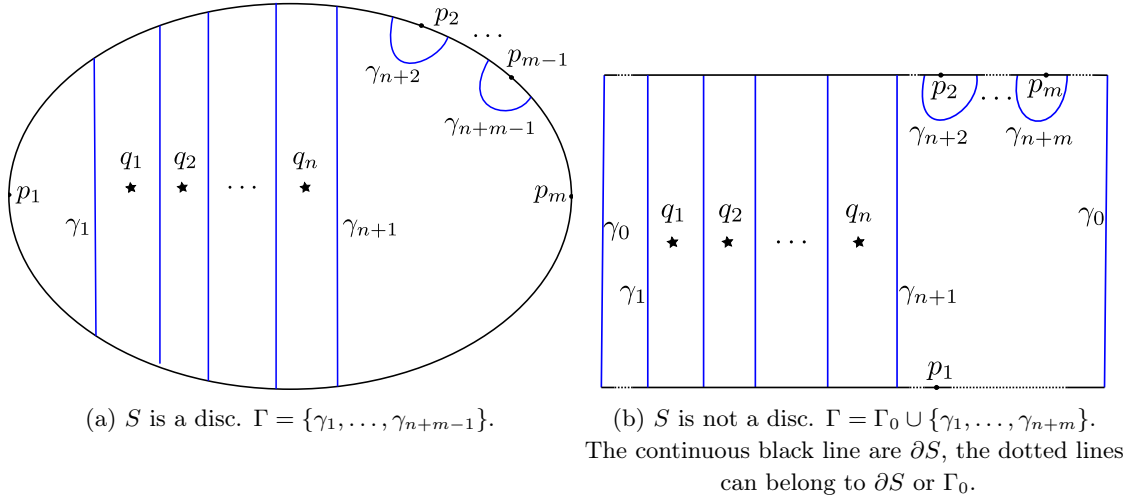


FIGURE 1. Full arc systems giving a dg-category. $\Lambda = \{p_1, \dots, p_m\}$, $D = \{q_1, \dots, q_n\}$, the arcs are in blue.

Proof. We prove (1). We can deal with each component of S separately, so we assume S is connected. If S is a disc, then we choose the full arc system as in Fig. 1(a). Now suppose S is not a disc and $\Lambda \neq \emptyset$. Then we can cut S by an arc system Γ_0 to get a single disc and then glue back

along $\gamma_0 \in \Gamma_0$ to obtain an annulus. We then add to Γ_0 the arcs as in Fig. 1(b) to obtain a full arc system Γ . If S is an annulus and $\Lambda = \emptyset$, then $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ in Fig. 1(b) is a full arc system.

In each case we obtained a full arc system Γ in S such that each region cut out by the arcs either has a point from Λ or is a quadrilateral or is a bigon (the bigons exist only when S is a disc with at most one point from Λ). Let $\mathbf{\Gamma}$ be a full arc brane system supported on Γ . As quadrilaterals and bigons can only glue to quadrilaterals and bigons, the only contributions to $\mu_{\mathbf{\Gamma}}$ from (1.6) and (1.7) are to μ^2 and μ^1 . We conclude that $\mu^n = 0$ for $n \geq 3$ in $\mathcal{V}_{\mathbf{\Gamma}}$.

To prove (2) observe that μ^1 in $\mathcal{V}_{\mathbf{\Gamma}}$ only comes from (1.6) and (1.7) for bigons or (1.4) hence vanishes by the assumption. $\square$

1.2.4. Classical topological Fukaya category. Our definition of the topological Fukaya category $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ is a direct generalisation of the classical topological Fukaya category introduced by Haiden–Katzarkov–Kontsevich [25]. To specialise back to [25] take $Q = \Lambda \subset \partial S$, k a field and $\mathcal{C}$ a trivial constant system (i.e. all fibres are the same dg-category, all transition functors are the identity dg-functor, and all weights are the identity morphism of dg-functors) with fibre k viewed as a dg-category with one object.

Note that the signs in our definition and in [25, §3.3] are different. The discrepancy comes from the fact that in [25] the arcs are not oriented and the orientation at an end of an arc is taken to point out of S , this convention does not extend naturally to the case when arcs are allowed to intersect. Since in our setup we eventually extend the category to include all immersed Lagrangian branes, we choose a different convention, where the orientation is induced by the orientation of the domain.

Classical topological wrapped Fukaya categories are equivalent to other categories coming from algebraic geometry and representation theory. These include

- (1) Categories of matrix factorisations, see Abouzaid–Auroux–Efimov–Katzarkov–Orlov [3] and Pascaleff–Sibilla [46] for Landau–Ginzburg models, and Bocklandt [6] for a non-commutative version.
- (2) Derived categories of (stacky) curves, see Kontsevich [32, Pictures], Lekili–Polishchuk [37], and Habermann [24].
- (3) Perfect derived categories of dg-modules over graded gentle algebras, see [25, §3.4], Lekili–Polishchuk [38], and Oppen–Plamondon–Schroll [41].
- (4) Global sections of constructible cosheaves or of parametrised perverse schobers on ribbon graphs, see [25, §3.6] and Christ–Haiden–Qiu [14, §7.2].

The first examples of implementing ideas from (2) in our setting are given in Theorem 2, (1) and (2). (3) extends to our setting: it is the content of Proposition 1.9. We do not address (1) and (4) in this paper.

1.2.5. Relative wrapped Fukaya category of a surface. Relative Fukaya categories were introduced by Seidel [51] to relate a Fukaya category of a closed symplectic manifold to a Fukaya category of an open submanifold in order to establish HMS, e.g. for the quartic surface [53]. A similar idea of weighting disc counts according to how many times a disc passes through a point goes back to the work of Ozsváth–Szabó on Heegaard Floer homology [43]. Relative Fukaya categories are usually defined in a formal neighbourhood of a point, i.e. k is a complete local ring, see e.g. Perutz–Sheridan [47].

Relative wrapped Fukaya categories of surfaces were introduced by Lekili–Perutz [35] and then developed by Lekili–Polishchuk [36] to prove HMS between the ‘ n -Tate curve’ and the n -punctured torus over $\mathbb{Z}[[t_1, \dots, t_n]]$. Bocklandt–van de Kreeke [7, 8] gave an algebraic definition of relative wrapped Fukaya category linear over $\mathbb{C}[[q_1, \dots, q_n]]$ where $q_1, \dots, q_n$ are punctures. Evans–Lekili [22] gave a combinatorial definition of the relative wrapped Fukaya category over an arbitrary commutative ring, but were only able to prove that the resulting category does not depend on choices after completion. Theorem 1 resolves this issue.

To specialise to the setup of [22] from ours take $Q = D \subset \text{Int } S$ and $\mathcal{C}: \mathcal{P}_2^+(S) \rightarrow \text{dgc}_{\text{at}} k$ such that all fibres of $\mathcal{C}$ are k viewed as a dg-category with one object, all transition functors are the identity, and $\mathcal{C}$ factors through $\mathcal{H}_2^+(S)$ from Definition 1.8. Indeed, unwrapping the definitions, we

see that the data of such $\mathcal{C}$ is equivalent to specifying a weight $w(q) \in Z^0(\text{Fun}_{dg}(\mathbf{k}, \mathbf{k}))(\text{id}_{\mathbf{k}}, \text{id}_{\mathbf{k}}) = \mathbf{k}$ for each $q \in Q$.

2. MIRROR SYMMETRY EXAMPLES

In this section we prove Theorem 2. We prove part (1) in §2.1, part (2) in §2.2, and part (3) in §2.3.

2.1. Total spaces of projective completions of line bundles. To prove Theorem 2 (1) we use the fact that for a 2-dimensional vector bundle $\mathcal{E}$ on X the category $D_{\text{perf}}^{\text{dg}} \mathbb{P}(\mathcal{E})$ is the gluing of $D_{\text{perf}}^{\text{dg}} X$ with itself along a bimodule. When $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}$ we can realise this gluing as the topological Fukaya category of $\mathcal{S} = (S = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}, Q = \{1, -2\}, \eta \text{ radial})$ with coefficients in a system of dg-categories with fibre $D_{\text{perf}}^{\text{dg}} X$. To this end we recall a particular model for $D_{\text{perf}}^{\text{dg}} X$ in §2.1.1, recall the gluing of dg-categories and identify the relevant bimodule on the B-side in §2.1.2, and in §2.1.3 identify the bimodule on the A-side and prove Theorem 2 (1).

2.1.1. Dg-category of perfect complexes. We recall a Čech-style dg-enhancement of the derived category of perfect complexes on a separated Noetherian scheme X by Wei [57]. To use the enhancement as a fibre of a weighted system of dg-categories we need it to be small and to enhance the tensor product with a line bundle to a strict dg-autoequivalence.

Remark 2.1. One can probably do the same with a dg-enhancement via h-injective complexes by carefully choosing a small subcategory of that large dg-enhancement as in Kuznetsov–Lunts [34, §3.11] and then apply a dg-version of strictification of autoequivalences as in Shinder [54, Theorem 5.4]. That should work in larger generality, possibly for any algebraic stack X . We do not pursue this direction as it looks less geometric and enlightening.

Fix a separated Noetherian scheme X over $\mathbf{k}$ and a finite affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X . As usual, denote $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$.

Recall the small dg-category $\mathcal{U}\text{-proj}^\bullet$ from [57, §2.2] (we describe a simplified version sufficient for our setting, in particular, all products become direct sums). An object of $\mathcal{U}\text{-proj}^\bullet$ is a formal direct sum $\mathcal{E} = \bigoplus_{i \in I} \mathcal{E}_i$ where $\mathcal{E}_i$ is a graded $\mathcal{O}_{U_i}$ -module that is a direct summand subsheaf of $\bigoplus_{\alpha=-n}^n \mathcal{O}_{U_i}^{\oplus n}[-\alpha]$ for some $n \geq 0$. The morphism space is

$$(2.1) \quad \mathcal{U}\text{-proj}^\bullet(\mathcal{E}, \mathcal{F}) = \bigoplus_{p \geq 0} \bigoplus_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{O}_{U_{i_0 \dots i_p}}\text{-Mod}^\bullet \left(\mathcal{E}_{i_p}|_{U_{i_0 \dots i_p}}, \mathcal{F}_{i_0}|_{U_{i_0 \dots i_p}} \right).$$

For $u \in \mathcal{U}\text{-proj}^\bullet(\mathcal{E}, \mathcal{F})$ denote by $u_{i_0 \dots i_p}^{p,q}$ the corresponding component. The *total degree* is given by $|u_{i_0 \dots i_p}^{p,q}| = p + q$.

The differential is given by

$$(2.2) \quad (du)_{i_0 \dots i_{p+1}}^{p+1,q} = \sum_{k=1}^p (-1)^k u_{i_0 \dots \hat{i}_k \dots i_{p+1}}^{p,q} \Big|_{U_{i_0 \dots i_{p+1}}}.$$

Note that the index k starts from 1 and ends at p in the sum above. The composition is given by

$$(2.3) \quad (u \cdot v)_{i_0 \dots i_{p+r}}^{p+r,q+s} = (-1)^{qr} u_{i_0 \dots i_p}^{p,q} \Big|_{U_{i_0 \dots i_{p+r}}} \cdot v_{i_{p+1} \dots i_{p+r}}^{r,s} \Big|_{U_{i_0 \dots i_{p+r}}}.$$

Note that $\mathcal{U}\text{-proj}^\bullet$ is the additive enlargement of its full subcategory on objects $\bigsqcup_{i \in I} \text{Ob}(\mathcal{O}_{U_i}\text{-proj})$.

We now describe the triangulated dg-category $\text{Tw}_{\text{perf}}(\mathcal{U})$ of *twisted perfect complexes* from [57, §2.3]. Note that the construction is different from twisted complexes we consider elsewhere in the paper in that the differential of a twisted perfect complex is not upper-triangular and its diagonal entries satisfy a certain non-degeneracy condition.

Namely, an object of $\text{Tw}_{\text{perf}}(\mathcal{U})$ is a pair $(\mathcal{E}, a)$ where $\mathcal{E} \in \text{Ob} \mathcal{U}\text{-proj}^\bullet$ and $a \in \mathcal{U}\text{-proj}^1(\mathcal{E}, \mathcal{E})$ such that the Maurer–Cartan equation $\check{d}a + a \cdot a = 0$ is satisfied and $a_{ii}^{1,0}$ is chain homotopic to $\text{id}_{(\mathcal{E}_i, a_i^{0,1})}$ for each $i \in I$.

The morphism space is the dg-k-module

$$\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})((\mathcal{E}, a), (\mathcal{F}, b)) = \mathcal{U}\text{-proj}^\bullet(\mathcal{E}, \mathcal{F}) \text{ with } du = \check{d}u + b \cdot u - (-1)^{|u|} u \cdot a.$$

Denote by $\mathrm{Sh}_{\mathrm{perf}}(X)$ the dg-category of perfect complexes of $\mathcal{O}_X$ -modules. Recall the *sheafification dg-functor* $\mathbb{S}: \mathrm{Tw}_{\mathrm{perf}}(\mathcal{U}) \rightarrow \mathrm{Sh}_{\mathrm{perf}}(X)$ from [57, §3.1]. First, for $\mathcal{E} \in \mathrm{Ob} \mathcal{U}\text{-proj}^\bullet$ and an open V in X define

$$\check{C}(\mathcal{U}, \mathcal{E}; V) = \bigoplus_{p \geq 0} \bigoplus_{q \in \mathbb{Z}} \check{C}^p(\mathcal{U}, \mathcal{E}^q; V) \text{ where } \check{C}^p(\mathcal{U}, \mathcal{E}^q; V) = \bigoplus_{(i_0, \dots, i_p) \in I^{p+1}} \Gamma(U_{i_0 \dots i_p} \cap V, \mathcal{E}_{i_0}^q).$$

For $c \in \check{C}(\mathcal{U}, \mathcal{E}; V)$ denote by $c_{i_0 \dots i_p}^{p,q}$ the corresponding component. We also have $\mathcal{U}\text{-proj}^\bullet(\mathcal{E}, \mathcal{F}; V)$ defined as in (2.1) but with $U_{i_0 \dots i_p} \cap V$ in place of $U_{i_0 \dots i_p}$. The same modification of (2.2) and (2.3) gives the dg-category $\mathcal{U}\text{-proj}^\bullet(-; V)$.

Moreover, $\check{C}(\mathcal{U}, \mathcal{E}; V)$ has a differential given by

$$(\check{d}c)_{i_0 \dots i_{p+1}}^{p+1,q} = \sum_{k=1}^{p+1} (-1)^k c_{i_0 \dots \hat{i}_k \dots i_{p+1}}^{p,q} \Big|_{U_{i_0 \dots i_{p+1}} \cap V}$$

and the map

$$(2.4) \quad \mathcal{U}\text{-proj}^\bullet(\mathcal{E}, \mathcal{F}; V) \otimes \check{C}(\mathcal{U}, \mathcal{E}; V) \rightarrow \check{C}(\mathcal{U}, \mathcal{F}; V)$$

given by (2.3) with c in place of v and $U_{i_0 \dots i_p} \cap V$ in place of $U_{i_0 \dots i_p}$ satisfies the graded Leibniz rule.

Now, for a twisted perfect complex $(\mathcal{E}, a)$ set

$$\mathbb{S}(\mathcal{E}, a) = \left(\bigoplus_{n \in \mathbb{Z}} \mathbb{S}^n(\mathcal{E}), \check{d} + a \cdot \right) \text{ where } \mathbb{S}^n(\mathcal{E})(V) = \bigoplus_{p+q=n} \check{C}^p(\mathcal{U}, \mathcal{E}^q; V)$$

and define $\mathbb{S}$ on morphisms via restriction and (2.4). By [57, Theorem 3.32, Corollary 6.6, and Appendix 2] the induced exact functor

$$\mathbb{S}: H^0(\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})) \rightarrow \mathrm{D}_{\mathrm{perf}}(X)$$

is an equivalence for any finite affine open cover $\mathcal{U}$. That is, $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$ is a dg-enhancement of $\mathrm{D}_{\mathrm{perf}}(X)$. We write $\mathrm{D}_{\mathrm{perf}}^{\mathrm{dg}} X$ for $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$ when we do not need to specify the cover.

Now let $\mathcal{L}$ be a line bundle on X . Take $\mathcal{U}$ trivialising $\mathcal{L}$. Let $s \in \check{C}^0(\mathcal{U}, \mathcal{L})$ be a Čech cochain of non-vanishing sections and $l \in \check{Z}^1(\mathcal{U}, \mathcal{O}_X^\times)$ the corresponding Čech cocycle with $l_{i_0 i_1} = s_{i_1}/s_{i_0}$. Define $F_l: \mathcal{U}\text{-proj}^\bullet \rightarrow \mathcal{U}\text{-proj}^\bullet$ by identity on objects and

$$(2.5) \quad F_l(u)_{i_0 \dots i_p}^{p,q} = l_{i_0 i_p} u_{i_0 \dots i_p}^{p,q}$$

on morphisms.

Lemma 2.2.

- (1) F_l is a strict dg-autoequivalence of $\mathcal{U}\text{-proj}^\bullet$ with $F_l^{-1} = F_{l^{-1}}$.
- (2) The induced strict dg-autoequivalence F_l of $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$ enhances $- \otimes \mathcal{L}$. Namely,

$$\varphi_{(\mathcal{E}, a)}: \mathbb{S}(\mathcal{E}, a) \otimes \mathcal{L} \rightarrow \mathbb{S}(\mathcal{E}, F_l(a)), \quad c_{i_0 \dots i_p}^{p,q} \otimes \sigma_V \mapsto \frac{\sigma_V}{s_{i_0}} c_{i_0 \dots i_p}^{p,q}$$

defines an isomorphism of dg-functors $\mathbb{S}(-) \otimes \mathcal{L} \rightarrow \mathbb{S} \circ F_l$ from $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$ to $\mathrm{Sh}_{\mathrm{perf}} X$.

- (3) For $\sigma \in \Gamma(X, \mathcal{L})$ the Čech cochain $\sigma/s \in \check{C}^0(\mathcal{U}, \mathcal{O}_X)$ defines a morphism of dg-functors $\sigma/s: \mathrm{id}_{\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})} \rightarrow F_l$ which φ identifies with the multiplication by $\sigma: \mathbb{S} \rightarrow \mathbb{S}(-) \otimes \mathcal{L}$.

Proof. (1) First prove F_l is a dg-functor. Compatibility with composition (2.3) follows from the cocycle condition $l_{i_0 i_p} l_{i_p i_{p+r}} = l_{i_0 i_{p+r}}$. Compatibility with $\check{d}$ holds since (2.2) does not change the outer indices. Next, $F_{l^{-1}}$ is a Čech cocycle corresponding to $\mathcal{L}^{-1}$, so is also a dg-functor. Clearly, $F_l \circ F_{l^{-1}} = \mathrm{id}_{\mathcal{B}} = F_{l^{-1}} \circ F_l$.

(2) Since F_l is a strict dg-autoequivalence and $F_l(a)_{ii}^{1,0} = a_{ii}^{1,0}$, it preserves $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$. Note that $\mathbb{S}$ and $\mathbb{S} \circ F_l$ assign the same graded sheaf $\mathbb{S}^\bullet(\mathcal{E})$ to the twisted perfect complex $(\mathcal{E}, a)$. The map $\varphi_{\mathcal{E}, a}$ commutes with $\check{d}$ since $\check{d}$ keeps i_0 , and it intertwines $a \cdot$ with $F_l(a) \cdot$ since $\frac{s_{i_p}}{s_{i_0}} \cdot \frac{\sigma_V}{s_{i_p}} = \frac{\sigma_V}{s_{i_0}}$,

so it is a map of complexes. The same computation with u in place of a gives naturality of φ . Each $\varphi_{\varepsilon,a}$ is a local isomorphism as it multiplies each component by a unit.

(3) The cochain σ/s is $\bar{d}$ -closed since (2.2) vanishes in Čech degree 0, and $F_l(u) \cdot \sigma/s = \sigma/s \cdot u$ since $\frac{s_{i_p}}{s_{i_0}} \cdot \frac{\sigma}{s_{i_p}} = \frac{\sigma}{s_{i_0}}$. Taking $u = a$ shows that σ/s is closed in $\mathrm{Tw}_{\mathrm{perf}}(\mathcal{U})$, and arbitrary u gives naturality. Finally, $\varphi(c \otimes \sigma) = \sigma/s \cdot c$ by definition. $\square$

2.1.2. Gluing of dg-categories. We recall the gluing of dg-categories by Tabuada [56, Definition 3.57] and Orlov [45, Definitions 3.4, 3.5]. Let $\mathcal{A}$ and $\mathcal{B}$ be two small dg-categories over a commutative ring k and $\mathcal{T}$ an $\mathcal{A}$ - $\mathcal{B}$ -bimodule (i.e. a dg-functor $\mathcal{A}^{op} \otimes \mathcal{B} \rightarrow \mathrm{Ch}(k)$). The *upper triangular dg-category* $\mathcal{A} \sqcup_{\mathcal{T}} \mathcal{B}$ is given by $\mathrm{Ob} \mathcal{A} \sqcup_{\mathcal{T}} \mathcal{B} = \mathrm{Ob} \mathcal{A} \sqcup \mathrm{Ob} \mathcal{B}$ and

$$\mathcal{A} \sqcup_{\mathcal{T}} \mathcal{B}(M, N) = \begin{cases} \mathcal{A}(M, N) & \text{if } M, N \in \mathrm{Ob} \mathcal{A}; \\ \mathcal{B}(M, N) & \text{if } M, N \in \mathrm{Ob} \mathcal{B}; \\ \mathcal{T}(M, N) & \text{if } M \in \mathrm{Ob} \mathcal{A} \text{ and } N \in \mathrm{Ob} \mathcal{B}; \\ 0 & \text{otherwise} \end{cases}$$

with composition given by that in $\mathcal{A}$ and $\mathcal{B}$, and the bimodule structure on $\mathcal{T}$. The *gluing of $\mathcal{A}$ and $\mathcal{B}$ along $\mathcal{T}$* is the triangulated envelope $\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B} = \mathrm{Tw}(\mathcal{A} \sqcup_{\mathcal{T}} \mathcal{B})$.

We recast [45, Example 3.9] for our setting.

Lemma 2.3. *Let $\mathcal{L}$ be a line bundle on a separated Noetherian scheme X over k and $\mathcal{U}^X = \{U_i\}_{i \in I}$ a finite affine open cover of X trivialising $\mathcal{L}$. Then $D_{\mathrm{perf}}^{\mathrm{dg}} \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ is quasi-equivalent to $\mathcal{B} \oplus_{\mathcal{T}} \mathcal{B}$ where $\mathcal{B} = \mathrm{Tw}_{\mathrm{perf}}(\mathcal{U}^X)$ and the $\mathcal{B}$ - $\mathcal{B}$ -bimodule $\mathcal{T}$ is given by*

$$\mathcal{T}(M, N) = \mathcal{B}(M, N) \oplus \mathcal{B}(F_l(M), N) \text{ for } M, N \in \mathrm{Ob} \mathcal{B}.$$

Proof. Let $\pi: Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \rightarrow X$ be the projection and

$$x_{\mathcal{O}} \in \Gamma(Y, \mathcal{O}_Y(1)), \quad x_{\mathcal{L}} \in \Gamma(Y, \mathcal{O}_Y(1) \otimes \pi^* \mathcal{L})$$

the coordinates of the tautological inclusion $\mathcal{O}_Y(-1) \hookrightarrow \mathcal{O}_Y \oplus \pi^* \mathcal{L}$. By Orlov's semi-orthogonal decomposition [44, Theorem 2.6], which holds in our setting by Bergh–Schnürer [5, Corollary 6.8], we have

$$D_{\mathrm{perf}}(Y) = \langle \pi^* D_{\mathrm{perf}}(X), \pi^* D_{\mathrm{perf}}(X) \otimes \mathcal{O}_Y(1) \rangle.$$

By [45, Proposition 3.8] (it is stated over a field, but the proof works verbatim over any commutative ring k) it is enough to find a finite affine cover $\mathcal{U}^Y$ of Y , two dg-functors $\pi_0^*, \pi_1^*: \mathcal{B} \rightarrow \mathrm{Tw}_{\mathrm{perf}}(\mathcal{U}^Y)$ enhancing π^* and $\pi^*(-) \otimes \mathcal{O}_Y(1)$ respectively, and a quasi-isomorphism of $\mathcal{B}$ - $\mathcal{B}$ -bimodules

$$(2.6) \quad \mathcal{B}(-_0, -_1) \oplus \mathcal{B}(F_l(-_0), -_1) \rightarrow \mathrm{Tw}_{\mathrm{perf}}(\mathcal{U}^Y)(\pi_0^*(-_0), \pi_1^*(-_1)).$$

For $i \in I$ define $U_{i,\mathcal{O}}^Y = \pi^{-1}(U_i) \cap \{x_{\mathcal{L}} \neq 0\}$ and $U_{i,\mathcal{L}}^Y = \pi^{-1}(U_i) \cap \{x_{\mathcal{O}} \neq 0\}$ and set $\mathcal{U}^Y = \{U_{i,\mathcal{O}}^Y, U_{i,\mathcal{L}}^Y\}_{i \in I}$. Let $s \in \check{C}^0(\mathcal{U}, \mathcal{L})$ be a Čech cochain of non-vanishing sections. Then $s^Y \in \check{C}^0(\mathcal{U}^Y, \mathcal{O}_Y(1))$ given by $s_{i,\mathcal{O}}^Y = x_{\mathcal{L}} \cdot \pi^* s_i^{-1}$ and $s_{i,\mathcal{L}}^Y = x_{\mathcal{O}}$ also consists of non-vanishing sections. Let $l \in \check{Z}^1(\mathcal{U}, \mathcal{O}_X^\times)$ and $l^Y \in \check{Z}^1(\mathcal{U}^Y, \mathcal{O}_Y^\times)$ be the corresponding Čech cocycles, and F_l, F_{l^Y} the dg-functors from (2.5).

Define π_0^* by pulling back the twisted perfect complexes along $U_{i,\mathcal{O}}^Y, U_{i,\mathcal{L}}^Y \rightarrow U_i$. Set $\pi_1^* = F_{l^Y} \circ \pi_0^*$. Lemma 2.2 shows that π_0^* and π_1^* enhance π^* and $\pi^*(-) \otimes \mathcal{O}_Y(1)$ respectively.

The morphism of $\mathcal{B}$ - $\mathcal{B}$ -bimodules in (2.6) is given by

$$(2.7) \quad (u, v) \mapsto \pi_1^*(u) \cdot \frac{x_{\mathcal{O}}}{s^Y} \pi_0^* + \pi_1^*(v) \cdot \frac{x_{\mathcal{L}}}{s^Y \cdot \pi^* s} \pi_0^*$$

where

$$\frac{x_{\mathcal{O}}}{s^Y} \pi_0^*: \pi_0^* \rightarrow F_{l^Y} \circ \pi_0^* = \pi_1^* \text{ and } \frac{x_{\mathcal{L}}}{s^Y \cdot \pi^* s} \pi_0^*: \pi_0^* \rightarrow F_{l^Y} \circ F_{\pi^* l} \circ \pi_0^* = \pi_1^* \circ F_l$$

are morphisms of dg-functors given by Lemma 2.2(3). (2.7) defines a quasi-isomorphism since on cohomology it is the adjunction isomorphism composed with the isomorphism $\mathcal{O}_X \oplus \mathcal{L}^{-1} \rightarrow \pi_* \mathcal{O}_Y(1)$ given by $x_{\mathcal{O}}$ and $x_{\mathcal{L}}$. $\square$

2.1.3. *Proof of Theorem 2(1).* Before proving Theorem 2(1) we give an abstract dg-category construction realising the gluing as in Lemma 2.3 as a topological Fukaya category. Let $\mathcal{B}$ be a small dg-category over k and $F: \mathcal{B} \rightarrow \mathcal{B}$ a strict dg-autoequivalence. Define the $\mathcal{B}$ - $\mathcal{B}$ -bimodule $\mathcal{T} = \mathcal{T}(\text{id}_{\mathcal{B}} \oplus F)$ by $\mathcal{T}(M, N) = \mathcal{B}(M, N) \oplus \mathcal{B}(F(M), N)$ and the corresponding upper-triangular dg-category $\mathcal{D} = \mathcal{B} \underset{\mathcal{T}}{\sqcup} \mathcal{B}$. We describe the weighted system $\mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgc}at_k$ on

$$\mathcal{S} = (S = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}, Q = \{1, -2\}, \eta \text{ radial})$$

such that $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$ is homotopy equivalent to $\text{Tw } \mathcal{D}$.

The fibres of $\mathcal{C}$ are equal to $\mathcal{B}$. For a lazy 1-path $\gamma: [0, 1] \rightarrow S \setminus Q$ lift the polar coordinate on $S \subset \mathbb{C}^\times$ along γ to $\theta_\gamma: [0, 1] \rightarrow \mathbb{R}$ and set

$$n(\gamma) = \left\lfloor \frac{\theta_\gamma(1)}{2\pi} \right\rfloor - \left\lfloor \frac{\theta_\gamma(0)}{2\pi} \right\rfloor \text{ and } \mathcal{C}(\gamma) = F^{n(\gamma)}.$$

Note that $n(\gamma)$ is independent of the lift, invariant under homotopy-rel-endpoints, and additive with respect to concatenation of paths. Hence, by weighting all 2-morphisms in $\mathcal{P}_2^+(\mathcal{S})$ by the identity, we obtain a weighted system of dg-categories $\mathcal{C}$. Note that $\mathcal{C}$ factors through the directed relative fundamental 2-category $\Pi_2^+(\mathcal{S})$ from Definition 1.7.

Lemma 2.4. *In the above notation $\text{Tw } \mathcal{D}$ is homotopy equivalent to $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$.*

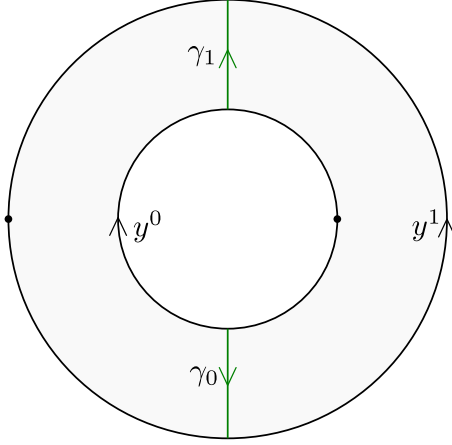


FIGURE 2. A full arc system $\{\gamma_0, \gamma_1\}$ in $\mathcal{S}$.

Proof. Let $\Gamma = \{\gamma_0, \gamma_1\}$ be the full arc system in $\mathcal{S}$ from Fig. 2 and consider the full arc brane system

$$\mathbf{\Gamma} = \{\mathbb{L}_i(M) = (\gamma_i, \text{const}, M) \mid i \in \{0, 1\}, M \in \text{Ob } \mathcal{B}\}$$

(*const* above means the constant path as $\gamma_i^* \eta = \dot{\gamma}$). A straightforward computation shows that for any $M, N \in \text{Ob } \mathcal{B}$ we have

$$\begin{aligned} \mathcal{V}_{\mathbf{\Gamma}}(\mathbb{L}_i(M), \mathbb{L}_i(N)) &= \mathcal{B}(M, N) \text{ for } i = 0, 1; \\ \mathcal{V}_{\mathbf{\Gamma}}(\mathbb{L}_0(M), \mathbb{L}_1(N)) &= \mathcal{V}_{\mathbf{\Gamma}}(y^0) \oplus \mathcal{V}_{\mathbf{\Gamma}}(y^1) = \mathcal{B}(M, N) \oplus \mathcal{B}(F(M), N); \\ \mathcal{V}_{\mathbf{\Gamma}}(\mathbb{L}_1(M), \mathbb{L}_0(N)) &= 0 \end{aligned}$$

The A_∞ -operations in $\mathcal{V}_{\mathbf{\Gamma}}$ only come from (1.4) and (1.5) and, using dg-category notation for $\mathcal{V}_{\mathbf{\Gamma}}$, i.e. $da = (-1)^{|a|} \mu^1(a)$ and $a \cdot b = (-1)^{|b|} \mu^2(a, b)$, we identify $\mathcal{V}_{\mathbf{\Gamma}}$ with $\mathcal{D}$. Thus $\text{Tw } \mathcal{D} = \text{Tw } \mathcal{V}_{\mathbf{\Gamma}}$ is homotopy equivalent to $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$. $\square$

Proof of Theorem 2(1). In the notation of Lemma 2.3 $D_{\text{perf}}^{\text{dg}} \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ is quasi-equivalent to $\mathcal{B} \underset{\mathcal{T}}{\oplus} \mathcal{B}$ and in the notation of Lemma 2.4 $\mathcal{B} \underset{\mathcal{T}}{\oplus} \mathcal{B} = \text{Tw } \mathcal{D}$ is quasi-equivalent to $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$. $\square$

2.2. Bamboo forest blow-ups of affine plane. Suppose we are in the setup of Theorem 2 (2), i.e. X is a bamboo forest blow-up of $\mathbb{A}_k^2$ for an infinite field k .

First, we introduce some notation for X . Let $\pi: X \rightarrow \mathbb{A}^2$ be the projection, $\{p_1, \dots, p_n\} \subset \mathbb{A}^2(k)$ the image of the exceptional locus, and $\pi^{-1}(p_i) = C_i^1 \cup \dots \cup C_i^{k_i}$ with

$$C_i^{j_0} \cdot C_i^{j_1} = \begin{cases} -2 & \text{if } j_0 = j_1 < k_i; \\ -1 & \text{if } j_0 = j_1 = k_i; \\ 1 & \text{if } |j_0 - j_1| = 1; \\ 0 & \text{otherwise} \end{cases}$$

and $\iota_i^j: \mathbb{P}^1 \hookrightarrow X$ paramtrising C_i^j in such a way that $\iota_i^1(\infty) = C_i^1 \cap C_i^2 = \iota_i^2(0)$, $\dots$, $\iota_i^{k_i-1}(\infty) = C_i^{k_i-1} \cap C_i^{k_i} = \iota_i^{k_i}(0)$.

Since k is infinite we can find linear coordinates (t, x) on $\mathbb{A}^2$ such that the t -coordinate λ_i of p_i is different for different i and $\iota_i^1(\infty)$ does not lie on the strict transform $\tilde{X}_{\lambda_i}$ of $\{t = \lambda_i\}$. Composing with an automorphism $(t, x) \mapsto (t, x - \varphi(t))$, where $\varphi \in k[t]$ has a prescribed $(k_i - 1)$ -jet at each λ_i , we can assume that $p_i = (\lambda_i, 0)$, $\tilde{X}_{\lambda_i}$ meets $\pi^{-1}(p_i)$ only at $\iota_i^1(0)$, the strict transform $\tilde{T}$ of $\{x = 0\}$ meets $\pi^{-1}(p_i)$ only at $\iota_i^{k_i}(\infty)$, and all ι_i^j are the natural paramtrisations coming from the successive blow-up structure. See Fig. 3(a) for the picture of the projection $\pi_t = \text{pr}_1 \circ \pi: X \rightarrow \mathbb{A}_t^1$ that makes X into a relative curve (i.e. a flat morphism with 1-dimensional fibres) over $\mathbb{A}^1$.

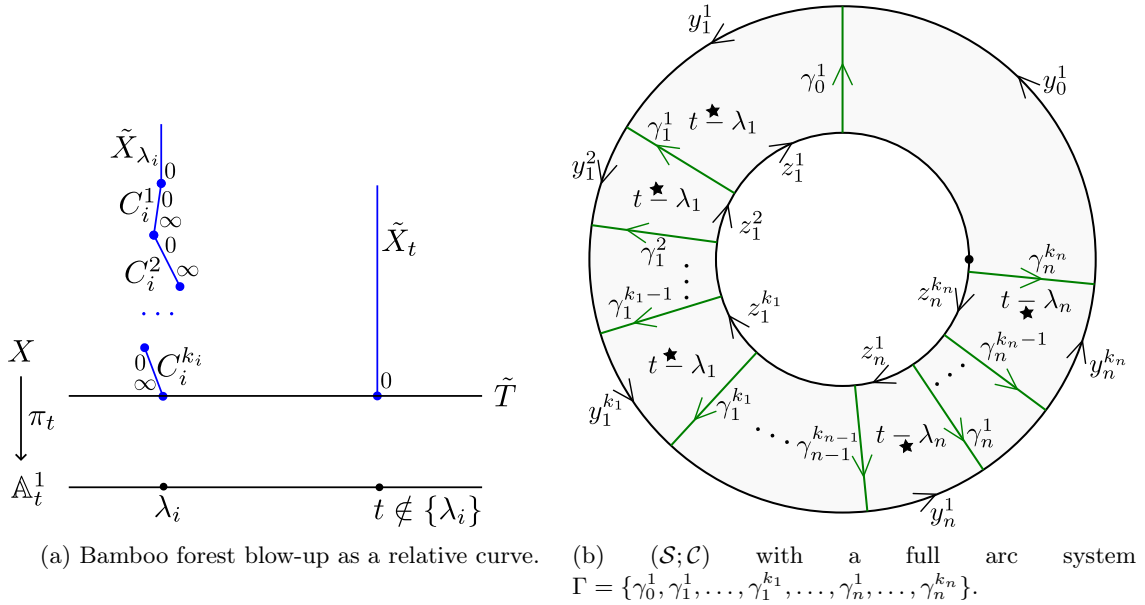


FIGURE 3

Secondly, we specify the graded marked surface $\mathcal{S}$ and the weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(S, Q) \rightarrow \text{dgc}at_k$ from Theorem 2 (2) corresponding to X . Namely,

$$\mathcal{S} = (S = \{z \in \mathbb{C}: 1 \leq |z| \leq 2\}, Q, \eta \text{ radial})$$

where $\Lambda = Q \cap \partial S = \{1\}$ and $D = Q \cap \text{Int } S = \{q_1^1, \dots, q_1^{k_1}, \dots, q_n^1, \dots, q_n^{k_n}\}$. The weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(S, Q) \rightarrow \text{dgc}at_k$ factors through $\mathcal{H}_2^+(S, Q)$ from Definition 1.8, has fibres $k[t]$ viewed as a dg-category with one object, all transition functors are the identity, and the weight of $v: D \rightarrow \mathbb{Z}_{\geq 0}$ is

$$\prod_{i=1}^n \prod_{j=1}^{k_i} (t - \lambda_i)^{v(q_i^j)} \in k[t] = Z^0(\text{Fun}_{dg}(k[t], k[t]))(\text{id}, \text{id}).$$

See Fig. 3(b) for the picture of $(S; \mathcal{C})$.

Thirdly, we describe a $k[t]$ -linear category $\mathcal{B}$ such that $\mathrm{Tw} \mathcal{B}$ is quasi-equivalent to $D_{\mathrm{perf}} X$ and $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C})$. $\mathcal{B}$ is the quotient of the $k[t]$ -linear path category of the quiver

(2.8)

by the ideal generated by

(2.9) $y_i^j \cdot z_i^j = (t - \lambda_i) \mathrm{id}$ and $z_i^j \cdot y_i^j = (t - \lambda_i) \mathrm{id}$ for $i = 1, \dots, n, j = 1, \dots, k_i$.

Note that the paths that contain only y -arrows or only z -arrows give a $k[t]$ -basis of morphisms in $\mathcal{B}$.

We are now ready to prove the theorem.

Proof of Theorem 2(2). We first show that $\mathrm{Tw} \mathcal{B}$ is homotopy equivalent to $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C})$. Consider the full arc system $\Gamma = \{\gamma_0^1, \gamma_1^1, \dots, \gamma_1^{k_1}, \dots, \gamma_n^1, \dots, \gamma_n^{k_n}\}$ in $\mathcal{S}$ from Fig. 3(b). Adding the constant grading and $k[t]$ to each $\gamma \in \Gamma$ we obtain a full arc brane system $\mathbf{\Gamma} = \{\mathbb{L}_0^1, \mathbb{L}_1^1, \dots, \mathbb{L}_1^{k_1}, \dots, \mathbb{L}_n^1, \dots, \mathbb{L}_n^{k_n}\}$ in $(\mathcal{S}; \mathcal{C})$. Note that all boundary paths have degree 0 and the only non-zero μ in $\mathcal{V}_{\mathbf{\Gamma}}$ is μ^2 . We therefore view $\mathcal{V}_{\mathbf{\Gamma}}$ as the ordinary $k[t]$ -linear category with $a \cdot b = \mu^2(a, b)$. The evident bijections between $\mathrm{Ob} \mathcal{B}$ and $\mathbf{\Gamma}$, and between the basis of paths in the quiver (2.8) and boundary paths between arcs from $\mathbf{\Gamma}$ then establish an isomorphism $\mathcal{B} \cong \mathcal{V}_{\mathbf{\Gamma}}$ as path concatenation in the quiver corresponds to (1.5) and (2.9) corresponds to (1.6) and (1.7).

We now prove that $\mathrm{Tw} \mathcal{B}$ is quasi-equivalent to $D_{\mathrm{perf}}^{\mathrm{dg}} X$. Construct X from $X_0^1 = \mathbb{A}^2$ by successive blow-ups at one point that create $C_1^1, \dots, C_1^{k_1}, \dots, C_n^1, \dots, C_n^{k_n}$ in this order. Denote the varieties we get in the process X_i^j , note that they are all bamboo forest blow-ups of $\mathbb{A}^2$. Denote by $\tilde{C}_i^j = C_i^j + \dots + C_i^{k_i}$ the total transform of $C_i^j \subset X_i^j$ in X . Define line bundles $\mathcal{L}_0^1 = \mathcal{O}_X$ and $\mathcal{L}_i^j = \mathcal{O}_X(\tilde{C}_1^1 + \dots + \tilde{C}_i^j)$ for $i = 1, \dots, n, j = 1, \dots, k_i$.

We have the bijection

$$F: \mathrm{Ob} \mathcal{B} \rightarrow \{\mathcal{L}_0^1, \dots, \mathcal{L}_n^{k_n}\}, \binom{j}{i} \mapsto \mathcal{L}_i^j.$$

Observe that the principal divisor of $(t - \lambda_i) \in \Gamma(X, \mathcal{O}_X)$ is $\tilde{X}_{\lambda_i} + \tilde{C}_i^1$ and of $x \in \Gamma(X, \mathcal{O}_X)$ is $\tilde{T} + \tilde{C}_1^1 + \dots + \tilde{C}_n^{k_n}$. We therefore can define F on the arrows of the quiver (2.8) by

$$F(y_0^1) = x \in \Gamma(X, \mathcal{O}_X(-\tilde{C}_1^1 - \dots - \tilde{C}_n^{k_n})) = \mathrm{Hom}_X(\mathcal{L}_n^{k_n}, \mathcal{L}_0^1)$$

and for $i = 1, \dots, n, j = 1, \dots, k_i$ by

$$\begin{aligned} F(y_i^j) &= s_{\tilde{C}_i^j} \in \Gamma(X, \mathcal{O}_X(\tilde{C}_i^j)) = \mathrm{Hom}_X(\mathcal{L}_i^{j-1}, \mathcal{L}_i^j) \text{ where } s_{\tilde{C}_i^j} \text{ is the canonical section;} \\ F(z_i^j) &= (t - \lambda_i) \in \Gamma(X, \mathcal{O}_X(-\tilde{C}_i^j)) = \mathrm{Hom}_X(\mathcal{L}_i^j, \mathcal{L}_i^{j-1}). \end{aligned}$$

where we use the notation $\mathcal{L}_i^0 = \mathcal{L}_{i-1}^{k_{i-1}}$ and $k_0 = 1$. Note that

$$s_{\tilde{C}_i^j} \cdot (t - \lambda_i) = (t - \lambda_i) \cdot s_{\tilde{C}_i^j} = (t - \lambda_i) \in \Gamma(X, \mathcal{O}_X),$$

so F respects the relations (2.9) hence defines a functor from $\mathcal{B}$ to the Hom_X -category on objects $\mathcal{L}_0^1, \dots, \mathcal{L}_n^{k_n}$. The following claim finishes the proof since $\mathcal{B}$ is intrinsically formal.

Claim.

- (1) F is an isomorphism from $\mathcal{B}$ to the Ext^* -category on objects $\mathcal{L}_0^1, \dots, \mathcal{L}_n^{k_n}$.
- (2) The smallest full subcategory of $\text{D}_{\text{perf}} X$ containing $\mathcal{L}_0^1, \dots, \mathcal{L}_n^{k_n}$ and closed under shifts, cones, and isomorphisms is $\text{D}_{\text{perf}} X$.

□

Proof of Claim. We argue by induction on the number of blow-ups. All X_i^j are bamboo forest blow-ups in normal form with respect to the same coordinates (t, x) , so the construction above gives $k[t]$ -linear categories $\mathcal{B}_i^j$, line bundles on X_i^j , and functors F_i^j . The pullback along $X \rightarrow X_i^j$ is fully faithful on D_{perf} and sends these line bundles to the corresponding $\mathcal{L}$'s, so it is enough to prove the claim for every X_i^j . For $X_0^1 = \mathbb{A}^2$ we have $\mathcal{B}_0^1 = k[t][y_0^1] \cong k[t, x] = \text{Ext}_{\mathbb{A}^2}^*(\mathcal{O}, \mathcal{O})$, and every perfect complex on $\mathbb{A}^2$ is quasi-isomorphic to a bounded complex of free modules by the Quillen–Suslin theorem.

Let $\rho: Y' = X_i^j \rightarrow Y = X_i^{j-1}$ (where $X_i^0 = X_{i-1}^{k_{i-1}}$) be the blow-up at p with exceptional curve $E = C_i^j$, let $\mathcal{L}$ be the last line bundle on Y , so that $\mathcal{L}' = \rho^* \mathcal{L}(E)$ is the last line bundle on Y' , and assume the claim for Y .

We first prove (2). By Orlov's blow-up formula [44, Theorem 4.3], which holds in our setting by [5, Corollary 6.10], we have $\text{D}_{\text{perf}} Y' = \langle \mathcal{O}_E(-1), \rho^* \text{D}_{\text{perf}} Y \rangle$, so every object of $\text{D}_{\text{perf}} Y'$ is a cone of a morphism between an object of $\rho^* \text{D}_{\text{perf}} Y$ and a finite direct sum of shifts of $\mathcal{O}_E(-1)$. As $(\rho^* \mathcal{L})|_E$ is trivial, $\mathcal{O}_E(-1) = \mathcal{L}'|_E$ is the cone of $s_E: \rho^* \mathcal{L} \rightarrow \mathcal{L}'$, and we are done by the induction hypothesis.

Now prove (1). Since $\mathcal{B}_i^{j-1}$ and $\mathcal{B}_i^j$ have bases of y - and z -paths, sending y_0^1 to $y_0^1 \cdot y_i^j$ identifies $\mathcal{B}_i^{j-1}$ with the full subcategory of $\mathcal{B}_i^j$ on all objects except $\binom{j}{i}$, compatibly with F_i^{j-1} , F_i^j , and ρ^* . It remains to compare morphisms to and from $\mathcal{L}'$. The short exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(E) \rightarrow \mathcal{O}_E(-1) \rightarrow 0$ gives $\rho_* \mathcal{O}(E) = \mathcal{O}$ and $\rho_* \mathcal{O}(-E) = \mathcal{I}_p$, the ideal sheaf of p (all functors are derived). Hence, by the projection formula, twisting by $\mathcal{O}(E)$ induces an isomorphism

$$(2.10) \quad \text{Ext}_{Y'}^*(\mathcal{O}_{Y'}, \mathcal{O}_{Y'}) = \text{Ext}_{Y'}^*(\mathcal{L}', \mathcal{L}') \cong \text{Ext}_Y^*(\mathcal{L}, \mathcal{L}) = \text{Ext}_Y^*(\mathcal{O}_Y, \mathcal{O}_Y),$$

and for every line bundle $\mathcal{M}$ on Y from our collection the multiplication by $F(y_i^j) = s_E$ induces isomorphisms

$$(2.11) \quad \text{Ext}_{Y'}^*(\rho^* \mathcal{M}, \mathcal{L}') \cong \text{Ext}_Y^*(\mathcal{M}, \mathcal{L})$$

and

$$(2.12) \quad \text{Ext}_{Y'}^*(\mathcal{L}', \rho^* \mathcal{M}) \cong \text{Ext}_Y^*(\mathcal{L}, \mathcal{M} \otimes \mathcal{I}_p).$$

(2.10) matches $\mathcal{B}_i^j$ since both sides are $k[t, x]$ with the y -loop at $\binom{j}{i}$ mapping to x . (2.11) matches $\mathcal{B}_i^j$ since every path to $\binom{j}{i}$ from another vertex ends with y_i^j . For (2.12), every path from $\binom{j}{i}$ starts with y_0^1 or z_i^j , so $- \cdot y_i^j$ identifies $\mathcal{B}_i^j(\binom{j}{i}, -)$ with the $k[t]$ -span of the y -paths and of $(t - \lambda_i)$ times the z -path in $\mathcal{B}_i^{j-1}(\binom{j-1}{i}, -)$. On the other hand, $\text{Ext}_Y^*(\mathcal{L}, \mathcal{M} \otimes \mathcal{I}_p)$ is the fibre of the evaluation map $\text{ev}_p: \text{Hom}(\mathcal{L}, \mathcal{M}) \rightarrow (\mathcal{M} \otimes \mathcal{L}^{-1})|_p \cong k$. It thus suffices to show that ev_p vanishes on the images of the y -paths but not on the image of the z -path. The images of the y -paths are divisible by x and hence vanish on $p \in \tilde{T}$. The image of a z -path is a product of the $(t - \lambda_l)$, so its divisor is supported on the $\tilde{X}_{\lambda_l}$ and on $C_i^1, \dots, C_i^{j-2}$, none of which contains p by the choice of coordinates. □

2.3. Cotangent bundles of spheres. Abouzaid [2] proved that the wrapped Fukaya category $\mathcal{W}(T^*S^n)$ is quasi-equivalent to the triangulated envelope $\text{Tw } C_{-*} \Omega S^n$ of the chains on the loop space of S^n . The Adams–Hilton model for $C_{-*} \Omega S^n$ for $n \geq 2$ is the dg-algebra $\mathcal{B}_m = (k\langle z \rangle, dz = 0, |z| = m)$ with $m = 1 - n$. Therefore, to prove Theorem 2 (3) we need to find a weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgcat}_k$ on

$$\mathcal{S} = \{S = \{z \in \mathbb{C}: |z| \leq 2\}, Q = \{1, -1\}, \eta \text{ vertical}\}$$

such that $\mathrm{Tw} \mathcal{B}_{1-n}$ is quasi-equivalent to $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C})$. We first do it for any $\mathcal{B}_m$ with $m \in \mathbb{Z}$ in §2.3.1 and then give a geometric heuristic explaining why the quasi-equivalence is not a coincidence in §2.3.2.

2.3.1. *From $\mathcal{B}_m$ to $\mathcal{B}_{m-1}$.* We describe $\mathcal{C}_m: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \mathrm{dgc}at_k$ that gives $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C}_m)$ quasi-equivalent to $\mathrm{Tw} \mathcal{B}_{m-1}$. All fibres of $\mathcal{C}_m$ are $\mathrm{Tw} \mathcal{B}_m$. Define a strict dg-autoequivalence $T = [m] \circ \mathrm{Tw}(z \mapsto (-1)^m z)$ of $\mathrm{Tw} \mathcal{B}_m$ and a morphism of dg-functors $\eta: \mathrm{id} \Rightarrow T$ whose component from $\mathcal{B}_m[n]$ to $\mathcal{B}_m[n+m]$ is $(-1)^{mn} z[m]$. A straightforward computation shows that $\eta T = T\eta$. $\mathcal{C}_m$ is obtained from the trivial system of dg-categories on $S \setminus ([-2, -1] \cup [1, 2])$ by gluing with T along $[-2, -1] \cup (1, 2]$ and weighting ± 1 by η , see Fig. 4.

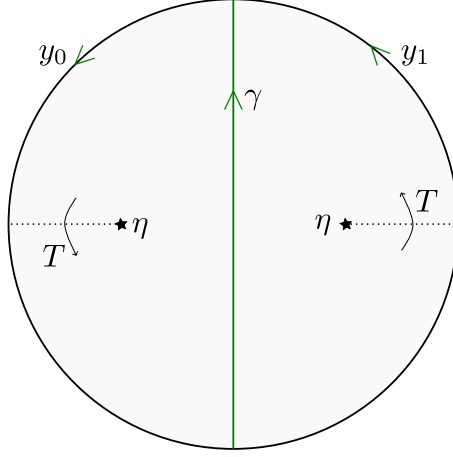


FIGURE 4. $(\mathcal{S}; \mathcal{C}_m)$ and the full arc system $\{\gamma\}$.

Crossing a dotted cut in the direction of the arrow gives the transition functor T , and the weights at ± 1 are η .

Namely, for a lazy 1-path $\gamma: [0, 1] \rightarrow S \setminus Q$ let $\theta_\gamma^{\pm 1}: [0, 1] \rightarrow \mathbb{R}$ be the lift of the polar coordinate of $\gamma \mp 1$ along γ . Set

$$n(\gamma) = \left\lfloor \frac{\theta_\gamma^{+1}(1)}{2\pi} \right\rfloor - \left\lfloor \frac{\theta_\gamma^{+1}(0)}{2\pi} \right\rfloor + \left\lfloor \frac{\theta_\gamma^{-1}(1) - \pi}{2\pi} \right\rfloor - \left\lfloor \frac{\theta_\gamma^{-1}(0) - \pi}{2\pi} \right\rfloor$$

and note that $n(\gamma)$ is independent of the lift, invariant under homotopy-rel-endpoints, and additive with respect to concatenation of paths. The transition functor is $\mathcal{C}_m(\gamma) = T^{n(\gamma)}$.

For a positive 2-path σ from γ_0 to γ_1 we have $n(\gamma_1) - n(\gamma_0) = \deg_\sigma(-1) + \deg_\sigma(1) =: \deg(\sigma)$. The weight $\mathcal{C}_m(\sigma)$ is

$$\eta^{\deg(\sigma)} T^{n(\gamma_0)}: T^{n(\gamma_0)} \Rightarrow T^{n(\gamma_1)}$$

where the power of η is with respect to the horizontal composition.

Lemma 2.5. $\mathcal{W}^{\mathrm{top}}(\mathcal{S}; \mathcal{C}_m)$ is quasi-equivalent to $\mathrm{Tw} \mathcal{B}_{m-1}$.

Proof. $\{\gamma\}$ from Fig. 4 is a full arc system in $\mathcal{S}$ and hence $\Gamma = \{\mathbb{L} = (\gamma, \mathrm{const}, \mathcal{B}_m)\}$ is a full arc brane system in $(\mathcal{S}; \mathcal{C}_m)$. The only non-zero μ in $\mathcal{V}_\Gamma$ is μ^1 and μ^2 and $\mathcal{V}_\Gamma$ has one object, so we describe $\mathcal{V}_\Gamma$ as a dg-algebra with $da = (-1)^{|a|} \mu^1(a)$ and $ab = (-1)^{|b|} \mu^2(a, b)$. Denote by $b \in \mathcal{B}_m = \mathcal{V}_\Gamma(c)$ the generator of $\mathcal{B}_m$, where c is a constant boundary path, and by a_i the element $1[-|y_i|] \in \mathrm{Tw} \mathcal{B}_m(T\mathcal{B}_m, \mathcal{B}_m)[-|y_i|] = \mathcal{V}_\Gamma(y_i)[-|y_i|]$ for $i = 0, 1$. Then $|b| = m$, $|a_0| = |a_1| = m - 1$ and a straightforward computation using (1.5) and (1.7) shows

$$\mathcal{V}_\Gamma = \frac{k\langle b, a_0, a_1 \rangle}{(a_0^2, a_1^2, ba_0 - (-1)^m a_0 b, ba_1 - (-1)^m a_1 b)}, \quad db = 0, \quad da_0 = -da_1 = (-1)^{m-1} b,$$

where the sign of da_1 comes from γ not being oriented as the boundary of the region to its right, and the sign in $ba_i = (-1)^m a_i b$ from the sign in T .

Observe that $\mathcal{V}_{\mathbf{r}}$ has a basis given by the prefixes of the words $b^n a_i a_{1-i} a_i \dots$ for $n \geq 0$ and $i = 0, 1$. We have $db^n = 0$, $d(b^n a_i) = \pm b^{n+1}$, and on a basis word with $k \geq 2$ letters a we have $d(b^n a_i \dots a_j) = \pm b^{n+1}(a_i \dots + \dots a_j)$. Therefore,

$$\mathcal{B}_{m-1} = (\mathbf{k}\langle z \rangle, dz = 0, |z| = m-1) \rightarrow \mathcal{V}_{\mathbf{r}}, z \mapsto a_0 + a_1$$

is a quasi-isomorphism and we are done. $\square$

Proof of Theorem 2 (3). By Lemma 2.5 with $m = 2 - n$ we have

$$\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C}_{2-n}) \simeq \text{Tw } \mathcal{B}_{1-n} \simeq \text{Tw } C_{-*} \Omega S^n,$$

which is quasi-equivalent to $\mathcal{W}(T^*S^n; \mathbf{k})$ by [2]. The fibres of $\mathcal{C}_{2-n}$ are $\text{Tw } \mathcal{B}_{2-n} \simeq \mathcal{W}(T^*S^{n-1}; \mathbf{k})$ for $n \geq 3$ by the same argument. $\square$

2.3.2. Lefschetz fibration and symplectic reduction. We now give the geometric heuristic. Let $n \geq 3$ (the case $n = 2$ is covered in [39] and does not quite fit in the discussion below since $\mathcal{W}(T^*S^1) \cong \text{Tw } \mathbf{k}[z, z^{-1}] \not\cong \text{Tw } \mathcal{B}_0$). Identify T^*S^n with the quadric $V = \{x_0^2 + \dots + x_n^2 = 1\} \subset \mathbb{C}^{n+1}$. The projection onto the first coordinate $\pi: V \rightarrow \mathbb{C}_t$ is a Lefschetz fibration with regular fibres $V_t \cong T^*S^{n-1}$ for $t \neq \pm 1$ and singular fibres $V_{\pm 1}$ which are T^*S^{n-1} with the zero section collapsed to a point. It is not hard to see that

- (1) the zero section S_0^n of T^*S^n is the matching sphere $V \cap \mathbb{R}^{n+1}$ above $[-1, 1]$;
- (2) the set

$$S_t^{n-1} = \{t\} \times \sqrt{1 - t^2} S^{n-1}$$

is the zero section of T^*S^{n-1} under the identification $T^*S^{n-1} \cong V_t$ for $t \neq \pm 1$ and for $t = \pm 1$ it is the collapsed zero section;

- (3) for $r \in \mathbb{R}$ and $y \in S^{n-1} \subset \mathbb{R}^n$ the cotangent fibre $T_y^*S^{n-1}$ under the identification $T^*S^{n-1} \cong V_{ir}$ is

$$V_{ir,y} = \{(ir, x) \in V : \text{Re } x \in \mathbb{R}_{>0} \cdot y\};$$

- (4) for $y \in S^{n-1}$ the cotangent fibre $T_{(0,y)}^*S^n$ under the identification $T^*S^n \cong V$ is $\bigcup_{r \in \mathbb{R}} V_{ir,y}$.

Fix the regular fibre V_0 and vanishing paths $[0, \pm 1]$. The monodromy around $+1$ and -1 is the Dehn twist along the zero section S_0^{n-1} . On $\mathcal{W}(T^*S^{n-1})$ the Dehn twist gives the spherical twist along the zero section. Under $\mathcal{W}(T^*S^{n-1}) \cong \text{Tw } \mathcal{B}_{2-n}$ the spherical twist becomes T , which is the spherical twist along the twisted complex $\mathbf{S} = (\mathcal{B}_{2-n}[n-1] \oplus \mathcal{B}_{2-n}, z[1-n])$ and η fits into the triangle

$$\text{Tw } \mathcal{B}_{2-n}(\mathbf{S}, X) \otimes \mathbf{S} \rightarrow X \xrightarrow{\eta_X} T(X).$$

Also observe that the standard action of $\text{GL}(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1}$ restricts to a Hamiltonian action of $\text{SO}(n)$ on V (the action is on the last n coordinates) with moment map

$$\mu: V \rightarrow \mathfrak{so}(n)^* \cong \Lambda^2 \mathbb{R}^n, \mu(x_0, \dots, x_n) = (\text{Im } \bar{x}_j x_k)_{1 \leq j < k \leq n}.$$

This action preserves the fibres V_t and

$$\mu^{-1}(0) = V \cap (\mathbb{C} \times (\mathbb{C} \cdot S^{n-1})) = \bigcup_{t \in \mathbb{C}} S_t^{n-1}.$$

The action is transitive on $\mu^{-1}(0) \cap V_t$ with stabilisers conjugate to $\text{SO}(n-1)$ and fixes the node for $t = \pm 1$. Hence set-theoretically we have $(\mu^{-1}(0) \cap \pi^{-1}(S))/\text{SO}(n) = S$.

Moreover, a cotangent fibre $T_{(0,y)}^*S^n$ from (4) above intersects $\mu^{-1}(0)$ transversally and we get a bijection

$$\pi: T_{(0,y)}^*S^n \cap \mu^{-1}(0) \cap \pi^{-1}(S) \rightarrow \text{Im } \gamma$$

where γ is as in Fig. 4. That is, $(\mathcal{S}; \mathcal{C})$ can be thought of as a ‘singular symplectic reduction’ of $(T^*S^n, \mathbf{k})$ with respect to the action of $\text{SO}(n)$ under which the cotangent fibre generating $\mathcal{W}(T^*S^n; \mathbf{k})$ reduces to the arc brane $\mathbb{L}$ generating $\mathcal{W}^{\text{top}}(\mathcal{S}; \mathcal{C})$.

3. GEOMETRIC PREPARATIONS

In this section we set up the geometric underpinning for Fukaya categories of surfaces with coefficients.

Let $S, Q = \Lambda \sqcup D$ be as in Definition 1.3, i.e. S is a compact oriented surface with $\partial S' \neq \emptyset$ for each $S' \in \pi_0 S$, and $\Lambda \subset \partial S$ and $D \subset \text{Int } S$ are finite sets. We call such data (S, Q) a *marked surface*.

In §3.1 we define a compatible Liouville form on a marked surface; the choice of such form is contractible (see Proposition 3.2) and the diffeomorphism group of the marked surfaces retracts onto the exact symplectomorphism group (see Proposition 3.4). In §3.2 we first define Lagrangians in a Liouville marked surface, which are certain immersed curves with additional data, these will give rise to objects in the Fukaya category; then we describe the self-intersection information of a Lagrangian, consisting of its double points and Reeb chords between its ends, these will give morphisms in the Fukaya category. We characterise Lagrangian structures on a given curve in terms of linear inequalities on actions and areas in Propositions 3.14 and 3.15. In Proposition 3.17 we characterise isotopies of Lagrangians topologically. In §3.3 we describe the polygons bounded by a Lagrangian, their counts will give rise to A_∞ -operations. In §3.4 we add the grading information to surfaces and curves to make the Fukaya category $\mathbb{Z}$ -graded.

3.1. Surface. In §3.1.1 we define Liouville structures on a marked surface (S, Q) , i.e. Liouville forms on S compatible with Q . In §3.1.2 we prove that the space of markings of a given surface with prescribed numbers of points on each boundary component and in the interior is aspherical, and that the space of Liouville structures on a given marked surface is non-empty and convex, see Proposition 3.2. In §3.1.3 we define Hamiltonian diffeomorphisms of a Liouville marked surface and prove that they form the identity component of the group of exact symplectomorphisms in Lemma 3.3. In §3.1.4 we prove that exact symplectomorphisms of Liouville marked surfaces form a strong deformation retract of diffeomorphisms of the underlying marked surfaces in Proposition 3.4; in particular, any two Liouville structures on the same marked surface are related by an exact symplectomorphism isotopic to the identity. In §3.1.5 we describe the two contact manifolds associated to a Liouville marked surface (S, Q, λ) : the boundary $\partial S \setminus \Lambda$ and the contactisation $(S \setminus Q) \times \mathbb{R}$.

3.1.1. Liouville structure on a marked surface. We describe a Liouville structure on S compatible with the marking Q .

Recall that a 1-form λ on a $2n$ -dimensional manifold M is called a *Liouville form* if $(d\lambda)^n$ is a volume (i.e. nowhere vanishing) form on M . The *Liouville vector field* $Z = Z_\lambda$ is defined by $d\lambda(Z, -) = \lambda$, and its flow is called the *Liouville flow*. Note that if M admits a Liouville form, then it is orientable (by the volume form) and M cannot be closed (by the Stokes theorem).

Let λ be a Liouville form on the surface S . We impose the following conditions on λ along ∂S , Λ , and D .

The first set of conditions ensures that the Liouville flow Φ_Z^t is defined for all $t \leq 0$ and that Q does not move under the flow.

- (1) λ (equivalently, Z) vanishes on D .
- (2) $\Lambda = \{p \in \partial S: \lambda(p) = 0\}$.
- (3) Z points outwards along $\partial S \setminus \Lambda$.

Note that there can be zeroes of λ in $\text{Int } S$ not lying in D , but all zeroes of λ on ∂S lie in Λ . Observe that condition (3) is equivalent to $\lambda|_{\partial S \setminus \Lambda}$ being a volume form that induces the orientation of $\partial S \setminus \Lambda$ agreeing with the orientation of ∂S as the boundary of S .

We impose one more condition to ensure that introducing λ does not affect the mapping class group. On one hand, the natural notion of automorphism of the marked surface (S, Q) is an orientation-preserving diffeomorphism Φ of S such that $\Phi(Q) = Q$. Let $\text{Diff}(S, Q)$ be the group of such maps Φ . On the other hand, an automorphism of the exact symplectic manifold (S, λ) is a diffeomorphism Φ of S such that $\Phi^*\lambda - \lambda$ is exact. Let $\text{Symp}_{\text{ex}}(S, \lambda)$ be the group of such maps

Φ . It is therefore natural to define

$$(3.1) \quad \text{Symp}_{\text{ex}}(S, Q, \lambda) = \text{Diff}(S, Q) \cap \text{Symp}_{\text{ex}}(S, \lambda).$$

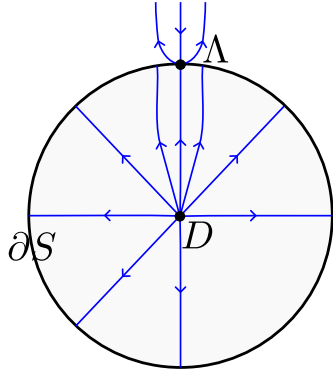
We would like the mapping class group $\pi_0 \text{Diff}(S, Q) = \text{MCG}(S, Q)$ to be equal to the mapping class group $\pi_0 \text{Symp}_{\text{ex}}(S, Q, \lambda) = \text{MCG}(S, Q, \lambda)$, but there is an obstruction to that. Namely, if C_0 and C_1 are two components of ∂S with $|C_0 \cap Q| = |C_1 \cap Q|$, then there exists $\Phi \in \text{Diff}(S, Q)$ mapping C_0 to C_1 , but if $\Phi \in \text{Symp}_{\text{ex}}(S, \lambda)$, then $\int_{C_0} \lambda = \int_{C_1} \lambda$. We require this obstruction to vanish.

(4) $\int_C \lambda = 1$ for each connected component C of ∂S .

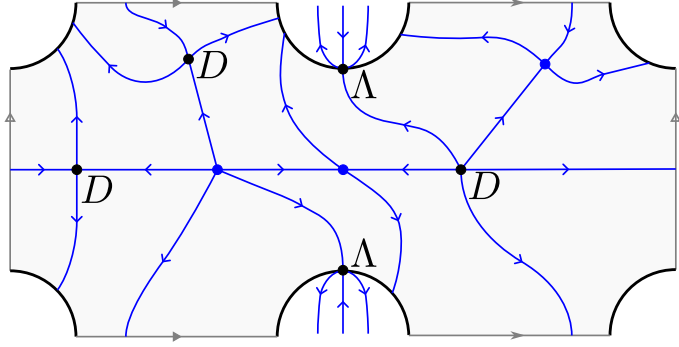
This is enough for $\text{Symp}_{\text{ex}}(S, Q, \lambda)$ to be a strong deformation retract of $\text{Diff}(S, Q)$, see Proposition 3.4.

We summarise the discussion above into the following definition.

Definition 3.1. A *Liouville structure* on the marked surface (S, Q) is a Liouville form λ on S satisfying conditions (1)–(4) above. $S = (S, Q, \lambda)$ is then called a *Liouville marked surface*.



(a) S is a disc and $|\Lambda| = |D| = 1$.



(b) S is a surface of genus 1 glued along the grey lines marked with the same arrow, $|\pi_0 \partial S| = 2$, $|\Lambda| = 2$, and $|D| = 3$.

FIGURE 5. Examples of Liouville marked surfaces.

The dots are the zeroes and the blue lines are the integral lines of the Liouville vector field. The zeroes that belong to Λ or D are marked by the corresponding letter.

In Fig. 5 the zeroes and integral lines of the Liouville vector field are sketched for two Liouville marked surfaces.

If S is orientable and $\partial S \neq \emptyset$ (which is necessary for the existence of a Liouville form on S), then the marked surface (S, Q) admits a Liouville structure. Moreover, the space of Liouville structures on (S, Q) such that $d\lambda$ induces a given orientation of S is convex in $\Omega^1(S)$, see Proposition 3.2.

3.1.2. Liouville marked structures on a given surface. We study the space of Liouville marked surface structures on a connected surface S , the generalisation to disconnected case is straightforward.

For a non-negative integer n and a function σ from the set of connected components of ∂S to $\mathbb{Z}_{\geq 0}$ let

$$(3.2) \quad \mathcal{M}(S; \sigma, n) = \left\{ \begin{array}{l} \Lambda \subset \partial S \quad |\Lambda \cap C| = \sigma(C) \text{ for all } C \in \pi_0 \partial S \\ D \subset \text{Int } S \quad \text{and } |D| = n \end{array} \right\}$$

be the set parametrising the choice of a marked surface structure (S, Q) with $Q = \Lambda \sqcup D$ on S .

For a finite set $Q \subset S$ define

$$(3.3) \quad \mathcal{LS}(S, Q) = \left\{ \lambda \in \Omega^1(S) : \begin{array}{l} \lambda \text{ is a Liouville structure on } (S, Q) \\ \text{and } d\lambda \text{ induces the orientation of } S \end{array} \right\}.$$

Proposition 3.2. *Let S be a compact connected oriented surface with non-empty boundary. Then*

- (1) $\mathcal{M}(S; \sigma, n)$ *is a smooth aspherical manifold of dimension $2n + \sum_{C \in \pi_0 \partial S} \sigma(C)$ for any $\sigma: \pi_0 \partial S \rightarrow \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$.*
- (2) $\mathcal{LS}(S, Q)$ *is a non-empty convex subset of $\Omega^1(S)$ for any finite set $Q \subset S$.*

Proof. We prove (1). $\mathcal{M} = \mathcal{M}(S; \sigma, n)$ is the orbit space of the trivial-stabiliser locus of the action of the product of symmetric groups

$$S_n \times \prod_{C \in \pi_0 \partial S} S_{\sigma(C)} \quad \text{on} \quad (\text{Int } S)^n \times \prod_{C \in \pi_0 \partial S} C^{\sigma(C)}$$

where each symmetric group acts on the corresponding factor by permuting entries. Since the trivial-stabiliser locus (i.e. the complement of the big diagonal) is open, $\mathcal{M}$ is a smooth manifold of dimension $2n + \sum_{C \in \pi_0 \partial S} \sigma(C)$. $\mathcal{M}$ is connected since both the complement of finitely many embedded closed intervals in S and the configuration space of a fixed number of (unordered) points in S^1 are connected. That $\pi_k(\mathcal{M}) = 0$ for $k \geq 2$ follows by induction on $n + \sum_{C \in \pi_0 \partial S} \sigma(C)$ from the homotopy sequence of fibration and the fact that the complement of finitely many points in S (resp. in a component of ∂S) is aspherical (resp. a disjoint union of contractible spaces).

To prove (2) we first show $\mathcal{LS} = \mathcal{LS}(S, Q)$ is non-empty. Recall that S admits a Stein structure, i.e. a complex structure J together with a Morse subharmonic function $\phi: S \rightarrow \mathbb{R}$ such that ∂S is a regular level set of ϕ where ϕ attains its maximum. Moreover, up to reparametrisation of $\mathbb{R}$ we can choose ϕ to be arbitrary as long as it does not have critical points of index 2. In particular, we can ensure that ϕ attains a non-degenerate local minimum at each point of D , see Cieliebak–Eliashberg [15, Chapter 8] for details. The 1-form $\lambda' = -d\phi \circ J$ is almost what we need, except the behaviour of λ' along the boundary. By pushing S inside its Liouville completion and rescaling we can ensure that $\int_C \lambda' = 1$ for all components C of ∂S . Next, for each $C \in \pi_0 \partial S$ fix an orientation-preserving diffeomorphism

$$\tau_C: C \rightarrow \mathbb{R}/\mathbb{Z} \text{ such that } \lambda'|_C = \tau_C^*(dt)$$

and pick a function $g_C: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ such that $g'_C(t) \geq -1$ for all $t \in \mathbb{R}/\mathbb{Z}$ and the solution set of $g'_C(t) = -1$ is $\tau_C(C \cap \Lambda)$ (this can be done since we can ensure $\int_{\mathbb{R}/\mathbb{Z}} g'_C(t) dt = 0$). Extend the collection of functions $\{g_C \circ \tau_C\}_{C \in \pi_0 \partial S}$ to a function $g: S \rightarrow \mathbb{R}$ supported near ∂S such that $(\lambda' + dg)(p) = 0$ for each $p \in \Lambda$ (this can be done since $(\lambda' + dg)|_{\partial S}(\Lambda) = 0$ and we can prescribe the derivative of g along ∂S in the normal direction). We obtain the form $\lambda = \lambda' + dg$ that lies in $\mathcal{LS}$.

Now show that $\mathcal{LS}$ is convex. Observe that $\lambda \in \Omega(S)$ belongs to $\mathcal{LS}$ if and only if

- (1) λ vanishes on Q ;
 - (2) $\int_C \lambda = 1$ for each $C \in \pi_0 \partial S$;
 - (3) $d\lambda$ is a volume form on S inducing the orientation of S ;
 - (4) $\lambda|_{\partial S \setminus \Lambda}$ is a volume form on $\partial S \setminus \Lambda$ inducing the orientation of ∂S .
- (1) is linear in λ , (2) is affine in λ , and (3), (4) are convex in λ , hence the result. $\square$

3.1.3. Hamiltonian diffeomorphisms. Fix a Liouville marked surface $\mathcal{S} = (S, Q, \lambda)$. Recall from (3.1) that $\text{Diff}(S)$ is the group of diffeomorphisms of S that preserve Q setwise and $\text{Symp}_{\text{ex}}(S)$ is its subgroup consisting of exact symplectomorphisms. We describe the identity component $\text{Symp}_{\text{ex}}(S)_0$ of $\text{Symp}_{\text{ex}}(S)$.

Let $H = \{H_s: S \rightarrow \mathbb{R}\}_{s \in [0,1]}$ be a smooth family such that dH_s vanishes on Q and $H_s|_{\partial S}$ is locally constant. Then, since $d\lambda$ is non-degenerate, the equation $d\lambda(X_s, -) = dH_s$ defines a family of vector fields

$$[0, 1] \rightarrow C^\infty(S, TS), s \mapsto X_s^H.$$

As X_s^H vanishes on Q and is tangent to ∂S , it integrates to a family

$$[0, 1] \rightarrow \text{Symp}_{\text{ex}}(S), s \mapsto \Phi_s^H.$$

We therefore obtain a map

$$C_{[0,1]}^\infty(S, \mathbb{R}) \rightarrow \text{Symp}_{\text{ex}}(S), H = \{H_s\}_{s \in [0,1]} \mapsto \Phi_1^H$$

where $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ denotes the space of smooth families of functions $\{H_s\}_{s \in [0,1]}$ described above. The image of this map is a connected subgroup $\text{Ham}(\mathcal{S})$ of $\text{Symp}_{\text{ex}}(\mathcal{S})$ consisting of *Hamiltonian diffeomorphisms* of $\mathcal{S}$.

Lemma 3.3. $\text{Ham}(\mathcal{S}) = \text{Symp}_{\text{ex}}(\mathcal{S})_0$.

Proof. $\text{Ham}(\mathcal{S})$ is connected hence we only need to prove that the right hand side is contained in the left hand side. Let $\Phi_s \in \text{Symp}_{\text{ex}}(\mathcal{S})$, $s \in [0, 1]$ be a family of symplectomorphisms generated by a family of vector fields $X_s \in C^\infty(S, TS)$, $s \in [0, 1]$. Since a function on S is determined up to a constant (on each component of S) by its differential, we get a family $g_s \in C^\infty(S, \mathbb{R})$, $s \in [0, 1]$ such that

$$\Phi_s^* \lambda - \lambda = dg_s \text{ for } s \in [0, 1].$$

Differentiating with respect to s and using Cartan's magic formula we obtain

$$\Phi_s^* (d(\lambda(X_s)) + d\lambda(X_s, -)) = d \left(\frac{dg_s}{ds} \right).$$

Therefore,

$$d\lambda(X_s, -) = dH_s \text{ where } H_s = \frac{dg_s}{ds} \circ \Phi_s^{-1} - \lambda(X_s).$$

Since Φ_s is a diffeomorphism of S fixing Q pointwise, X_s vanishes on Q and is tangent to ∂S . Therefore, dH_s vanishes on Q and H_s is locally constant on ∂S . That is, $H = \{H_s\}_{s \in [0,1]} \in C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ and hence Φ_s is a path in $\text{Ham}(\mathcal{S})$. $\square$

3.1.4. *Deforming diffeomorphisms to exact symplectomorphisms.* The purpose of proving Lemma 3.3 was mostly to warm up before the following statement.

Proposition 3.4. *Let $\mathcal{S}_i = (S_i, Q_i, \lambda_i)$, $i = 0, 1$ be two Liouville marked surfaces. Then $\text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$ is a strong deformation retract of $\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$.*

$\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ in Proposition 3.4 is the space of orientation-preserving diffeomorphisms Φ from S_0 to S_1 such that $\Phi(Q_0) = Q_1$, with the C^∞ -topology. $\text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$ is the subspace of $\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ consisting of exact symplectomorphisms, i.e. such that $\Phi^* \lambda_1 - \lambda_0 = dg$ for some $g \in C^\infty(S_0; \mathbb{R})$.

Applying Proposition 3.4 to $\mathcal{S}_0 = \mathcal{S}_1$ we get

Corollary 3.5. *If $\mathcal{S}$ is a Liouville marked surface, then $\text{Symp}_{\text{ex}}(\mathcal{S})$ is a strong deformation retract of $\text{Diff}(\mathcal{S})$.*

Applying Proposition 3.4 to Liouville structures on the same marked surface we get

Corollary 3.6. *If λ_0 and λ_1 are Liouville structures on a marked surface (S, Q) , then there exists a smooth family $\Phi_s \in \text{Diff}(S, Q)$ such that $\Phi_0 = \text{id}_S$ and $\Phi_1 \in \text{Symp}_{\text{ex}}((S, Q, \lambda_0), (S, Q, \lambda_1))$.*

Remark 3.7. A related result by Bourgeois–Ekholm–Eliashberg [10, Lemma 1.1] is that a symplectomorphism between Liouville manifolds with cylindrical ends can be deformed to an exact symplectomorphism.

Note that for a Liouville domain M the natural map $\pi_0 \text{Symp}_{\text{ex}}(M) \rightarrow \pi_0 \text{Symp}(M)$ is not necessarily surjective. For instance, on the cylinder $M = [-1, 3]_r \times (\mathbb{R}/\mathbb{Z})_\theta$ with the Liouville form $\lambda = rd\theta$ the symplectomorphism $(r, \theta) \mapsto (2 - r, -\theta)$ swaps the components of the boundary, but there is no exact symplectomorphism Φ swapping the components since we would have

$$0 = \int_{\{3\} \times \mathbb{R}/\mathbb{Z}} (\Phi^* \lambda - \lambda) = \int_{-\{1\} \times \mathbb{R}/\mathbb{Z}} -d\theta - \int_{\{3\} \times \mathbb{R}/\mathbb{Z}} 3d\theta = -2.$$

On the other hand, a slight adaptation of our proof yields that for any Liouville domain M the group $\text{Symp}_{\text{ex}}(M)$ is a strong deformation retract of the group

$$\{\Phi \in \text{Symp}(M) : (\Phi^* \lambda - \lambda)|_{\partial M} \text{ is exact}\}$$

of symplectomorphisms exact on the boundary.

To prove Proposition 3.4 we use a Moser argument and the following lemma.

Lemma 3.8. *Let (S, Q) be a marked surface. Then there exists a continuous (in the C^∞ -topology) map $\mu \mapsto g_\mu$ from*

$$(3.4) \quad \tilde{\Omega}^1(S) = \left\{ \mu \in \Omega^1(S) : \int_C \mu = 0 \text{ for all } C \in \pi_0 \partial S \right\}$$

to $C^\infty(S, \mathbb{R})$ such that $\mu|_{\partial S} = dg_\mu|_{\partial S}$ for any $\mu \in \tilde{\Omega}^1(S)$, $\mu(p) = d_p g_\mu$ for any $p \in Q$, and $dg_\mu = \mu$ whenever μ is exact.

Proof of Proposition 3.4. For any $\Phi \in \text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ we have $\Phi^* \lambda_1 - \lambda_0 \in \tilde{\Omega}^1(S)$ and hence a function $g_{\Phi^* \lambda_1 - \lambda_0}$ provided by Lemma 3.8 that depends on Φ continuously. The equation

$$\Phi^* \lambda_1 - \lambda_0 + d\tilde{\lambda}_s(X_s, -) = dg_{\Phi^* \lambda_1 - \lambda_0} \text{ where } \tilde{\lambda}_s = s\Phi^* \lambda_1 + (1-s)\lambda_0$$

defines a family of vector fields

$$[0, 1] \rightarrow C^\infty(S, TS), \quad s \mapsto X_s^\Phi = X_s$$

since $d\tilde{\lambda}_s$ is non-degenerate. As $d\tilde{\lambda}_s$ is skew-symmetric and $d\tilde{\lambda}_s(X_s, -)|_{\partial S}$ vanishes, X_s is tangent to ∂S . Since $\Phi^* \lambda_1 - \lambda_0$ and hence $dg_{\Phi^* \lambda_1 - \lambda_0}$ vanish on Q , so does X_s . Therefore, X_s integrates to a family of diffeomorphisms

$$[0, 1] \rightarrow \text{Diff}(\mathcal{S}_0), \quad s \mapsto \Psi_s^\Phi = \Psi_s; \quad \Psi_0 = \text{id}_{\mathcal{S}_0}.$$

Integrating

$$\frac{d}{ds} \Psi_s^* \tilde{\lambda}_s = \Psi_s^* \left(\frac{d}{ds} \tilde{\lambda}_s + d(\tilde{\lambda}_s(X_s)) + d\tilde{\lambda}_s(X_s, -) \right) = \Psi_s^* (d(\tilde{\lambda}_s(X_s)) + dg_{\Phi^* \lambda_1 - \lambda_0})$$

we obtain

$$(\Phi \circ \Psi_1)^* \lambda_1 - \lambda_0 = \Psi_1^* \tilde{\lambda}_1 - \lambda_0 = dh \text{ where } h = \int_0^1 (\tilde{\lambda}_s(X_s) + g_{\Phi^* \lambda_1 - \lambda_0}) \circ \Psi_s ds.$$

Therefore, $\Phi \circ \Psi_1 \in \text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$. If $\Phi \in \text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$, then $\Psi_s^\Phi = \text{id}_{\mathcal{S}_0}$ since in that case $dg_{\Phi^* \lambda_1 - \lambda_0} = \Phi^* \lambda_1 - \lambda_0$. Thus the map

$$\text{Diff}(\mathcal{S}_0, \mathcal{S}_1) \times [0, 1] \rightarrow \text{Diff}(\mathcal{S}_0, \mathcal{S}_1), \quad (\Phi, s) \mapsto \Phi \circ \Psi_s^\Phi$$

defines the desired strong deformation retraction. $\square$

Proof of Lemma 3.8. We perform the construction in three steps. The first step takes care of exactness using the Helmholtz decomposition, the second of the behaviour on Q and ∂S using a partition of unity argument, and the third combines the two.

Step 1. Fix a Riemannian metric on S and recall that the Helmholtz decomposition on S yields for any $\mu \in \Omega^1(S)$ an L^2 -orthogonal splitting $\mu = \mu_{\text{ex}} + \mu_{\text{co}}$ where μ_{ex} is exact and μ_{co} is co-closed with vanishing normal component, see e.g. Schwarz [49, p. 87]. Since $g \in C^\infty(S, \mathbb{R})$ is determined by its differential up to a constant we obtain a continuous linear map

$$\Omega^1(S) \rightarrow C^\infty(S, \mathbb{R}), \quad \mu \mapsto g_\mu^0$$

such that $\mu = dg_\mu^0$ if and only if μ is exact.

Step 2. Note that the integral condition in (3.4) is equivalent to $\mu|_{\partial S}$ being exact. Therefore, the behaviour of $\mu \in \tilde{\Omega}^1(S)$ on Q and ∂S can be recorded by a continuous linear map

$$\mu \mapsto ((\mu(q))_{q \in D}, (\mu(p))_{p \in \Lambda}, f_\mu)$$

from $\tilde{\Omega}^1(S)$ to

$$\mathcal{T} = \left\{ ((\eta_q)_{q \in D}, (\eta_p)_{p \in \Lambda}, f) : \begin{array}{l} d_p f = \eta_p|_{T_p \partial S} \text{ for all } p \in \Lambda \end{array} \right\} \subset \prod_{q \in D} T_q^* S \times \prod_{p \in \Lambda} T_p^* S \times C^\infty(\partial S, \mathbb{R})$$

satisfying $\mu|_{\partial S} = df_\mu$.

A standard partition of unity argument gives a continuous linear map

$$\mathcal{T} \rightarrow C^\infty(S, \mathbb{R}), \quad \mathbf{t} \mapsto g_{\mathbf{t}}$$

such that $g_{\mathbf{t}}|_{\partial S} = f$ and $d_p g_{\mathbf{t}} = \eta_p$ for all $p \in Q$ for any $\mathbf{t} = ((\eta_q)_{q \in D}, (\eta_p)_{p \in \Lambda}, f)$.

Composing the two maps above we get

$$\tilde{\Omega}^1(S) \rightarrow C^\infty(S; \mathbb{R}), \quad \mu \mapsto g_\mu^1$$

satisfying $\mu|_{\partial S} = dg_\mu^1|_{\partial S}$ and $\mu(p) = d_p g_\mu^1$ for any $p \in Q$.

Step 3. Define the continuous linear map

$$\tilde{\Omega}^1(S) \rightarrow C^\infty(S, \mathbb{R}), \quad \mu \mapsto g_\mu = g_\mu^0 + g_{\mu - dg_\mu^0}^1.$$

Then

$$d_p g_\mu = d_p g_\mu^0 + (\mu - dg_\mu^0) = \mu \text{ for any } p \in Q$$

and $dg_\mu|_{\partial S} = dg_\mu^0|_{\partial S} + (\mu - dg_\mu^0)|_{\partial S} = \mu|_{\partial S}$. Moreover, if μ is exact, then $dg_\mu^0 = \mu$ and hence $dg_\mu = \mu$. Thus $\mu \mapsto g_\mu$ satisfies all the requirements and we are done. $\square$

3.1.5. Contactisation. Liouville structures are closely connected to contact structures. Recall that a 1-form α on a $(2n+1)$ -dimensional manifold N is called a *contact form* if $\alpha \wedge (d\alpha)^n$ is a volume form on N . The pair (N, α) is then called a *(co-oriented) contact manifold*. The *Reeb vector field* R is defined by $d\alpha(R, -) = 0$ and $\alpha(R) = 1$, its integral curves are called *Reeb trajectories*.

An n -dimensional submanifold of a $(2n+1)$ -dimensional contact manifold (N, α) is called a *Legendrian* if it is everywhere tangent to the *contact distribution* $\ker \alpha$. A Reeb trajectory connecting two points of a Legendrian is called a *Reeb chord*. The *length* of a Reeb chord $y: [0, l] \rightarrow N$ is $l = \int_0^l y^* \alpha$. Unless stated otherwise, we only consider non-constant Reeb chords.

Given a Liouville marked surface $\mathcal{S} = (S, Q, \lambda)$ we obtain two contact manifolds from it. The first one is $(\partial S \setminus \Lambda, \lambda|_{\partial S \setminus \Lambda})$, which is a disjoint union of open intervals and circles. The second is the contact 3-manifold

$$(3.5) \quad (S^+, \lambda^+) = ((S \setminus Q) \times \mathbb{R}_z, \lambda + dz),$$

called the *contactisation* of $\mathcal{S}$.

3.2. Curves. Fix a Liouville marked surface $\mathcal{S} = (S, Q, \lambda)$. We describe the curves in S that will give rise to objects in the Fukaya category.

In §3.2.1 we define Lagrangians in $\mathcal{S}$, which are immersed curves together with a potential, i.e. a Legendrian lift to the contactisation. In §3.2.2 we describe the self-intersection information of a Lagrangian, consisting of its double points and Reeb chords between its ends together with their actions, and introduce slightly degenerate Lagrangians, whose ends are allowed to collide. In §§3.2.3 and 3.2.4 we characterise in terms of linear inequalities on actions and areas when an isotopy class of immersed curves contains an exact curve and which Reeb signs and Reeb chords of positive action a Lagrangian supported on a given exact curve can have, see Propositions 3.14 and 3.15. In §3.2.5 we parametrise a neighbourhood of a Lagrangian in the space of Lagrangians and characterise isotopies of Lagrangians topologically. In §3.2.6 we prove that any Lagrangian is isotopic through slightly degenerate Lagrangians to one without Reeb chords of positive action in Lemma 3.18 and describe what happens to a Lagrangian when an isotopy passes through a slightly degenerate one; this will allow us to exchange Reeb chords for double points in Proposition 4.8.

3.2.1. Lagrangians. Let I be a 1-dimensional compact manifold, i.e. a finite disjoint union of circles and closed segments. To avoid set-theoretic issues we tacitly only work with

$$(3.6) \quad I = I_{m,n} = \bigsqcup_{i=1}^m \mathbb{R}/\mathbb{Z} \sqcup \bigsqcup_{i=1}^n [0, 1] \text{ for } m, n \in \mathbb{Z}_{\geq 0}.$$

As a byproduct, we obtain a nowhere vanishing 1-form dt on I , which, in particular, determines the orientation of I .

Consider an immersion $\gamma: I \rightarrow S$. The first set of conditions we impose is that γ is a generic properly immersed curve in $S \setminus Q$:

- (1) $\gamma(I) \cap Q = \emptyset$.
- (2) $\gamma^{-1}(\partial S) = \partial I$ and γ is transverse to ∂S ;
- (3) $\gamma|_{\partial I}$ is an embedding into ∂S ;

(4) all singularities of γ are transverse double points.

The curve γ is called an *exact immersion* if there exists a function $P: I \rightarrow \mathbb{R}$ such that

(5) $\gamma^*\lambda + dP = 0$.

Note that this implies that the *Legendrian lift* $\gamma^+ = (\gamma, P): I \rightarrow S^+$ is indeed Legendrian in the contactisation (S^+, λ^+) , i.e. γ^+ is tangent to the contact distribution $\ker \lambda^+$.

We impose further general position requirements, now on P :

(6) for any double point of γ the values of P at its two preimages are different;

(7) for any Reeb chord $y: [0, l] \rightarrow \partial S \setminus \Lambda$ connecting $\gamma(t_0) = y(0)$ and $\gamma(t_1) = y(l)$, the value $P(t_1)$ is different from $P(t_0) - l$.

Note that (6) ensures the Legendrian lift γ^+ is embedded and that (7) is in a sense a variant of (6): for $t_0 \in \partial I$ if we attach the maximally extended Reeb trajectory $y: [0, l_0] \rightarrow \partial S \setminus \Lambda$ with $y(0) = t_0$ to γ and extend the potential to satisfy (5), then the potential at a point $l \in [0, l_0]$ will be equal to $P(t_0) - \int_0^l y^*\lambda = P(t_0) - l$.

Combining all conditions together we obtain the following definition.

Definition 3.9. A (exact immersed) *Lagrangian* L (with a fixed Legendrian lift) in the Liouville marked surface $\mathcal{S}$ is the data of

- a 1-dimensional oriented compact manifold I , called the *domain*;
- an immersion $\gamma: I \rightarrow S$, called the *underlying curve*;
- a function $P: I \rightarrow \mathbb{R}$, called the *potential*

satisfying conditions (1)–(7) above.

Note that the potential is part of the data. For a generic exact immersion $\gamma: I \rightarrow S$ the choice of potential P is parametrised by the complement of a hyperplane arrangement in $\mathbb{R}^{\pi_0 I}$, see §6.1.1. Choices of the potential from different connected components of this complement can give non-equivalent obstruction algebras and hence different objects in the Fukaya category.

Observe that the orientation of I was not used in Definition 3.9. We only choose it for the sake of convenience, different orientations of I will give isomorphic obstruction algebras and equivalent objects in the Fukaya category.

3.2.2. Double points and Reeb chords. Let $L = (I, \gamma, P)$ be a Lagrangian in a Liouville marked surface $\mathcal{S}$. We describe the self-intersection information of L . It consists of double points of γ and certain Reeb chords connecting the ends of L . They will give rise to the obstruction algebra and to morphism spaces in the Fukaya category.

Let $X = X_L$ be the set of double points of γ . Fix $x \in X$ and let

$$(3.7) \quad x_+, x_- \in I \text{ be such that } \gamma(x_+) = x = \gamma(x_-) \text{ and } P(x_+) > P(x_-).$$

This makes sense by condition (6). The *action* of x is

$$(3.8) \quad A(x) = P(x_+) - P(x_-) > 0.$$

Note that double points of L are in bijection with Reeb chords of its Legendrian lift L^+ and that the action of a double point is equal to the length of the corresponding Reeb chord.

The image of a small neighbourhood of x_+ (resp. x_-) under γ is called the *overstrand* (resp. *understrand*) at x . To arrange the counts of discs later it is convenient to encode the over- and understrand information at x by *Reeb signs*. These are the assignments of $+$ or $-$ to each of the four quadrants cut out by the tangent lines $\dot{\gamma}(x_+)$ and $\dot{\gamma}(x_-)$ from $T_x S$. Namely, we assign $+$ (resp. $-$) to the two quadrants that lie to the left of the curve that first follows the understrand (resp. overstrand) and then jumps to the overstrand (resp. understrand) at x , see Fig. 6(a). The quadrants with Reeb sign $+$ (resp. $-$) are called *positive* (resp. *negative*).

Note that $\gamma|_{\partial I}$ gives a Legendrian submanifold in the contact manifold $(\partial S \setminus \Lambda, \lambda|_{\partial S \setminus \Lambda})$. Let $Y = Y_L$ be the set of Reeb chords $y: [0, l_y] \rightarrow \partial S \setminus \Lambda$ that connect two ends of L . Consider a Reeb chord $y \in Y$. Let

$$(3.9) \quad y_+, y_- \in \partial I \text{ be such that } \gamma(y_+) = y(0) \text{ and } \gamma(y_-) = y(l_y).$$

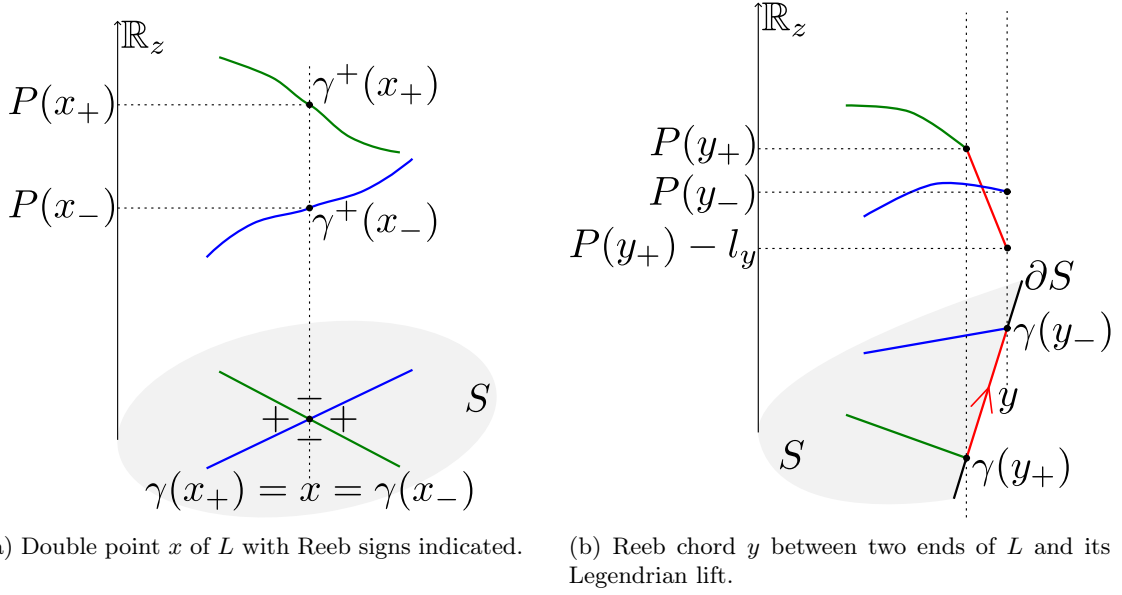


FIGURE 6. Lifts of a double point and a Reeb chord.

Lagrangian $L = (I, \gamma, P)$ in S and its Legendrian lift $\gamma^+ = (\gamma, P)$ in the contactisation S^+ around a double point and a Reeb chord.

The *action* of y is

$$(3.10) \quad A(y) = P(y_+) - l_y - P(y_-).$$

Note that $A(y)$ is always non-zero by condition (7).

The definitions of action for double points and Reeb chords are compatible in the following sense. Given a Reeb chord $y \in Y$, consider the Reeb trajectory $\tilde{y}: [0, l) \rightarrow \partial S \setminus \Lambda$ that maximally extends y . Extend $\gamma: I \rightarrow S$ to a piece-wise smooth curve $\tilde{\gamma} = \gamma \cup \tilde{y}: I \cup_{y_+=0} [0, l) \rightarrow S$ and $P: I \rightarrow \mathbb{R}$ to a continuous function $\tilde{P}: I \cup_{y_+=0} [0, l) \rightarrow \mathbb{R}$ satisfying (5) everywhere except $y_+ = 0$.

Then $\tilde{\gamma}(y_-) = \tilde{\gamma}(l_y)$ is a self-intersection point of $\tilde{\gamma}$ and $A(y) = \tilde{P}(l_y) - \tilde{P}(y_-)$, see Fig. 6(b).

Allow for a moment $\gamma|_{\partial I}$ not to be an embedding and let $z_+, z_- \in \partial I$ be such that $\gamma(z_+) = \gamma(z_-) = z \in \partial S$ and $P(z_+) > P(z_-)$. Then z can be considered both as a self-intersection point of L and as a constant Reeb chord between the ends of L . The two definitions of the action clearly agree. Motivated by this observation, we give the following definition.

Definition 3.10. A *slightly degenerate Lagrangian* L in the Liouville marked surface S is the same thing as a Lagrangian from Definition 3.9 except that $\gamma|_{\partial I}$ is not required to be an embedding and at each double point $z \in \partial S$ the unique quadrant of z contained in S is negative.

Given a slightly degenerate Lagrangian L we denote by X_L the set of double points of L in $\text{Int } S$, by Y_L the set of Reeb chords of positive length, and by XY_L the set of double points of L in ∂S , which we call *point-chords* of L .

Let $y^i: [0, l_i] \rightarrow \partial S \setminus \Lambda$, $i = 0, 1$ be two Reeb chords connecting the ends of L such that $y^0(l_0) = y^1(0)$. Their *concatenation* is the Reeb chord

$$(3.11) \quad y^1 y^0: [0, l_0 + l_1] \rightarrow \partial S \setminus \Lambda, \quad s \mapsto \begin{cases} y^0(s), & \text{if } s \in [0, l_0]; \\ y^1(s - l_0), & \text{if } s \in [l_0, l_0 + l_1]. \end{cases}$$

The action is additive with respect to concatenation.

We make the following simple observation

Lemma 3.11. *Let B be a real number. Then*

(1) The set

$$Y^{>B} = \{y \in Y : A(y) > B\}$$

is finite.

(2) The set

$$Y^{<B} = \{y \in Y : A(y) < B\}$$

is finite if and only if for any $t \in \partial I$ there is a stop on the component C of ∂S with $\gamma(t) \in C$.

Proof. ∂I is finite and for fixed $y_+, y_- \in \partial I$ the set of Reeb chords from $\gamma(y_+)$ to $\gamma(y_-)$ either contains at most one element or is equal to $\{yz^k \mid k \in \mathbb{Z}_{\geq 0}\}$ where y is the shortest Reeb chord from $\gamma(y_+)$ to $\gamma(y_-)$ and z is the Reeb chord from $\gamma(y_+)$ to itself that goes around the component of ∂S once. As $A(yz^{k+1}) - A(yz^k) = -l_z < 0$, only finitely many of these chords will have action larger than B . On the other hand, if $\gamma(t) \in C$ for some $t \in \partial I$ and a component $C \in \pi_0 \partial S$ with $C \cap \Lambda = \emptyset$, then $A(z^k) < B$ for $k \gg 0$ where z is the Reeb chord from $\gamma(t)$ to itself going around C once. $\square$

The sets X and XY are finite by compactness and so is the set $Y^{>0}$ of Reeb chords of positive action by Lemma 3.11. We denote $Z = Z_L = X \cup Y^{>0} \cup XY$.

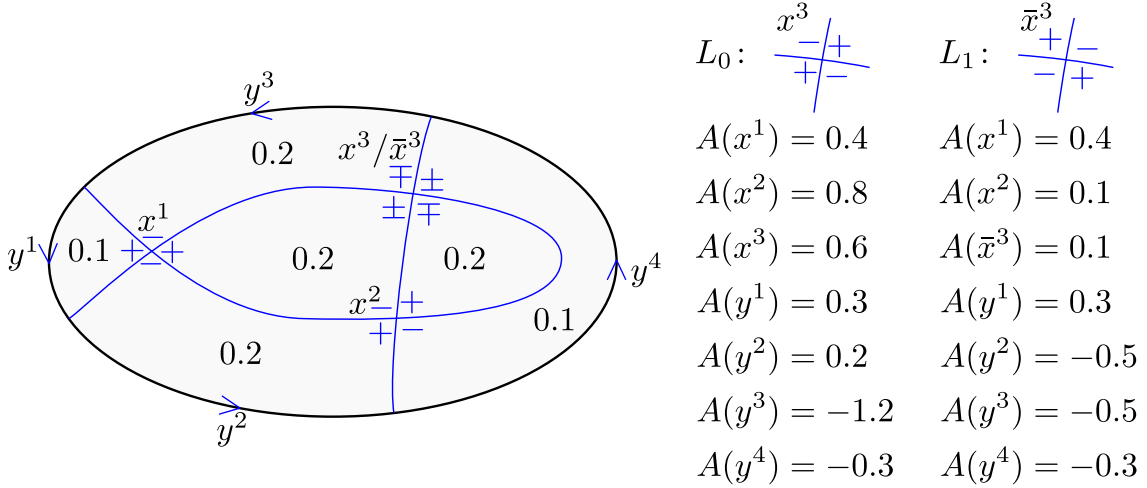


FIGURE 7. Two Lagrangians with the same underlying curve.

The blue curve is the underlying curve of both L_0 and L_1 , the blue signs at each double point are Reeb signs, the numbers inside the regions cut out from the disc by the curve are their areas.

Example 3.12. In Fig. 7 two Lagrangians L_0 and L_1 in a disc with some Liouville form λ are depicted. The domain of both Lagrangians is the disjoint union of two segments and their underlying curves are the same. L_1 is obtained from L_0 by changing the potential on one of the segments by a constant. We have $Z_{L_0} = \{x^1, x^2, x^3, y^1, y^2, y^2 y^1, y^4 y^2 y^1\}$ and $Z_{L_1} = \{x^1, x^2, \bar{x}^3, y^1\}$.

This example is continued in Example 3.22.

The Liouville form in Fig. 7 is only represented by areas (i.e. integrals of $d\lambda$) and actions. This numerical information is enough to determine whether a Lagrangian with these areas and actions exists as we explain in Propositions 3.14 and 3.15 below. Moreover, the Reeb signs and the signs of the actions of Reeb chords between the ends of a Lagrangian L determine it uniquely up to isotopy, see Proposition 3.17.

3.2.3. Exact curves isotopic to a given curve. Fix a marked surface (S, Q) . Let I be a compact 1-dimensional manifold and $\gamma: I \rightarrow S \setminus Q$ a generic enough properly immersed curve, i.e. γ satisfies conditions (1)–(4) from Definition 3.9. Let $\text{Imm}_I(S, Q)$ be the space of all such immersions with C^∞ -topology. We say that two curves in $\text{Imm}_I(S)$ are *isotopic* if they lie in the same component of $\text{Imm}_I(S)$.

Note that $\text{Diff}(\mathcal{S})$ acts on $\text{Imm}_I(\mathcal{S})$ on the left by post-composition and $\text{Diff}(I)$ acts on the right by pre-composition. Make the following simple observation.

Lemma 3.13.

$$\pi_0 \text{Imm}_I(\mathcal{S}) = \text{Diff}(\mathcal{S})_0 \setminus \text{Imm}_I(\mathcal{S}) / \text{Diff}(I)_0.$$

Proof. On one hand, each orbit is contained in one component since $\text{Diff}(\mathcal{S})_0$ and $\text{Diff}(I)_0$ are connected. On the other hand, if we have an isotopy $\gamma_s \in \text{Imm}_I(\mathcal{S})$, $s \in [0, 1]$, then we can first make the double points fixed throughout the isotopy by an isotopy $\Phi_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$, $\Phi_0 = \text{id}_S$, then make the preimage of each double point in I fixed throughout the isotopy by an isotopy $\psi_s \in \text{Diff}(I)$, $s \in [0, 1]$, $\psi_0 = \text{id}_I$, and then extend the time-dependent vector field $\frac{d\gamma_s(t)}{ds} \Big|_{s=s_0} \in C^\infty(I, \gamma^*TS)$, $s_0 \in [0, 1]$ to a time-dependent vector field on S that integrates to an isotopy $\tilde{\Phi}_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$, $\tilde{\Phi}_0 = \text{id}_S$. $\square$

Now let λ be a Liouville structure on $\mathcal{S}$. We give a criterion for the existence of an exact (i.e. such that $\gamma'^*\lambda$ is exact) immersion γ' isotopic to a given $\gamma \in \text{Imm}_I(\mathcal{S})$, cf. Chekanov [13, Lemma 11.4].

Note that if $\gamma \in \text{Imm}_I(\mathcal{S})$ is exact, then the restriction of γ to any union of connected components of I is exact and that we can add a number of embedded segments to γ and it will remain exact. It therefore suffices to study the case when $I = I_0 \sqcup I_1 \sqcup I_2$ where I_0 is a disjoint union of circles, I_1 and I_2 are disjoint unions of segments, and $\gamma|_{I_2}$ is an embedded curve such that $S \setminus \gamma(I_2)$ is contractible. We call such γ *totally separating*. Observe that in a given connected component $\mathcal{C}$ of $\text{Imm}_I(\mathcal{S})$ either all curves are totally separating or none are. Note that I_2 has $l = 2g + b - 1$ components where g is the genus and b is the number of boundary components of S . In particular, $I_2 = \emptyset$ if and only if S is a disc.

Let $\mathcal{R}$ be the set of regions cut out by γ from S . Let $\Sigma: \mathbb{R}^{\mathcal{R}} \rightarrow \mathbb{R}$ be the linear map $(\alpha_R)_{R \in \mathcal{R}} \mapsto \sum_{R \in \mathcal{R}} \alpha_R$. Note that if (I, γ, P) is a Lagrangian, then

$$\Sigma \left(\int_R d\lambda \right)_{R \in \mathcal{R}} = b$$

by the Stokes theorem and condition (4) in Definition 3.1.

For each $t_+ \in \partial I_2$ let $y_{t_+}: [0, 1] \rightarrow \partial S$ be a path on the boundary that starts at $\gamma(t_+)$ and ends at $\gamma(t_-)$ where $t_- \in \partial I_2$ is the point following t_+ in the order of the orientation of ∂S (y_{t_+} can pass through Λ and t_- can be equal to t_+). Let Y_2 be the set of such paths, it has order $2l$. For $y \in Y_2$ denote by $y_{\pm} \in \partial I$ the points satisfying $\gamma(y_+) = y(0)$ and $\gamma(y_-) = y(1)$ as in §3.2.2. If $(I_2, \gamma|_{I_2}, P_2)$ is a Lagrangian, then we can define the actions $A(y) = P_2(y_+) - P_2(y_-) - \int_0^1 y^* \lambda$ for $y \in Y_2$.

Let $B: \mathbb{R}^{Y_2} \rightarrow \mathbb{R}^{\pi_0 \partial S}$ be the linear map

$$(3.12) \quad (A_y)_{y \in Y_2} \mapsto \left(\sum_{y \in Y_2: \text{Im } y \subset C} A_y \right)_{C \in \pi_0 \partial S}.$$

Note that if $(I_2, \gamma|_{I_2}, P_2)$ is a Lagrangian, then $B(A(y))_{y \in Y_2} = (-1, \dots, -1)$ by condition (4) in Definition 3.1. If $Y_2 = \emptyset$ we define $B: \{0\} \rightarrow \mathbb{R}$ by $B(0) = -1$.

Fix an orientation of I . Then we have a linear map $E: \mathbb{R}^{\mathcal{R}} \oplus \mathbb{R}^{Y_2} \rightarrow \mathbb{R}^{\pi_0 I_0}$ that, given the areas $(\alpha_R)_{R \in \mathcal{R}}$ of the regions and the actions $(A_y)_{y \in Y_2}$ of the paths computes the integrals $(\int_C \gamma^* \lambda)_{C \in \pi_0 I_0}$ by the Stokes theorem. Note that the definitions of Σ , B , and E only depends on the component of γ in $\text{Imm}_I(\mathcal{S})$.

Proposition 3.14. *Let $\mathcal{C}$ be a component of $\text{Imm}_I(\mathcal{S})$ consisting of totally separating immersions. Then there exists an exact immersion in $\mathcal{C}$ if and only if there exist $\alpha_R > 0$, $R \in \mathcal{R}$ and $A_y \in \mathbb{R}$, $y \in Y_2$ such that*

$$\Sigma(\alpha_R)_{R \in \mathcal{R}} = b, \quad B(A_y)_{y \in Y_2} = (-1, \dots, -1), \quad \text{and} \quad E((\alpha_R)_{R \in \mathcal{R}}, (A_y)_{y \in Y_2}) = 0.$$

Proof. The ‘only if’ part is clear from the discussion above. Assume we have $(\alpha_R)_{R \in \mathcal{R}}$ and $(A_y)_{y \in Y_2}$ as in the statement. We want to find an immersion $\gamma \in \mathcal{C}$ and a function $P: I \rightarrow \mathbb{R}$ such that $\gamma^* \lambda + dP = 0$.

Pick an immersion $\gamma_0 \in \mathcal{C}$. By the condition on B we can pick $2l$ numbers $P(t)$, $t \in \partial I_2$ such that $A(y) = P(y_+) - P(y_-) - \int_0^1 y^* \lambda = A_y$ for each $y \in Y_2$. Recall that $\gamma_0|_{I_2}$ cuts S to form one polygon. By deforming the interiors of two sides of this polygon corresponding to a component C of I we can achieve any value of $\int_C \gamma^* \lambda$. Since this deformation can be realised by an isotopy in $\text{Diff}(S)$, we can obtain a new immersion $\gamma_1 \in \mathcal{C}$ such that $P(t_1) - P(t_0) + \int_{t_0}^{t_1} \gamma_1^* \lambda = 0$ for each component $C \in \pi_0 I_2$ where $\partial C = \{t_0, t_1\}$. Hence we extend P to I_2 to satisfy $(\gamma^* \lambda + dP)|_{I_2} = 0$.

We now proceed by removing the regions R from $\mathcal{R}$ one by one after ensuring that $\int_R d\lambda = \alpha_R$. We start with the polygon $T_1 = (T, D)$ cut out by $\gamma_1|_{I_2}$ from S , the curve $\tilde{\gamma}_1 = \gamma_1|_{I \setminus I_2}$, and $\mathcal{R}_1 = \mathcal{R}$. At each step we have a polygon $T_k = (T_k \subset T, D_k = D \cap T_k)$ and a curve $\tilde{\gamma}_k$ cutting T_k into regions $\mathcal{R}_k \subset \mathcal{R}$ such that

$$\sum_{R \in \mathcal{R}_k} \int_R d\lambda = \sum_{R \in \mathcal{R}_k} \alpha_R.$$

We take a region $R \in \mathcal{R}_k$ adjacent to ∂T_k such that $T_k \setminus R$ is connected. If $\partial R \subset \partial T_k$, then $R = T_k$, $\int_R d\lambda = \alpha_R$ and we are done. Otherwise, we can find an isotopy

$$\Phi_s \in \text{Diff}(T) = \{\Phi \in \text{Diff}(T): \Phi|_{\partial S} = \text{id}, \Phi(D_k) = D_k\}, \quad s \in [0, 1], \quad \Phi_0 = \text{id}_T$$

such that $\int_{\Phi_1(R)} d\lambda = \alpha_R$. We set T_{k+1} to be the closure (in T_k) of $T_k \setminus \Phi_1(R)$ and $\tilde{\gamma}_{k+1}$ to be the part of $\Phi_1 \circ \tilde{\gamma}_k$ that lies inside T_{k+1} .

After the procedure above we obtain an immersion $\gamma \in \mathcal{C}$ such that $\int_R d\lambda = \alpha_R$ for each $R \in \mathcal{R}$ and $A(y) = A_y$ for each $y \in Y_2$. The vanishing of E now implies that γ is exact and we are done. $\square$

3.2.4. Lagrangian structures on a given exact curve. Let $\gamma \in \text{Imm}_I(S)$ be an exact immersed curve. We study Lagrangians in $\mathcal{S}$ with underlying curve γ .

Let $L = (I, \gamma, P)$ be such Lagrangian.

Given L we obtain Reeb signs at each double point of γ (recall that this is the same information as specifying under- and over-crossings) and the set $Y_L^{>0}$ of Reeb chords of positive action.

Let $\gamma \in \text{Imm}_I(S)$ be a totally separating exact curve with Reeb signs at each double point $x \in X$ and a finite set $Y^{>0} \subset Y$ of Reeb chords between the ends of γ specified. We give a criterion for when there exists a Lagrangian supported on γ inducing this information, cf. Chekanov [13, Proposition 11.2].

Let Y_{short} be the set of paths $y: [0, 1] \rightarrow \partial S$ that follow the orientation of ∂S , start at $\gamma(y_+)$, end at $\gamma(y_-)$ for some $y_{\pm} \in \partial I$, and do not have the ends of γ in the interior. The number of paths in Y_{short} is twice the number of segment connected components of I .

Recall that $\mathcal{R}$ is the set of regions cut out by γ from S . We have a linear map $\text{Area}: \mathbb{R}^{X \cup Y_{\text{short}}} \rightarrow \mathbb{R}^{\mathcal{R}}$ that computes the area of the region R by the Stokes theorem (3.15). We also have the map $F: \mathbb{R}^{Y_{\text{short}}} \rightarrow \mathbb{R}^Y$ computing the action of each $y \in Y$ from the actions on Y_{short} by the additivity with respect to concatenation and the map $\tilde{B}: \mathbb{R}^{Y_{\text{short}}} \rightarrow H^1(\gamma(I_2) \cup \partial S; \mathbb{R})$ that, given a closed curve σ in $\gamma(I_2) \cup \partial S$ thought of as a 1-cycle, computes $\sum_{y \in Y_{\text{short}}} \deg_{\sigma}(y)$ where $\deg_{\sigma}(y)$ is the signed number of times σ covers y .

Proposition 3.15. *Let $\gamma \in \text{Imm}_I(S)$ be a totally separating exact curve with Reeb signs at each double point $x \in X$ and a finite set $Y^{>0} \subset Y$ of Reeb chords between the ends of γ specified. Then there exists a potential $P: I \rightarrow \mathbb{R}$ such that the Lagrangian $L = (I, \gamma, P)$ induces the same Reeb signs and $Y_L^{>0} = Y^{>0}$ if and only if there exists $A_x > 0$, $x \in X$ and $A_y \in \mathbb{R}$, $y \in Y_{\text{short}}$ such that*

- (1) $(F(A_y)_{y \in Y_{\text{short}}})_y > 0$ for $y \in Y^{>0}$;
- (2) $(F(A_y)_{y \in Y_{\text{short}}})_y < 0$ for $y \in Y \setminus Y^{>0}$;
- (3) $\text{Area}((A_x)_{x \in X}, (A_y)_{y \in Y_{\text{short}}}) = (\int_R d\lambda)_{R \in \mathcal{R}}$;
- (4) $\tilde{B}(A_y)_{y \in Y_{\text{short}}} = \lambda|_{\gamma(I_2) \cup \partial S}$ in $H^1(\gamma(I_2) \cup \partial S; \mathbb{R})$.

Note that it is enough to check condition (2) on the shortest Reeb chord $yz^k \in Y \setminus Y^{>0}$ from the infinite set $\{yz^k \mid k \geq 0\}$ where z goes around a component of ∂S once. Hence there are finitely many conditions to check in Proposition 3.15.

Proof. The ‘only if’ part is clear. To prove the ‘if’ part we construct a function $P: I \rightarrow \mathbb{R}$ such that $\gamma^*\lambda + dP = 0$, $A(x) = P(x_+) - P(x_-) = A_x$ for each $x \in X$, and $A(y) = P(y_+) - P(y_-) - \int_0^1 y^*\lambda = A_y$ for each $y \in Y_{short}$. Then $L = (I, \gamma, P)$ is a Lagrangian since $A(z) \neq 0$ for each $z \in X \cup Y$, it induces the correct Reeb signs since $A_x > 0$ for $x \in X$ and $Y_L^{>0} = Y^{>0}$ by (1) and (2).

Define $P(t) = 0$ for some $t \in I$ and spread using $\gamma^*\lambda + dP = 0$ and A_z for $z \in X \cup Y_{short}$ to all I (if $\text{Im } \gamma \cup \partial S$ is not connected, then do this separately for each component). No conflict arises by conditions (3) and (4) since any closed curve in $\text{Im } \gamma \cup \partial S$ is a linear combination (as a 1-chain) of boundaries of the regions from Y and closed curves in $\gamma(I_2) \cup \partial S$. $\square$

3.2.5. Isotopy of Lagrangians. For a compact 1-dimensional manifold I denote by $\text{Lag}_I(\mathcal{S})$ the space of Lagrangians (in the sense of Definition 3.9) in $\mathcal{S}$ with the C^∞ -topology. Two Lagrangians with the same domain I are *isotopic* if they lie in the same component of $\text{Lag}_I(\mathcal{S})$.

The group $\text{Symp}_{\text{ex}}(\mathcal{S})$ acts on $\text{Lag}_I(\mathcal{S})$ by

$$\Phi_*(I, \gamma, P) = (I, \Phi \circ \gamma, P - \gamma^*g)$$

where $g \in C^\infty(S, \mathbb{R})$ is such that $\Phi^*\lambda - \lambda = dg$ and $g(q_0) = 0$ for some fixed basepoint $q_0 \in S$. Note that Φ_*L is isotopic to L for $\Phi \in \text{Ham}(\mathcal{S})$, but the converse is not true (even after allowing vertical shifts) since the actions of double points and Reeb chords are the same for the corresponding points or chords of L and Φ_*L , but can change under an isotopy.

Fix $L_0 = (I, \gamma_0, P_0)$. We describe a neighbourhood of L_0 in $\text{Lag}_I(\mathcal{S})$. By the Weinstein neighbourhood theorem there exists a local diffeomorphism $\Phi: I \times (-\varepsilon, \varepsilon) \rightarrow S \setminus Q$ such that $\Phi(t, 0) = \gamma_0(t)$, $\Phi^*d\lambda = dt \wedge ds$ and $\Phi^{-1}(\partial S) = \partial I \times (-\varepsilon, \varepsilon)$. As $\Phi^*\lambda + sdt$ is closed and restricts to an exact form $-dP_0$ on $I \times \{0\}$, there is a function $\tilde{P}: I \times (-\varepsilon, \varepsilon)$ such that $\tilde{P}(t, 0) = P_0(t)$ and $\Phi^*\lambda + sdt + d\tilde{P} = 0$.

Therefore, for $F \in C^\infty(I, \mathbb{R})$ with F' close to 0 we obtain a Lagrangian (I, γ_F, P_F) where

$$\gamma_F(t) = \Phi(t, F'(t)) \text{ and } P_F(t) = F(t) + \tilde{P}(t, F'(t)) \text{ for } t \in I.$$

Lemma 3.16. *There exists a neighbourhood $\mathcal{U}$ of $\{0\} \times \{\text{id}\}$ in $C^\infty(I, \mathbb{R}) \times \text{Diff}_0(I)$ such that the map*

$$\mathcal{U} \rightarrow \text{Lag}_I(\mathcal{S}), (F, \varphi) \mapsto L(F, \varphi) = (I, \gamma_F \circ \varphi, P_F \circ \varphi)$$

is a homeomorphism onto a neighbourhood of L_0 in $\text{Lag}_I(\mathcal{S})$.

Proof. We construct the inverse map. Let $L = (I, \gamma, P)$ be a Lagrangian sufficiently C^∞ -close to L_0 for γ to be contained in the image of Φ and $\text{pr}_1 \circ \Phi^{-1} \circ \gamma$ to be a diffeomorphism. As Φ is a local diffeomorphism, there exist a unique diffeomorphism $\varphi_L: I \rightarrow I$ and a map $h_L: I \rightarrow (-\varepsilon, \varepsilon)$ such that $\Phi(\varphi_L(t), h_L(\varphi_L(t))) = \gamma(t)$ for any $t \in I$. Set $F_L(t) = P(\varphi_L^{-1}(t)) - \tilde{P}(t, h_L(t))$ for $t \in I$. Since $\gamma^*\lambda + dP = 0$ and $\Phi^*\lambda + sdt + d\tilde{P} = 0$, we have

$$\varphi_L^*dF_L = dP - \varphi_L^*(t \mapsto (t, h_L(t)))^*d\tilde{P} = \varphi_L^*(t \mapsto (t, h_L(t)))^*(-\Phi^*\lambda - d\tilde{P}) = \varphi_L^*(h_L dt).$$

Therefore, $F'_L = h_L$ hence $L(F_L, \varphi_L) = L$, $\varphi_{L(F, \varphi)} = \varphi$, and $F_{L(F, \varphi)} = F$. $\square$

We give a topological characterisation of isotopy. The result is folklore in $\mathbb{R}^2$ (see e.g. Kálmán [27, p. 2023]) and not hard in our setting, but we could not find a suitable reference.

Proposition 3.17. *Let $L_i = (I, \gamma_i, P_i)$, $i = 0, 1$ be two Lagrangians in the Liouville marked surface $\mathcal{S}$. Then the following are equivalent.*

- (1) L_0 and L_1 are isotopic.
- (2) γ_0 and γ_1 lie in the same component of $\text{Imm}_I(S)$, the Reeb signs at the corresponding double points of L_0 and L_1 agree, and the actions of the corresponding Reeb chords between the ends of L_0 and between the ends of L_1 have the same sign.

Proof. We prove that (1) follows from (2), the converse is clear. By Lemma 3.13 and the fact that the isotopy of the domain induces an isotopy, we can assume that there exists an isotopy $\Phi_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$ such that $\Phi_0 = \text{id}$ and $\Phi_1 \circ \gamma_0 = \gamma_1$. To prove the proposition it suffices to find a path Φ_s , $s \in [1, 2]$ in $\text{Diff}(\mathcal{S})$ such that $\Phi_2 \in \text{Symp}_{\text{ex}}(\mathcal{S})_0$ and the Lagrangian $L_2 = (\Phi_2)_* L_0$ is isotopic to L_1 . Indeed, Corollary 3.5 then implies that there exists a path Θ_s , $s \in [0, 1]$ in $\text{Symp}_{\text{ex}}(\mathcal{S})$ such that $\Theta_0 = \text{id}_S$ and $\Theta_1 = \Phi_2$ hence an isotopy from L_0 to L_1 can be obtained as the concatenation of $s \mapsto (\Theta_s)_* L_0$ and the isotopy from L_2 to L_1 .

We now find the desired path Φ_s , $s \in [1, 2]$ in $\text{Diff}(\mathcal{S})$. Consider the equation

$$(\Phi_1^{-1})^* \lambda - \lambda + d\lambda_s(X_s, -) = dg_{(\Phi_1^{-1})^* \lambda - \lambda}, \text{ where } s \in [0, 1].$$

where $\lambda_s = s(\Phi_1^{-1})^* \lambda + (1-s)\lambda$ and $g_{(\Phi_1^{-1})^* \lambda - \lambda}$ is provided by Lemma 3.8. As in the proof of Proposition 3.4, this equation determines a time-dependent vector field X_s on S which integrates to a family of diffeomorphisms

$$[0, 1] \rightarrow \text{Diff}(\mathcal{S}), \quad s \mapsto \Psi_s, \quad \Psi_0 = \text{id}_S.$$

Cartan's magic formula yields

$$s(\Phi_1^{-1})^* \lambda + (1-s)\lambda = (\Psi_s^{-1})^* \lambda + dh_s \text{ where } h_s = sg_{(\Phi_1^{-1})^* \lambda - \lambda} + \int_0^s \lambda_t(X_t) dt$$

We define the isotopy $L_u = (I, \gamma_u, P_u)$, $u \in [1, 2]$ by

$$\gamma_{s+1} = \Psi_s^{-1} \circ \gamma_1, \quad P_{s+1} = sP_0 + (1-s)P_1 + \gamma_1^* h_s.$$

A straightforward calculation shows that $\gamma_{s+1}^* \lambda + dP_{s+1} = 0$. For any self-intersection point x of γ_{s+1} we have $\gamma_{s+1}(x_+) = x = \gamma_{s+1}(x_-)$ for some $x_+, x_- \in I$ such that $P_i(x_+) > P_i(x_-)$ for $i = 0, 1$ by assumption of the proposition. Therefore,

$$\begin{aligned} P_{s+1}(x_+) &= sP_0(x_+) + (1-s)P_1(x_+) + h_s(x) > \\ &= sP_0(x_-) + (1-s)P_1(x_-) + h_s(x) = P_{s+1}(x_-). \end{aligned}$$

For any Reeb chord $y: [0, l] \rightarrow \partial S \setminus \Lambda$ connecting $\gamma_{s+1}(y_+) = y(0)$ and $\gamma_{s+1}(y_-) = y(l)$ we have that $P_0(y_+) - l - P_0(y_-)$ and $P_1(y_+) - l - P_1(y_-)$ are non-zero and have the same sign by assumption, therefore, a computation similar to the above shows that $P_{s+1}(y_+) - l - P_{s+1}(y_-)$ is non-zero. Hence L_u , $u \in [1, 2]$ is indeed an isotopy.

We set $\Phi_{s+1} = \Psi_s^{-1} \circ \Phi_1$, $s \in [0, 1]$. Then

$$\Phi_2^* \lambda - \lambda = \Phi_1^* ((\Psi_1^{-1})^* \lambda - (\Phi_1^{-1})^* \lambda) = d(-\Phi_1^* h_1) \text{ hence } \Phi_2 \in \text{Symp}_{\text{ex}}(\mathcal{S})_0$$

and

$$(\Phi_2)_* L_0 = (I, \Psi_1^{-1} \circ \Phi_1 \circ \gamma_0, P_0 - \gamma_0^*(-\Phi_1^* h_1)) = (I, \Psi_1^{-1} \circ \gamma_1, P_0 + \gamma_1^* h_1) = L_2,$$

as required. $\square$

3.2.6. Isotopy of slightly degenerate Lagrangians. For a compact 1-dimensional manifold I denote by $\text{Lag}_I^{sg}(\mathcal{S})$ the space of slightly degenerate Lagrangians (see Definition 3.10) in $\mathcal{S}$ with the C^∞ -topology. Two slightly degenerate Lagrangians with the same domain I are *isotopic* if they lie in the same component of $\text{Lag}_I^{sg}(\mathcal{S})$.

The following lemma is the reason we introduce slightly degenerate Lagrangians. We call a Lagrangian L *sufficiently self-wrapped* if there are no Reeb chords of positive action with ends on L , i.e. $Y_L^{>0} = \emptyset$.

Lemma 3.18. *Any slightly degenerate Lagrangian L is isotopic to a sufficiently self-wrapped Lagrangian.*

Proof. This follows by induction on $|Y_L^{>0} \cup XY_L|$ since we can always exchange a chord by a double point. Indeed, if $Y_L^{>0} \cup XY_L \neq \emptyset$ and y^0 is the shortest Reeb chord in $Y_L^{>0} \cup XY_L$ then either it is a point-chord, or all its proper sub-chords must have negative action, but as the action is additive with respect to concatenation, y^0 does not have proper sub-chords hence no ends of L in the interior. In any case, L in a neighbourhood of y^0 looks like $L_0 = (I, \gamma_0, P_0)$ in Fig. 8 (if y^0 is

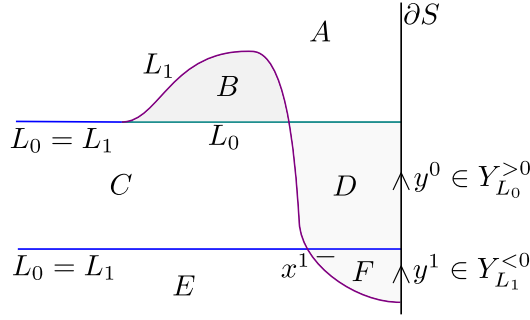


FIGURE 8. Replacing a Reeb chord by a double point.

a point-chord, then we have a slightly different non-diffeomorphic picture, but the reasoning stays the same). Now replace L_0 with a Lagrangian $L_1 = (I, \gamma_1, P_1)$ in Fig. 8 in such a way that the move happens on a subsegment of I (i.e. the teal segment is changed to the purple segment), the darker shaded area is equal to the lighter shaded area (i.e. $\int_B d\lambda = \int_{D \cup F} d\lambda$), and $\int_F d\lambda < A(y^0)$.

Note that for double points we have $X_{L_1} = X_{L_0} \cup \{x^1\}$. For the unique $p \in \partial I$ such that $\gamma_0(p) \neq \gamma_1(p)$ let $r_p = y^0 y^1$, otherwise let r_p be the constant path at $\gamma_0(p) = \gamma_1(p)$. Define the map $\psi: X_{L_0} \cup Y_{L_0} \rightarrow X_{L_1} \cup Y_{L_1}$ by

$$(3.13) \quad \psi(z) = \begin{cases} z, & \text{if } z \in X_{L_0}; \\ x^0, & \text{if } z = y^0; \\ z', & \text{such that } z r_{z_-} = r_{z_+} z' \text{ if } z \in Y_{L_0} \setminus \{y^0\}. \end{cases}$$

In other words, on $Y_{L_0} \setminus \{y^0\}$ the map ψ is defined by continuously deforming the Reeb chord by the isotopy from L_0 to L_1 .

It is clear that ψ is a bijection from $X_{L_0} \cup Y_{L_0}$ to $X_{L_1} \cup Y_{L_1} \setminus \{y^1\}$. If $z \in X_{L_0}$, then $\psi(z) = z$, so $A(z) = A(\psi(z))$. If $z \in Y_{L_0} \setminus \{y^0\}$, then

$$A(\psi(z)) = (P_0(z_+) - l_{r_{z_+}}) - (P_0(z_-) - l_{r_{z_-}}) - (l_z + l_{r_{z_-}} - l_{r_{z_+}}) = A(z).$$

Finally, applying the Stokes theorem to region D and using $\int_B d\lambda = \int_D d\lambda + \int_F d\lambda$, we get

$$A(x^0) = A(y^0) - \int_F d\lambda > 0.$$

In particular, ψ restrict to a bijection $Z_{L_0} \rightarrow Z_{L_1}$ and $|Y_{L_0}^{>0} \cup XY_{L_0}| - 1 = |Y_{L_1}^{>0} \cup XY_{L_1}|$, which finishes the proof of the inductive step. $\square$

We will also need the following observation.

Lemma 3.19. *Let $L_s = (I, \gamma_s, P_s)$, $s \in [0, 1]$ be a path of slightly degenerate Lagrangians such that L_s is a Lagrangian for $s \neq 1$ and L_1 has a unique point-chord $z^1 \in XY_{L_1}$ and L_s is transverse to the slightly degenerate locus. Then there exist smooth families of maps $\Phi^s: S \rightarrow S$, $\psi^s: I \rightarrow I$, $s \in [0, 1]$ such that $\Phi^0 = \text{id}$, $\psi^0 = \text{id}$, $\Phi^s \circ \gamma_0 = \gamma_s \circ \psi^s$ for $s \in [0, 1]$, $\Phi^s \in \text{Diff}(S)$, $\psi^s \in \text{Diff}(I)$ for $s \in [0, 1)$, and $\Phi^1|_{S \setminus (\Phi^1)^{-1}(z^1)}$, $\psi^1|_{I \setminus (\psi^1)^{-1}\{z^1, z^1\}}$ are diffeomorphisms onto their images.*

Proof. Let $y^s, s \in [0, 1]$ be the Reeb chord in L_s that collapses to the point-chord z^1 . There are two cases.

Case 1. $A(y^s) > 0$ for all $s \in [0, 1]$. In this case we can trace each double point to obtain a family $x^s \in X_{L_s}$. Correct γ_s by a family in $\text{Diff}(I)$ in such a way that the points $x^s_{\pm} \in I$ do not depend on s for all families of double points x^s . Then by a family of diffeomorphisms $\Phi_0^s \in \text{Diff}(S)$, $s \in [0, 1]$ we can identify neighbourhoods of y^s with a neighbourhood U of z^1 and make $\Phi_0^s \circ \gamma_0 = \gamma_s$ outside of U , see Fig. 9(a). Applying a family of compactly supported diffeomorphisms of U we can choose local coordinates (x, y) in U centred at z^1 with U given by $\{y \geq 0\}$ such that the two branches of γ_s are parametrised by $[0, 1)$ with $t \mapsto (s + t, t)$ and $t \mapsto (-s - t, t)$ near ∂S .

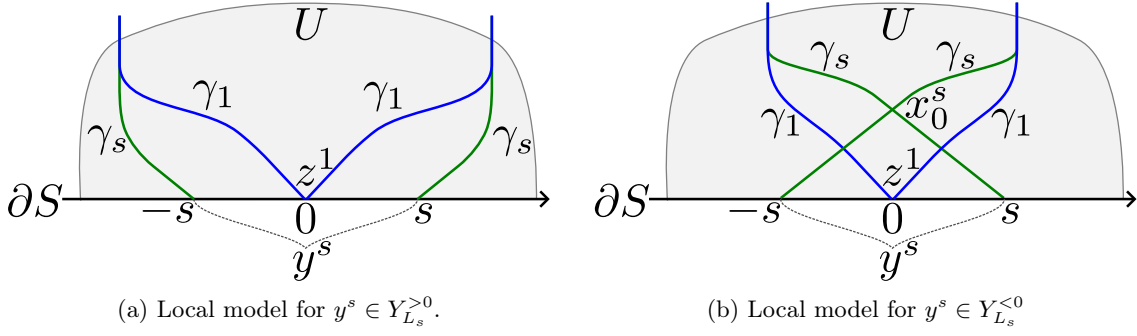


FIGURE 9

There is a family of smooth functions $\varphi^s: \mathbb{R} \rightarrow \mathbb{R}, s \in [0, 1]$ such that $\varphi^s(x) = sx$ for $x \in [-1, 1]$, $\varphi^s(x) = x$ for large $|x|$, φ^s is a diffeomorphism for $s < 1$, and $\varphi^1|_{\mathbb{R} \setminus [-1, 1]}$ is a diffeomorphism. We extend this to a family of smooth functions $\Phi_1^s: \mathbb{R}^2 \rightarrow \mathbb{R}^2, s \in [0, 1]$ such that $\Phi_1^s \circ \gamma_0 = \gamma_s$ near ∂S , $\Phi_1^s(x, y) = (x, y)$ for large $|(x, y)|$, Φ_1^s is a diffeomorphism for $s < 1$ and $\Phi_1^1|_{\mathbb{R}^2 \setminus [-1, 1] \times \{0\}}$ is a diffeomorphism. Finally, we correct by a family of diffeomorphisms compactly supported on $U \setminus \partial S$ to make γ_s agree with $\Phi^s \circ \gamma_0$ everywhere.

Case 2. $A(y^s) < 0$ for all $s \in [0, 1]$. In this cases there is also a double point $x_0^s \in X_{L_s}$ that collapses to the point-chord z^1 . Any other double points we can track to obtain a family $x^s \in X_{L_s} \setminus \{x_0^s\}$. Choose a family of smooth maps $\psi^s: I \rightarrow I, s \in [0, 1]$ such that $\psi^s(x_\pm^0) = x_\pm^s$ for any $x^s \in X_{L_s} \cup XY_{L_s}$, ψ^s is a diffeomorphism for $s < 1$, and $\psi^1|_{(\psi^1)^{-1}\{z_\pm^1, z_\pm^1\}}$ is a diffeomorphism. Then by a family of diffeomorphisms $\Phi_0^s \in \text{Diff}(\mathcal{S}), s \in [0, 1]$ we can identify neighbourhoods of y^s with a neighbourhood U of z^1 and make $\Phi_0^s \circ \gamma_0 = \gamma_s \circ \psi^s$ outside of U , see Fig. 9(b). Applying a family of compactly supported diffeomorphisms of U we can choose local coordinates (x, y) in U centred at z^1 with U given by $\{y \geq 0\}$ such that the two branches of γ_s are parametrised by $[0, 1]$ with $t \mapsto (s - t, t)$ and $t \mapsto (-s + t, t)$ near ∂S , and $x_0^s = (0, s)$. Let T_s be the triangle with vertices $(s, 0), (0, s), (-s, 0)$, i.e. the triangle shrinking to z^1 . There is a family of smooth functions $\Phi_1^s: \mathbb{R}^2 \rightarrow \mathbb{R}^2, s \in [0, 1]$ such that $\Phi_1^s \circ \gamma_0 = \gamma_s \circ \psi^s$ near T_s , $\Phi_1^s(x, y) = (x, y)$ for large $|(x, y)|$, Φ_1^s is a diffeomorphism for $s < 1$ and $\Phi_1^1|_{\mathbb{R}^2 \setminus T_1}$ is a diffeomorphism. Finally, we correct by a family of diffeomorphisms compactly supported on $U \setminus T_s$ to make $\gamma_s \circ \psi^s$ agree with $\Phi^s \circ \gamma_0$ everywhere. $\square$

3.3. Polygons. Fix a Lagrangian $L = (I, \gamma, P)$ in a Liouville marked surface $\mathcal{S} = (S, Q, \lambda)$. We describe immersed polygons bounded by L . Their counts will give A_∞ -operations.

In §3.3.1 we define immersed polygons bounded by L , their virtual dimension, and their positive and negative corners, and relate the area of a polygon to the actions of its corners. In §3.3.2 we define the sets $\Delta(L; z^0; z^n, \dots, z^1)$ of polygons with a positive corner at z^0 and negative corners at $z^1, \dots, z^n$, where we also allow Reeb chords as corners, and prove that only finitely many of them are non-empty in Lemma 3.24. In §3.3.3 we associate to each such polygon a 2-morphism in $\mathcal{P}_2^+(S, Q)$, which determines the contribution of the polygon to the A_∞ -operations via composition along a 2-morphism from §1.1.3.

3.3.1. Polygons bounded by a Lagrangian.

Definition 3.20. A polygon bounded by L is a tuple (Π, u, v) such that

- Π is the compact region in $\mathbb{R}^2$ bounded by a piecewise-smooth Jordan curve;
- $u: (\Pi, \partial\Pi) \rightarrow (S, \text{Im } \gamma \cup (\partial S \setminus \Lambda))$ is an orientation-preserving immersion;
- v is a vertex (i.e. non-smooth point) of Π .

Two such polygons $(\Pi_i, u_i, v_i), i = 0, 1$ are *equivalent* if there exists a diffeomorphism $\phi: \Pi_0 \rightarrow \Pi_1$ such that $u_1 \circ \phi = u_0$ and $\phi(v_0) = v_1$.

Let X be the set of self-intersection points of L and Y the set of Reeb chords between the ends of L , as in §3.2.2. Fix a polygon $\mathbf{u} = (\Pi, u, v^0)$ bounded by L . Note that the set of vertices V of Π is mapped by u to $X \cup \gamma(\partial I)$. Let $W(\mathbf{u}) = \{w^0, \dots, w^n\}$ be the set of connected components of $V \cup u^{-1}(\partial S)$, numbered in the counter-clockwise order on $\partial \Pi$, with $v^0 \in w^0$. For each $w \in W(\mathbf{u})$ either $u(w)$ is a double point of L or $u(w)$ is a Reeb chord between the ends of L , i.e. $u(w) \in X \cup Y$.

Let v be a vertex of Π . We say that v is *convex* (resp. *concave*) if the angle of the tangent cone of Π at v is in $(0, \pi)$ (resp. in $(\pi, 2\pi)$). Note that concave vertices of Π are necessarily mapped to X by u . Call the number of concave vertices of Π the *virtual dimension* of $\mathbf{u}$ and denote it by $\text{vdim } \mathbf{u}$. To define the A_∞ -operations we will only need polygons of virtual dimension zero. To prove the A_∞ -relations we will also need polygons of virtual dimension one.

Fix $w \in W(\mathbf{u})$ such that $u(w)$ is a double point of L (i.e. $u(w) \in X$). Then the image under u of a small neighbourhood of w in Π covers one quadrant of $u(w)$ if w is convex and three quadrants of $u(w)$ if w is concave. If the majority of these quadrants is positive (negative), then we say that $\mathbf{u}$ has a *positive* (*negative*) *corner* at w . We obtain the partition

$$(3.14) \quad \{w \in W(\mathbf{u}) : u(w) \in X\} = W^+(\mathbf{u}) \sqcup W^-(\mathbf{u})$$

into positive and negative corners. Our definition of positive and negative corners follows the convention in Legendrian contact homology, the convention in Lagrangian Floer theory is, unfortunately, the opposite.

The Stokes theorem gives

$$(3.15) \quad \sum_{w \in W^+(\mathbf{u})} A(u(w)) - \sum_{\substack{w \in W(\mathbf{u}) : \\ u(w) \in Y}} A(u(w)) - \sum_{w \in W^-(\mathbf{u})} A(u(w)) = \int_{\Pi} u^* d\lambda > 0.$$

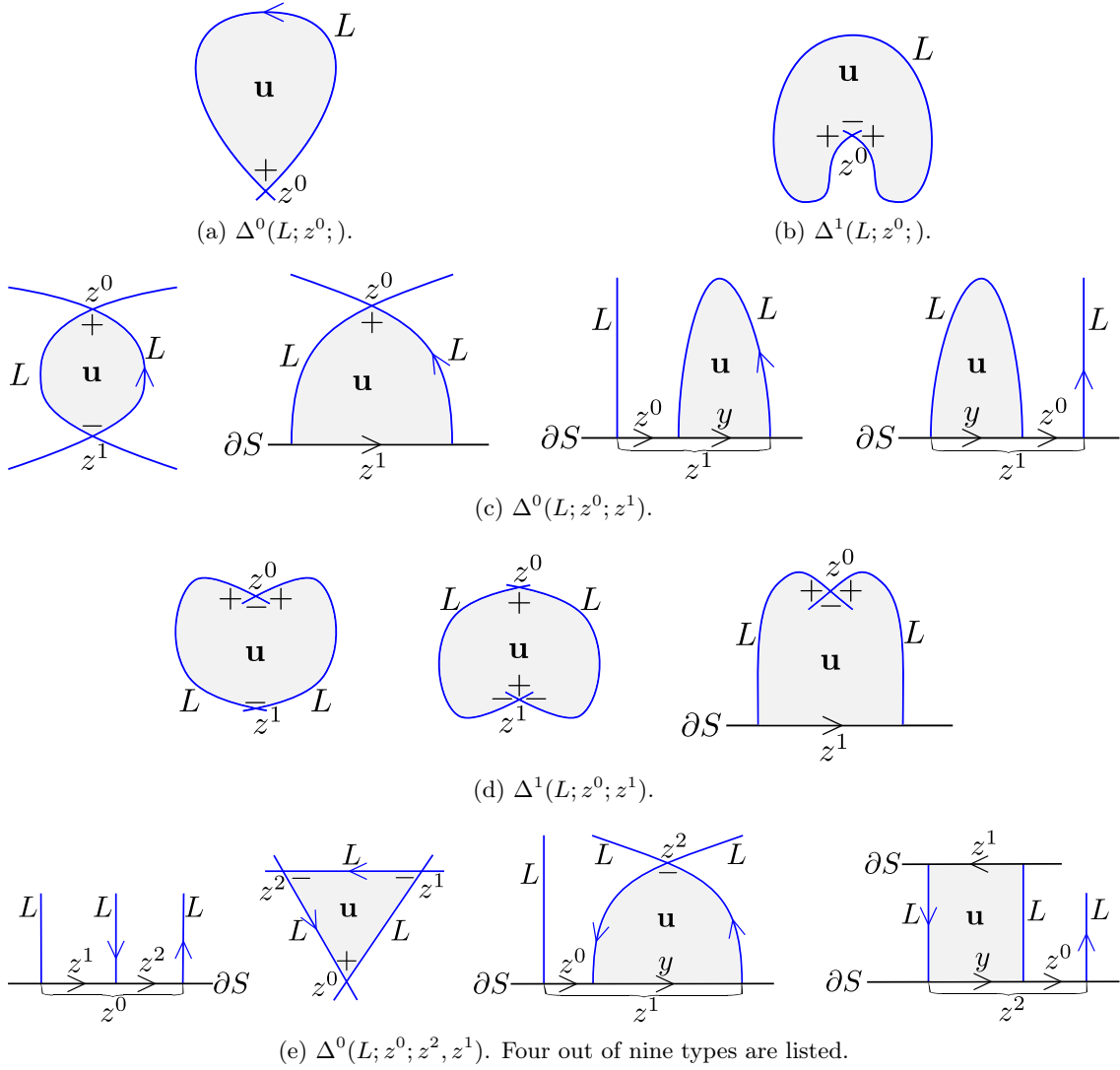
3.3.2. Polygons with one positive corner. Formula (3.15) suggests that the Reeb chords on the boundary of an immersed polygon can be thought of as negative corners. In fact, certain Reeb chords not on the boundary of an immersed polygon can be thought of as its positive corners. In the definition below we make this idea precise by grouping immersed polygons bounded by L based on where their negative and positive ‘corners’ are.

Definition 3.21. Fix $n \geq 0$, and $z^0, z^1, \dots, z^n \in Z = X \cup Y^{>0}$. We define the set $\Delta(L; z^0; z^n, \dots, z^1)$ of polygons bounded by L with a ‘positive corner’ at z^0 and ‘negative corners’ at $z^1, \dots, z^n$.

- (1) If z^0 is a double point of L (i.e. $z^0 \in X$), let $\Delta(L; z^0; z^n, \dots, z^1)$ be the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^n\}$, $W^+(\mathbf{u}) = w^0$, and $u(w^i) = z^i$ for $i = 0, \dots, n$.
- (2) If z^0 is a Reeb chord between the ends of L (i.e. $z^0 \in Y$), let $\Delta(L; z^0; z^n, \dots, z^1)$ be the union of the following:
 - (a) if $z^1 \in Y^{>0}$ and there exists (necessarily unique) $y \in Y$ such that the concatenation yz^0 is equal to z^1 , then the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^{n-1}\}$, $W^+(\mathbf{u}) = \emptyset$, $u(w^0) = y$, and $u(w^i) = z^{i+1}$ for $i = 1, \dots, n-1$;
 - (b) if $z^n \in Y^{>0}$ and there exists (necessarily unique) $y \in Y$ such that the concatenation z^0y is equal to z^n , then the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^{n-1}\}$, $W^+(\mathbf{u}) = \emptyset$, $u(w^i) = z^{i+1}$ for $i = 0, \dots, n-2$, and $u(w^{n-1}) = y$;
 - (c) if $n = 2$, $z^1, z^2 \in Y^{>0}$, and $z^0 = z^2z^1$, then a singleton $\{*\}$, we call its unique element a *degenerate polygon*. Set $\text{vdim}(*) = 0$.

For $k \geq 0$ denote by $\Delta^k(L; z^0; z^n, \dots, z^1)$ the subset of $\Delta(L; z^0; z^n, \dots, z^1)$ consisting of the elements of virtual dimension k .

See Fig. 10 for examples of polygons from Definition 3.21. The pictures of polygons bounded by L that we draw are of the pull-back of a neighbourhood of the polygon $\mathbf{u}$ and Reeb chords involved from S to $\mathbb{R}^2$ via an extension of u . For a polygon $\mathbf{u} = (\Pi, u, v^0) \in \Delta(L; z^0; z^n, \dots, z^1)$

FIGURE 10. Examples of polygons in $\Delta(L; z^0; z^n, \dots, z^1)$.

the vertex v^0 is uniquely determined by the rest of the data, so we suppress it from the notation and write $\mathbf{u} = (\Pi, u)$.

Example 3.22. We continue Example 3.12. For the Lagrangian L_0 in Fig. 7 only the following $\Delta^0(L_0; \dots)$ are non-empty:

$$\begin{aligned} &\Delta^0(L_0; x^1; .), \Delta^0(L_0; x^2; x^3), \\ &\Delta^0(L_0; x^1; y^1), \Delta^0(L_0; x^2; y^2 y^1), \Delta^0(L_0; x^3; y^2), \Delta^0(L_0; x^3; y^4 y^2 y^1), \\ &\Delta^0(L_0; y^2 y^1; y^4 y^2 y^1), \Delta^0(L_0; y^2 y^1; y^2, y^1), \Delta^0(L_0; x^2; y^2, x^1). \end{aligned}$$

For the Lagrangian L_1 in Fig. 7 only the following $\Delta^0(L_1; \dots)$ are non-empty:

$$\Delta^0(L_1; x^1; .), \Delta^0(L_1; x^1; y^1), \Delta^0(L_1; x^1; \bar{x}^3, x^2).$$

These non-empty sets are all singletons.

This example is continued in Example 3.28.

Remark 3.23. If we were to consider moduli spaces of pseudoholomorphic discs, then the set $\Delta^k(L; z^0; z^n, \dots, z^1)$ would match the set of connected components of the corresponding moduli space for $k = 0, 1$.

Below we list elementary properties of $\Delta(L; z^0; z^n, \dots, z^1)$.

Lemma 3.24. *Let L be a Lagrangian in a Liouville marked surface $\mathcal{S}$.*

(1) *Suppose that $\Delta(L; z^0; z^n, \dots, z^1) \neq \emptyset$. Then*

$$0 \leq \sum_{i=1}^n A(z^i) \leq A(z^0).$$

Moreover, the first inequality is strict unless $n = 0$ and the second inequality is strict unless $n = 2$ and $\Delta(L; z^0; z^2, z^1)$ contains a degenerate polygon. In particular, if $\Delta(L; z^0; z^2, z^1)$ contains a degenerate polygon, then $\Delta(L; z^0; z^2, z^1) = \{*\}$.

(2) *The set*

$$\bigcup_{\substack{n \geq 0, \\ z^0, z^1, \dots, z^n \in Z_L}} \Delta(L; z^0; z^n, \dots, z^1)$$

is finite.

Proof. Suppose the case (c) in Definition 3.21 holds, i.e. $z^0 = z^2 z^1$ for $z^0, z^1, z^2 \in Y^{>0}$. Then $0 < A(z^1) + A(z^2) = A(z^0)$.

We prove (1) in the remaining cases. Pick a polygon $\mathbf{u} = (\Pi, u) \in \Delta(L; z^0; z^n, \dots, z^1)$. From (3.15) and additivity of action with respect to the concatenation of Reeb chords we get

$$(3.16) \quad 0 < \int_{\Pi} u^* d\lambda = A(z^0) - \sum_{i=1}^n A(z^i) \leq A(z^0).$$

(1) follows since $A(z^i) > 0$ for all i .

To show (2), as Z_L is finite, (3.16) implies that it suffices to prove that for any $C > 0$ there are only finitely many (equivalence classes of) polygons bounded by L of area not exceeding C . This is indeed the case since each polygon bounded by L is obtained by gluing copies of the regions into which L divides S along the sides of these regions (a side is a smooth segment of the boundary of the region between two double points of L or between a double point of L and an end of L). There are finitely many such regions, each of them has positive area and finitely many sides, so there are only finitely many ways to glue a number of their copies to get the total area of no more than C . $\square$

3.3.3. 2-morphism induced by a polygon. We continue studying polygons with one positive corner bounded by the Lagrangian $L = (I, \gamma, P)$.

Note that each $z \in Z$ gives a path from $\gamma(z_+)$ to $\gamma(z_-)$, which we again denote by z . Namely, if z is a Reeb chord, then the path is z itself and if z is a double point, then the path is the constant path at z . We denote by $\gamma(z_+) \xrightarrow{[z]} \gamma(z_-)$ the corresponding 1-morphism in the path 2-category $\mathcal{P}_2^+(S, Q)$ (see Definition 1.2).

Fix $\mathbf{u} \in \Delta(L; z^0; z^n, \dots, z^1)$. We construct a 2-morphism $[z^0] \xrightarrow{\sigma_{\mathbf{u}}} [\gamma_{\mathbf{u}}]$.

If $\mathbf{u} = (\Pi, u)$ is a polygon bounded by L , then the curve $\gamma_{\mathbf{u}}$ is a piecewise-smooth curve from $\gamma(z_+^0)$ to $\gamma(z_-^0)$ given by the concatenation

$$\gamma(a^n)z^n \dots \gamma(a^1)z^1\gamma(a^0)$$

where $a^i = a_{\mathbf{u}}^i$ for $i = 0, \dots, n$ are paths in I defined as follows.

- If $\gamma(z_+^0) = \gamma(z_+^1)$, then a^0 is the constant path at z_+^0 , otherwise there is a side of $\mathbf{u}$ going from $\gamma(z_+^0)$ to $\gamma(z_+^1)$, and a^0 is the path in I parametrising this side by γ .
- For $i = 1, \dots, n-1$ if $\gamma(z_-^i) = \gamma(z_+^{i+1})$, then a^i is the constant path at z_-^i , otherwise there is a side of $\mathbf{u}$ going from $\gamma(z_-^i)$ to $\gamma(z_+^{i+1})$, and a^i is the path in I parametrising this side by γ .

- If $\gamma(z_-^n) = \gamma(z_-^0)$, then a^n is the constant path at z_-^n , otherwise there is a side of $\mathbf{u}$ going from $\gamma(z_-^n)$ to $\gamma(z_-^0)$ and a^n is the path in I parametrising this side by γ .

The 2-morphism $\sigma_{\mathbf{u}}: [z^0] \Rightarrow [\gamma_{\mathbf{u}}]$ is induced by $\mathbf{u}$. Namely,

- in case (1) of Definition 3.21 $u: \Pi \rightarrow S$ itself is (after a lazy reparametrisation) a 2-morphism from $[z^0]$ to $[\gamma_{\mathbf{u}}]$;
- in case (2a) of Definition 3.21 u gives a 2-morphism from $[\gamma(z_-^0)]$ to $[\gamma(a^n)z^n \dots \gamma(a^1)y]$ where $z^1 = yz^0$, and $\sigma_{\mathbf{u}}$ is obtained from that by pre-composition with $[z^0]$;
- in case (2b) of Definition 3.21 u gives a 2-morphism from $[\gamma(z_+^0)]$ to $[y\gamma(a^{n-1}) \dots \gamma(a^1)z^1\gamma(a^0)]$ where $z^n = z^0y$, and $\sigma_{\mathbf{u}}$ is obtained from that by post-composition with $[z^0]$.
- in case (2c) of Definition 3.21 we have $[\gamma_{\mathbf{u}}] = [z^0] = [z^2z^1]$ and $\sigma_{\mathbf{u}} = \text{id}_{[z^0]}$.

3.4. Grading. We describe the last piece of geometric data, which is needed to make obstruction algebras and Fukaya categories $\mathbb{Z}$ -graded.

Definition 3.25. A *grading* on a marked surface $(S; Q)$ is a *line field* η on S , i.e. an element of $C^\infty(S, \mathbb{P}TS)$.

A *grading* on an immersed curve $\gamma: I \rightarrow S \setminus Q$ is a homotopy class of paths from $\gamma^*\eta$ to the tangent line $\dot{\gamma}$, i.e. an element $\text{gr} \in \Pi_1 C^\infty(I, \gamma^*\mathbb{P}TS)(\gamma^*\eta, \dot{\gamma})$.

Following Lekili–Polishchuk [38, §1.1], we observe that $\pi_0 C^\infty(S, \mathbb{P}TS)$ is a torsor over $H^1(S; \mathbb{Z})$ and each component of $C^\infty(S, \mathbb{P}TS)$ is a $K(\mathbb{Z}, 1)$ space. For a line field η on S the generator of $\pi_1(C^\infty(S, \mathbb{P}TS), \eta) = \mathbb{Z}$ is given by a counter-clockwise rotation of η by π in each tangent space $T_p S$, $p \in S$.

In §3.4.1 we characterise the immersed curves admitting a grading in terms of winding numbers and prove that the gradings on such a curve form a torsor over $\mathbb{Z}^{\pi_0 I}$ in Lemma 3.26. In §3.4.2 we define the degrees of double points and Reeb chords of a graded Lagrangian, check that the two definitions agree on point-chords of slightly degenerate Lagrangians, and relate the degrees of the corners of a polygon to its virtual dimension. In §3.4.3 we recall graded diffeomorphisms and deduce from Proposition 3.4 that graded exact symplectomorphisms form a strong deformation retract of graded diffeomorphisms in Corollary 3.29.

3.4.1. Gradings on a given curve. Consider an immersed curve $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow S \setminus Q$. The tangent line $\dot{\gamma}$ is a section of $\gamma^*\mathbb{P}TS$ and η gives a trivialisation of $\mathbb{P}TS$, hence we obtain the *winding number* $w_\eta(\gamma) \in \mathbb{Z}$ as the degree of $\dot{\gamma}$. Note that the winding number is an invariant of the isotopy class of an immersed curve and of the homotopy class of the line field.

We have the following easy lemma characterising gradings on a given curve.

Lemma 3.26. *Let I be a compact 1-dimensional manifold and $\gamma: I \rightarrow S \setminus Q$ be an immersed curve in a graded surface (S, Q, η) . Then*

- (1) *γ admits a grading $\text{gr} \in \Pi_1 C^\infty(I, \gamma^*\mathbb{P}TS)(\gamma^*\eta, \dot{\gamma})$ if and only if the winding number $w_\eta(\gamma|_C)$ vanishes for each circle component C of I .*
- (2) *If γ admits a grading, then the set of gradings on γ is a torsor over $\mathbb{Z}^{\pi_0 I}$.*

Proof. (1) is clear since given a path gr_t from $\gamma^*\eta(t)$ to $\dot{\gamma}(t)$ for some $t \in I$ there is a homotopically unique way to extend it to a path $i^*\text{gr}$ from $i^*\gamma^*\eta$ to $i^*\dot{\gamma}$ for any immersion $i: [0, 1] \rightarrow C$ with $i(0) = t$. If $i: [0, 1] \rightarrow C$ is a parametrisation of a circle component C of I with $i(0) = i(1) = t$, then the winding number $w_\eta(\gamma|_C)$ is equal to

$$i^* \text{gr}(0)^{-1} \cdot i^* \text{gr}(1) \in \pi_1(\mathbb{P}T_{\gamma(t)} S, \eta(\gamma(t))) = \mathbb{Z}.$$

(2) follows from the above since

$$\pi_1(C^\infty(C, \gamma^*\mathbb{P}TS), \gamma^*\eta) = \pi_1(\mathbb{P}T_{\gamma(t)} S, \eta(\gamma(t))) = \mathbb{Z}$$

for any connected component C of I and $t \in C$. □

Lemma 3.26 implies that grading is discrete data defined for any immersed curve in $S \setminus Q$ hence it is transported along isotopies of such curves.

3.4.2. Degrees of double points and Reeb chords. Let $L = (I, \gamma, P; \text{gr})$ be a graded Lagrangian in a graded Liouville marked surface $\mathcal{S} = (S, Q, \eta, \lambda)$. We define the degree of double points of L and Reeb chords between the ends of L .

The *degree* of a double point $x \in X_L$ is an integer $|x|$ determined as follows. First, obtain an element in $\pi_1(\mathbb{P}T_x \mathcal{S}, \eta(x))$ as a concatenation of $\text{gr}(x_-)$ (which rotates $\eta(x)$ to $\dot{\gamma}(x_-)$), followed by the smallest counter-clockwise rotation from $\dot{\gamma}(x_-)$ to $\dot{\gamma}(x_+)$, and then followed by the inverse of $\text{gr}(x_+)$. Now identify $\pi_1(\mathbb{P}T_x \mathcal{S}, \eta(x))$ with $\mathbb{Z}$ by sending the counter-clockwise rotation by π to 1 to obtain the degree of x .

The *degree* of a Reeb chord $y \in Y_L$ is an integer $|y|$ determined as follows. First, obtain a path from $\eta(\gamma(y_-))$ to $\dot{y}(l_y)$ in $\mathbb{P}(T_{\gamma(y_-)} \mathcal{S})$ as a concatenation of $\text{gr}(y_-)$ and the smallest counter-clockwise rotation from $\dot{\gamma}(y_-)$ to $\dot{y}(l_y)$. Secondly, transport it along $[0, l_y]$ to a path from $\eta(\gamma(y_+))$ to $\dot{y}(0)$ in $\mathbb{P}(T_{\gamma(y_+)} \mathcal{S})$. Thirdly, compose it with the smallest clockwise rotation from $\dot{y}(0)$ to $\dot{\gamma}(y_+)$ and then with the inverse of $\text{gr}(y_+)$ to obtain an element in $\pi_1(\mathbb{P}T_{\gamma(y_+)} \mathcal{S}, \eta(\gamma(y_+)))$. Finally, identify $\pi_1(\mathbb{P}T_{\gamma(y_+)} \mathcal{S}, \eta(\gamma(y_+)))$ with $\mathbb{Z}$ by sending the counter-clockwise rotation by π to 1 to obtain the degree of y . Note that the degree is additive with respect to concatenation of Reeb chords.

Lemma 3.27. *Let L be a slightly degenerate Lagrangian (see Definition 3.10). Then for a point-chord $z \in XY_L$ the degrees of z as a double point and as a Reeb chord are equal.*

Proof. The two definitions of degree agree if and only if the composition of the smallest counter-clockwise rotation from $\dot{\gamma}(z_-^0)$ to $T_z \partial \mathcal{S}$ with the smallest clockwise rotation from $T_z \partial \mathcal{S}$ to $\dot{\gamma}(z_+^0)$ is homotopic to the smallest counter-clockwise rotation from $\dot{\gamma}(z_-^0)$ to $\dot{\gamma}(z_+^0)$. This is equivalent to the unique quadrant of z contained in S being negative. $\square$

Let $\mathbf{u} = (\Pi, u)$ be a polygon bounded by L as in Definition 3.20. The Poincaré–Hopf index theorem gives

$$(3.17) \quad \sum_{w \in W^+(\mathbf{u})} |u(w)| + \sum_{w \in W^-(\mathbf{u})} (1 - |u(w)|) + \sum_{\substack{w \in W(\mathbf{u}): \\ u(w) \in Y}} (1 - |u(w)|) = 2 + \text{vdim } \mathbf{u}.$$

Regrouping the terms for $\mathbf{u} \in \Delta^k(L; z^0; z^n, \dots, z^1)$ we obtain

$$(3.18) \quad |z^0| = 2 - n + |z^1| + \dots + |z^n| + k.$$

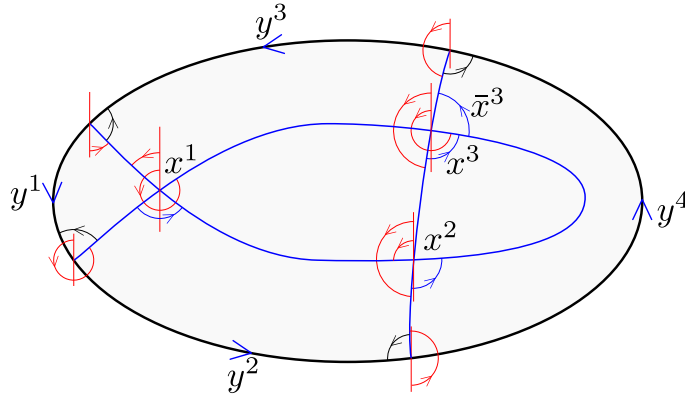


FIGURE 11. Graded Lagrangian in a graded surface.

The vertical red lines are the line field η , the red arcs are the paths from η to the tangent line, the blue arcs are the smallest counter-clockwise rotations of the understrand to the overstrand, the black arcs are the smallest counter-clockwise rotations from the curve to the boundary.

Example 3.28. We continue Example 3.22.

We fix a line field on the disc to be the vertical line at every point and pick the same grading for L_0 and L_1 , as indicated by the red arcs in Fig. 11. The only difference between L_0 and L_1

that affects the degree is that the understrand and the overstrand for x^3 and $\bar{x}^3$ are swapped, the corresponding smallest counter-clockwise rotations are marked with x^3 for L_0 and $\bar{x}^3$ for L_1 .

The degrees of the elements of Z_{L_0} are

$$|x^1| = 2, |x^2| = 1, |x^3| = 0, |y^1| = 1, |y^2| = -1, |y^2 y^1| = 0, |y^4 y^2 y^1| = -1.$$

The degrees of the elements of Z_{L_1} are

$$|x^1| = 2, |x^2| = 1, |\bar{x}^3| = 1, |y^1| = 1.$$

3.4.3. Graded diffeomorphisms. We recall the well-known way to incorporate gradings into diffeomorphisms and symplectomorphisms, see Seidel [50, §2.b] for a more general discussion.

Recall that a diffeomorphism of surfaces $\Phi: S_0 \rightarrow S_1$ induces a bijection

$$\Phi_*: C^\infty(S_0, \mathbb{P}TS_0) \rightarrow C^\infty(S_1, \mathbb{P}TS_1), (\Phi_*\eta)(\Phi(p)) = d_p\Phi(\eta(p)).$$

Let η_i be a line field on S_i for $i = 0, 1$. Then Φ is called *gradable* if $\Phi_*\eta_0$ and η_1 lie in the same component of $C^\infty(S_1, \mathbb{P}TS_1)$. Let Φ^s , $s \in [0, 1]$ be a family in $\text{Diff}(S_0, S_1)$ and $\tilde{\Phi}^0$ a path from $(\Phi^0)_*\eta_0$ to η_1 in $C^\infty(S_1, \mathbb{P}TS_1)$. Then we can get a path $\tilde{\Phi}^1$ from $(\Phi^1)_*\eta_0$ to η_1 by concatenating $s \mapsto \Phi_*^{1-s}\eta_0$ and $\tilde{\Phi}^0$. This implies that in a given component of $\text{Diff}(S_0, S_1)$ either all diffeomorphisms are gradable or none is.

Let $\text{Diff}^{\text{gr}}((S_0, \eta_0), (S_1, \eta_1))$ be the space of pairs $(\Phi, \tilde{\Phi})$ where $\Phi \in \text{Diff}(S_0, S_1)$ is gradable and $\tilde{\Phi}$ is a homotopy class of paths from $\Phi_*\eta_0$ to η_1 , i.e. an element of $\Pi_1 C^\infty(S_1, \mathbb{P}TS_1)(\Phi_*\eta_0, \eta_1)$. The composition is given by

$$(\Phi, \tilde{\Phi}) \circ (\Psi, \tilde{\Psi}) = (\Phi \circ \Psi, \tilde{\Phi} \cdot \Phi_*\tilde{\Psi})$$

where $\cdot$ denotes concatenation.

Consider a graded marked surface (S, Q, η) . Each component of S can be treated separately, so we assume S is connected. The observation after Definition 3.25 gives an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \text{Diff}^{\text{gr}}(S, Q, \eta) \longrightarrow \text{Diff}(S, Q) \longrightarrow H^1(S; \mathbb{Z})$$

where all maps except the one to $H^1(S; \mathbb{Z})$ are group homomorphisms and $H^1(S; \mathbb{Z})$ is considered as a pointed set. Moreover, the projection map $(\Phi, \tilde{\Phi}) \mapsto \Phi$ sends each component of $\text{Diff}^{\text{gr}}(S)$ homeomorphically onto a component of $\text{Diff}(S)$. In particular, the identity component $\text{Diff}(S)_0$ can be identified with the identity component $\text{Diff}^{\text{gr}}(S)_0$.

Now let $\mathcal{S}_i = (S, Q, \eta, \lambda_i)$ be two Liouville surfaces supported on the same (S, Q, η) . Denote by $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}_0, \mathcal{S}_1)$ the subset of $\text{Diff}^{\text{gr}}(S, Q, \eta)$ consisting of the pairs $(\Phi, \tilde{\Phi})$ with $\Phi \in \text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$. Proposition 3.4 and the above discussion yield a strong deformation retraction for graded diffeomorphisms.

Corollary 3.29. *$\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}_0, \mathcal{S}_1)$ is a strong deformation retract of $\text{Diff}^{\text{gr}}(S, Q, \eta)$.*

4. OBSTRUCTION ALGEBRA

In this section we define Lagrangian branes in a weighted Liouville surface, which will be the objects of the geometric Fukaya A_∞ -pre-category.

In §4.1 we define Lagrangian pre-branes and their obstruction algebras, which generalise the Chekanov–Eliashberg algebra to dg-coefficients and to Legendrians with ends on the boundary. In §4.2 we prove that the obstruction algebra does not change under isotopies of slightly degenerate Lagrangians, which in particular allow us to exchange Reeb chords for double points, see Proposition 4.8. In §4.3 we prove that the obstruction algebra is a structured curved A_∞ -algebra in Proposition 4.10. In §4.4 we define Lagrangian branes as Lagrangian pre-branes together with a bounding cochain in the obstruction algebra and relate bounding cochains to augmentations of the Chekanov–Eliashberg algebra.

4.1. Obstruction algebra of a Lagrangian pre-brane. We now combine the algebraic data in §1.1 with the geometric setup described in §3. Namely, we fix a *graded Liouville marked surface* $\mathcal{S} = (S, Q, \eta, \lambda)$ (see Definitions 1.3 and 3.1) and a weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgc}at_k$ (see Definition 1.4). Recall from Introduction that we call $(\mathcal{S}; \mathcal{C})$ a *weighted Liouville surface*.

In §4.1.1 we define Lagrangian pre-branes, which are graded Lagrangians together with a local system of objects over the pullback of $\mathcal{C}$. In §4.1.2 we recall curved A_∞ -algebras and introduce structured A_∞ -algebras, in which sufficiently nested expressions in μ vanish, so that no completion is needed over an arbitrary commutative ring k . In §4.1.3 we define the obstruction algebra $\mathcal{A}_{\mathcal{L}}$ of a Lagrangian pre-brane $\mathcal{L}$, whose entries are given by double points and Reeb chords of positive action and whose structure maps are given by compositions along the 2-morphisms induced by polygons with one positive corner, and relate it to the Chekanov–Eliashberg algebra.

4.1.1. Lagrangian pre-branes. Let $\mathcal{C}$ be a local system of dg-categories on a manifold I , i.e. a functor from the fundamental groupoid $\Pi_1(I)$ to $\text{dgc}at_k$. A *local system of objects* over $\mathcal{C}$ on I is the data of

- $\mathcal{M}(t) \in \text{Ob } \mathcal{C}_t$ for each $t \in I$ and
- $\mathcal{M}(a) \in Z^0 \mathcal{C}_{a(1)}(\mathcal{C}(a)\mathcal{M}(a(0)), \mathcal{M}(a(1)))$ for each path $a: [0, 1] \rightarrow I$

such that

- $\mathcal{M}(a)$ only depends on the homotopy-rel-endpoints class of a ;
- $\mathcal{M}(\text{constant path})$ is the identity;
- $\mathcal{M}(ba) = \mathcal{M}(b) \circ \mathcal{C}(b)\mathcal{M}(a)$ for any two paths $a, b: [0, 1] \rightarrow I$ with $a(1) = b(0)$.

Now let $L = (I, \gamma, P)$ be a Lagrangian in $\mathcal{S}$. Then $\mathcal{C}$ induces a local system $\gamma^*\mathcal{C}$ of dg-categories on I , namely, $\gamma^*\mathcal{C}_t = \mathcal{C}_{\gamma(t)}$ for $t \in I$ and the transition functor $\gamma^*\mathcal{C}(a): \gamma^*\mathcal{C}_{a(0)} \rightarrow \gamma^*\mathcal{C}_{a(1)}$ associated to a path $a: [0, 1] \rightarrow I$ only depends on the homotopy-rel-endpoints class of a since all lazy 2-paths in I are thin homotopies.

Definition 4.1. A *Lagrangian pre-brane* in $(\mathcal{S}; \mathcal{C})$ is the data of

- a graded Lagrangian $L = (I, \gamma, P; \text{gr})$ in $\mathcal{S}$ (see Definitions 3.9 and 3.25) and
- a local system $\mathcal{M}$ of objects over $\gamma^*\mathcal{C}$ on I .

Note that our Lagrangian pre-branes do not carry a Pin structure since the signs coming from the non-triviality of the Pin structure are incorporated into the transition maps of the local system.

4.1.2. Curved A_∞ -algebras. Our primary reference for curved A_∞ -algebras is Fukaya–Oh–Ohta–Ono [23, §§3.2, 3.6, 4.2, 4.3]. The difference with [23] is that there the curved A_∞ -algebras are filtered and supported on a free graded module over the Novikov ring, while our curved A_∞ -algebras are structured and supported on an arbitrary graded module over an arbitrary commutative ring k .

A ($\mathbb{Z}$ -graded) *curved A_∞ -algebra* (over k) is a $\mathbb{Z}$ -graded k -module

$$\mathcal{A} = \bigoplus_{d \in \mathbb{Z}} \mathcal{A}^d$$

together with a collection of linear maps $\mu^n: \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ of degree $2 - n$ for each $n \geq 0$ such that the A_n -relation

$$(4.1) \quad \sum_{n \geq j \geq i \geq 0} (-1)^\dagger \mu^{n-j+1+i} (a^n, \dots, a^{j+1}, \mu^{j-i}(a^j, \dots, a^{i+1}), a^i, \dots, a^1) = 0$$

where $\dagger = |a^i| + \dots + |a^1| - i$ holds for any $n \geq 0$ and homogeneous elements $a^1, \dots, a^n \in \mathcal{A}$.

$\mu^0()$ in (4.1) is interpreted as $\mu^0(1_k) \in \mathcal{A}^2$. The A_0 -relation (4.1) reads $\mu^1(\mu^0()) = 0$ and the non-vanishing of μ^0 is the obstruction to μ^1 being a differential as $\mu^1(\mu^1(a)) = (-1)^{|a|} \mu^2(\mu^0(), a) - \mu^2(a, \mu^0())$ by the A_1 -relation.

An A_∞ -algebra is a curved A_∞ -algebra with $\mu^0 = 0$. We sometimes use the term *uncurved A_∞ -algebra* for emphasis. In an uncurved A_∞ -algebra the A_0 -relation is vacuous and the first three non-trivial relations are

$$(A_1) \quad \mu^1(\mu^1(a_1)) = 0;$$

$$(A_2) \quad \mu^1(\mu^2(a_2, a_1)) + \mu^2(a_2, \mu^1(a_1)) + (-1)^{|a_1|-1} \mu^2(\mu^1(a_2), a_1) = 0;$$

$$(A_3) \quad \begin{aligned} & \mu^2(a_3, \mu^2(a_2, a_1)) + (-1)^{|a_1|-1} \mu^2(\mu^2(a_3, a_2), a_1) + \mu^1(\mu^3(a_3, a_2, a_1)) \\ & + \mu^3(a_3, a_2, \mu^1(a_1)) + (-1)^{|a_1|-1} \mu^3(a_3, \mu^1(a_2), a_1) + \\ & (-1)^{|a_1|+|a_2|-2} \mu^3(\mu^1(a_3), a_2, a_1) = 0. \end{aligned}$$

These relations imply that $a \cdot b = (-1)^{|b|} \mu^2(a, b)$ satisfies the graded Leibniz rule with respect to the differential $da = (-1)^{|a|} \mu^1(a)$ and descends to an associative product on the cohomology of $(\mathcal{A}, d)$.

In curved A_∞ -algebras without any additional structure many natural notions (such as bounding cochains, which we define in §4.4.1) do not make sense since they involve infinite sums. The additional structure we use is the following.

Definition 4.2. A *structured module* is a graded k -module $\mathcal{A}$ together with the data of

- a finite set Z and a graded k -module $\mathcal{A}(z)$ for each $z \in Z$;
- a *degree* function $|\cdot| : Z \rightarrow \mathbb{Z}$;
- a decomposition $\mathcal{A} = \bigoplus_{z \in Z} \mathcal{A}(z)[-|z|]$.

The data of a structured module $\mathcal{A}$ and a curved A_∞ -algebra $(\mathcal{A}, \mu)$ is called a *structured A_∞ -algebra* if there exists an *action* function $A : Z \rightarrow \mathbb{R}_{\geq 0}$ such that if the entry

$$\mu^{z^0; z^n, \dots, z^1} : \mathcal{A}(z^n) \otimes \dots \otimes \mathcal{A}(z^1) \rightarrow \mathcal{A}(z^0)$$

of μ^n is non-zero, then $A(z^0) \geq A(z^n) + \dots + A(z^1)$ and the inequality is strict except possibly in the following two cases:

- (1) $n = 1$ and $z^0 = z^1$;
- (2) $n = 2$.

Note that the action function is not part of the data in Definition 4.2, we explain the reason for that in Remark 4.9. Observe that in a structured A_∞ -algebra $\mu^n = 0$ for $n \gg 0$ and moreover any nested expression involving μ with sufficiently many occurrences of μ or sufficiently many inputs vanishes.

Recall that a curved A_∞ -algebra $(\mathcal{A}, \mu)$ is called *nilpotent* if it possesses a filtration $\mathcal{A} = F^1 \mathcal{A} \supset F^2 \mathcal{A} \supset \dots \supset F^N \mathcal{A} = 0$ such that

$$\mu^n(F^{i_1} \mathcal{A} \otimes \dots \otimes F^{i_n} \mathcal{A}) \subset F^{i_1 + \dots + i_n} \mathcal{A}.$$

The following example shows that a nilpotent A_∞ -algebra supported on a structured module is not necessarily structured.

Example 4.3. Let $k = \mathbb{Z}[x]/x^2$, $Z = \{a, b\}$, $\mathcal{A}(a) = \mathcal{A}(b) = k$, $|b| = |a| + 1 = 0$ and the only non-zero entries of μ be

$$\mu^{b;a} = x \text{ and } \mu^{a;b,b} = x.$$

The A_∞ -relations are satisfied since they are quadratic and $x^2 = 0$. $\mathcal{A}$ is nilpotent as witnessed by the filtration

$$\mathcal{A}(a)[1] \oplus \mathcal{A}(b) \supset x\mathcal{A}(a)[1] \supset x\mathcal{A}(a)[1] \supset 0.$$

However, $\mathcal{A}$ is not structured since we would have $A(b) > A(a) > 3A(b) > 0$ if it was.

4.1.3. *Obstruction algebra.* Let $\mathcal{L} = (L = (I, \gamma, P, \text{gr}), \mathcal{M})$ be a Lagrangian pre-brane in the weighted Liouville surface $(\mathcal{S}; \mathcal{C})$. We define the *obstruction algebra* $\mathcal{A}_{\mathcal{L}}$, which is a structured A_{∞} -algebra over k .

Let Z be the union of the set X of double points of L and the set $Y^{>0}$ of Reeb chords of positive action between the ends of L , as in §3.2.2. Recall that Z is a finite set by Lemma 3.11. We defined the action function $A: Z \rightarrow \mathbb{R}_{>0}$ in §3.2.2 and the degree function $|\cdot|: Z \rightarrow \mathbb{Z}$ in §3.4. As structured k -module we define

$$(4.2) \quad \mathcal{A}_{\mathcal{L}} = \bigoplus_{z \in Z} \mathcal{A}_{\mathcal{L}}(z)[-|z|]$$

where

$$(4.3) \quad \mathcal{A}_{\mathcal{L}}(z) = \begin{cases} \mathcal{C}_z(\mathcal{M}(z_+), \mathcal{M}(z_-)) & \text{if } z \text{ is a double point;} \\ \mathcal{C}_{\gamma(z_-)}(\mathcal{C}(z)\mathcal{M}(z_+), \mathcal{M}(z_-)) & \text{if } z \text{ is a Reeb chord.} \end{cases}$$

To avoid chasing cases in (4.3) we will write $\mathcal{C}(z)$ for the identity functor when z is a double point, so $\mathcal{A}_{\mathcal{L}}(z) = \mathcal{C}_{\gamma(z_-)}(\mathcal{C}(z)\mathcal{M}(z_+), \mathcal{M}(z_-))$ for any $z \in Z$.

The structure maps $\mu^n: \mathcal{A}_{\mathcal{L}}^{\otimes n} \rightarrow \mathcal{A}_{\mathcal{L}}[2-n]$ are defined entry-wise. For $n \geq 0$ and $z^0, \dots, z^n \in Z_L$ the entry

$$(4.4) \quad \mu^{z^0; z^n, \dots, z^1}: \mathcal{A}_{\mathcal{L}}(z^n) \otimes \dots \otimes \mathcal{A}_{\mathcal{L}}(z^1) \rightarrow \mathcal{A}_{\mathcal{L}}(z^0)$$

is given by

$$(4.5) \quad \mu^{z^0; z^0}(\varphi) = (-1)^{|\varphi|+|z^0|} d_{\mathcal{A}_{\mathcal{L}}(z^0)}(\varphi)$$

if $n = 1$ and $z^1 = z^0$, and in all other cases by

$$(4.6) \quad \mu^{z^0; z^n, \dots, z^1}(\varphi^n, \dots, \varphi^1) = (-1)^{\sum_{j>k} (|z^j|-1)|\varphi^k|} \sum_{\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)} \mu^{\mathbf{u}}(\varphi^n, \dots, \varphi^1) \cdot \prod_{i=0}^n (\varepsilon_i^{\mathbf{u}})^{|z^i|}$$

where for $\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)$ the map $\mu^{\mathbf{u}}$ and the signs $\varepsilon_i^{\mathbf{u}}$ are defined as explained below.

Note that $\Delta^0(L; z^0; z^n, \dots, z^1)$ is non-empty only if $z_+^0 \in C_0$, $z_-^0 \in C_n$ and $z_+^i \in C_{i-1}$, $z_-^i \in C_i$ for $i = 1, \dots, n$ for some components $C_0, \dots, C_n$ of I . We first describe a choice of orientation of $C_1, \dots, C_n$ induced by $\mathbf{u}$ that extends the one from Seidel [52, Figure 13.1]. The sign $\varepsilon_i^{\mathbf{u}}$ is equal to 1 (resp. (-1)) if the orientation of the component C with $z_-^i \in C$ as a component of I coincides with (resp. is opposite to) the orientation of C induced by $\mathbf{u}$.

Definition 4.4.

- Suppose $\mathbf{u}$ is a degenerate polygon, i.e. $n = 2$, $z^0, z^1, z^2 \in Y^{>0}$ and $z^0 = z^2 z^1$. The *orientation induced by $\mathbf{u}$* of C_2 (resp. C_1) is orientation induced by a tangent vector to $\gamma(C_2)$ at $\gamma(z_-^0) \in \partial S$ pointing inside S (resp. to $\gamma(C_1)$ at $\gamma(z_-^1) \in \partial S$ pointing outside S).
- Suppose $\mathbf{u} = (\Pi, u)$ is a polygon bounded by L . For $i = 1, \dots, n$ let the orientation of C_i *induced by $\mathbf{u}$* be the orientation of $\partial\Pi$ if there is a vertex of $\partial\Pi$ corresponding to z_-^i , or if this is not the case, i.e. we need to determine the orientation of C_n in the case (2b) of Definition 3.21, then the orientation of C_n induced by a tangent vector to $\gamma(C_n)$ at $\gamma(z_-^n) \in \partial S$ pointing inside S .

See the blue arrows in Fig. 10 for examples of orientations defined above.

Finally, given $\varphi^i \in \mathcal{C}_{\gamma(z_-^i)}(\mathcal{C}(z^i)\mathcal{M}(z_+^i), \mathcal{M}(z_-^i))$ for $i = 1, \dots, n$, we define $\mu^{\mathbf{u}}(\varphi^n, \dots, \varphi^1)$ as the composition along a 2-morphism (see §1.1.3)

$$(4.7) \quad \begin{aligned} & \mathcal{C}(z^0; \gamma(a^n), \quad z^n, \dots, \quad \gamma(a^1), z^1, \quad \gamma(a^0); \sigma_{\mathbf{u}}) \\ & (\mathcal{M}(a^n), \quad \varphi^n, \dots, \quad \mathcal{M}(a^1), \varphi^1, \quad \mathcal{M}(a^0)) \end{aligned}$$

where the curves $a^i = a_{\mathbf{u}}^i$ for $i = 0, \dots, n$, the curve $\gamma_{\mathbf{u}}$, and the 2-morphism $\sigma_{\mathbf{u}}: [z^0] \Rightarrow [\gamma_{\mathbf{u}}]$ in $\mathcal{P}_2^+(S, Q)$ are defined in §3.3.3.

We prove that $\mathcal{A}_{\mathcal{L}}$ is indeed a structured A_{∞} -algebra in Proposition 4.10.

Remark 4.5. Assume that $\mathcal{A}_{\mathcal{L}}(z)$ is a finitely generated projective graded k -module for each $z \in Z$. Applying the component-wise k -linear dual to the bar construction on the obstruction algebra $(\mathcal{A}_{\mathcal{L}}, \mu)$ we get a differential graded algebra structure on

$$B\mathcal{A}_{\mathcal{L}}^{\vee} = \bigoplus_{d=0}^{\infty} (\mathcal{A}_{\mathcal{L}}^{\vee}[-1])^{\otimes d}.$$

More concretely, the algebra structure comes from the tensor product and the differential is given by

$$\partial(az^0) = \pm(\mu^{z^0; z^n, \dots, z^1})^{\vee}(a) \cdot z^1 \dots z^n \text{ for } z^0, \dots, z^n \in Z_L.$$

and the graded Leibniz rule. The A_{∞} -relations translate to $\partial^2 = 0$.

What we get is a version of the Chekanov–Eliashberg algebra. Namely, if S is a disc, $k = \mathbb{F}_2$, $\mathcal{C}$ is the trivial 2-functor with fibre k (thought of as dg-category with one object), then

- when $Q = \emptyset$ and the domain of L is closed, we obtain the DGA of a Legendrian link;
- when $D = \emptyset$, $|\Lambda| = 2$, the domain of L is a segment, and the two ends of L lie in different components of $\partial S \setminus \Lambda$, we obtain the DGA of a long Legendrian knot

both defined by Chekanov [13]. More generally, modulo sign conventions, when $Q = \emptyset$, k is any commutative ring, $\mathcal{C}$ is the trivial 2-functor with fibre k , and the domain of L is closed, we obtain the DGA defined by Ekholm–Etnyre–Sullivan [18, 19].

4.2. Invariance of the obstruction algebra. We prove that the obstruction algebra is invariant under isotopies of slightly degenerate Lagrangian pre-branes (see Definition 3.10). For this first observe that in §4.1.3 the definition goes through if we allow Reeb chords of zero length, so we define the obstruction algebra of a slightly degenerate Lagrangian pre-brane by treating the point-chords as Reeb chords. Throughout the subsection we work with a fixed weighted Liouville surface $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C})$.

In §4.2.1 we define naïve isomorphisms, the sense in which the obstruction algebra is invariant. In §4.2.2 we transport the grading and the local system of a Lagrangian pre-brane along an isotopy of slightly degenerate Lagrangians and prove that the obstruction algebra is invariant in Proposition 4.8.

Remark 4.6. We expect that, similarly to the classical Chekanov–Eliashberg algebra, the obstruction algebra is invariant with respect to more general isotopies of the Legendrian lift of a Lagrangian pre-brane. The invariance in that case should be understood as up to an A_{∞} -isomorphism. We will not treat it in this paper.

4.2.1. Naïve isomorphisms. By invariance of the obstruction algebra we mean that all data defining it is preserved in the naïve sense. Namely, a *naïve morphism* of two structured modules with higher products (which are not required to satisfy the A_{∞} -relations) $(\mathcal{A}_i, Z_i, \mu_i^0, \mu_i^1, \dots)$, $i = 0, 1$ is

- a map $\psi: Z_0 \rightarrow Z_1$;
- a morphism of graded k -modules $\mathcal{F}^z: \mathcal{A}_0(z) \rightarrow \mathcal{A}_1(\psi(z))$ for each $z \in Z_0$

such that

- $|\psi(z)| = |z|$ for each $z \in Z_0$;
- $\mu_1^{\psi(z^0); \psi(z^n), \dots, \psi(z^1)}(\mathcal{F}^{z^n}(-), \dots, \mathcal{F}^{z^1}(-)) = \mathcal{F}^{z^0} \circ \mu_0^{z^0; z^n, \dots, z^1}(-, \dots, -)$ for all $z^0, \dots, z^n \in Z_0$.

Structured modules with higher products together with naïve morphisms form a category under the component-wise composition and a *naïve isomorphism* is an isomorphism in this category. We have the following standard observation.

Lemma 4.7. *A naïve morphism $(\psi, (\mathcal{F}^z)_{z \in Z_0})$ is an isomorphism if and only if ψ is a bijection and $\mathcal{F}^z$ is an isomorphism for each $z \in Z_0$.*

4.2.2. Invariance under isotopies. Let $\mathcal{L}_0 = (I, \gamma_0, P_0, \text{gr}_0, \mathcal{M}_0)$ be a Lagrangian pre-brane and $L_1 = (I, \gamma_1, P_1)$ a Lagrangian isotopic to $L_0 = (I, \gamma_0, P_0)$ as a slightly degenerate Lagrangian, see §3.2.6.

We want to transport gr_0 and $\mathcal{M}_0$ to get a Lagrangian pre-brane $\mathcal{L}_1 = (L_1, \text{gr}_1, \mathcal{M}_1)$ and prove that obstruction algebras $(\mathcal{A}_{\mathcal{L}_0}, \mu_0)$ and $(\mathcal{A}_{\mathcal{L}_1}, \mu_1)$ are isomorphic. Since a generic isotopy

intersects the slightly degenerate locus transversely, it is enough to consider the case when $L_s = (I, \gamma_s, P_s)$, $s \in [0, 1]$ is a path of slightly degenerate Lagrangians such that L_s is a Lagrangian for $s \neq 1$ and L_1 is either a Lagrangian or has a unique point-chord $z^1 \in XY_{L_1}$, i.e. we are in the setting of Lemma 3.19.

By Lemma 3.26 the grading gr_0 transports to a grading gr_1 on L_1 . We define the local system $\mathcal{M}_1$ by $\mathcal{M}_1(p) = \mathcal{C}[s \mapsto \gamma_s(p)]\mathcal{M}_0(p)$ for $p \in I$ and for a path $a: [0, 1] \rightarrow I$, the transition map

$$\mathcal{M}_1(a) \in \mathcal{C}_{\gamma_1 a(1)}(\mathcal{C}(\gamma_1 a)\mathcal{M}_1(a(0)), \mathcal{M}_1(a(1)))$$

is the composition along a 2-morphism (see §1.1.3)

$$(4.8) \quad \mathcal{C}(\gamma_1 a; s \mapsto \gamma_s a(1), \gamma_0 a, s \mapsto \gamma_{1-s} a(0); \sigma_a)(\text{id}_{\mathcal{M}_1(a(1))}, \mathcal{M}_0(a), \text{id}_{\mathcal{M}_0(a(0))})$$

where σ_a is the 2-morphism in $\mathcal{P}_2^+(S, Q)$ given by $\gamma_s a(t)$.

A naïve isomorphism $(\psi, \mathcal{F}): \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ is constructed as follows. Recall from Lemma 3.19 that we have the Reeb chords $y_0^s, s \in [0, 1]$ of L_s collapsing to the point-chord z^1 and that there are two cases.

Case 1. $A(y_0^s) > 0$ for all $s \in [0, 1]$. In this case we can track each double point $x^0 \in X_{L_0}$ to obtain a family $x^s \in X_{L_s}, s \in [0, 1]$ and can track each Reeb chord $y^0 \in Y_{L_0} \setminus \{y_0^0\}$ to get a family $y^s \in Y_{L_s}, s \in [0, 1]$, and moreover $A(y^s)$ does not change sign in the family. We then define $\psi(x^0) = x^1$ for each $x^0 \in X_{L_0}$, $\psi(y^0) = y^1$ for each $y^0 \in Y_{L_0}^{>0} \setminus \{y_0^0\}$ and $\psi(y_0^0) = z^1$.

Case 2. $A(y_0^s) < 0$ for all $s \in [0, 1]$. In this case there is a family of double points $x_0^s \in X_{L_s}$ that collapses to the point-chord z^1 . We can track each double point $x^0 \in X_{L_0} \setminus \{x_0^0\}$ to obtain a family $x^s \in X_{L_s}, s \in [0, 1]$ and can track each Reeb chord $y^0 \in Y_{L_0} \setminus \{y_0^0\}$ to get a family $y^s \in Y_{L_s}, s \in [0, 1]$, and moreover $A(y^s)$ does not change sign in the family. We then define $\psi(x^0) = x^1$ for each $x^0 \in X_{L_0} \setminus \{x_0^0\}$, $\psi(y^0) = y^1$ for each $y^0 \in Y_{L_0}^{>0}$ and $\psi(x_0^0) = z^1$.

Fix $z^0 \in Z_0$. From the reasoning above we can always track z^0 to obtain the family $z^s \in Z_s, s \in [0, 1]$. To define

$$\mathcal{F}^{z^0}: \mathcal{C}_{\gamma_0(z^0)}(\mathcal{C}(z^0)\mathcal{M}_0(z_+^0), \mathcal{M}_0(z_-^0)) \rightarrow \mathcal{C}_{\gamma_1(z^1)}(\mathcal{C}(z^1)\mathcal{M}_1(z_+^1), \mathcal{M}_1(z_-^1))$$

first take into account the moving of $z_\pm^s$ in I . Namely, consider the morphism

$$\begin{aligned} \mathcal{M}(s \mapsto \gamma_s(z_-^s)) &= \mathcal{C}(s \mapsto \gamma_s(z_-^s); s \mapsto \gamma_s(z_-^1) \quad , s \mapsto \gamma_0(z_-^s) \quad ; [\gamma_s(z_-^t), s \leq t]) \\ &\quad (\text{id} \quad , \mathcal{M}_0(s \mapsto z_-^s) \quad) \end{aligned}$$

in $\mathcal{C}_{\gamma_1(z^1)}(\mathcal{C}(s \mapsto \gamma_s(z_-^s))\mathcal{M}_0(z_-^0), \mathcal{M}_1(z_-^1))$ and the morphism

$$\begin{aligned} \mathcal{M}(s \mapsto \gamma_{1-s}(z_+^{1-s})) &= \mathcal{C}(s \mapsto \gamma_{1-s}(z_+^{1-s}); s \mapsto \gamma_0(z_+^{1-s}) \quad , s \mapsto \gamma_{1-s}(z_+^1) \quad ; [\gamma_{1-t}(z_+^{1-s}), s \leq t]) \\ &\quad (\mathcal{M}_0(s \mapsto z_+^{1-s}) \quad , \text{id} \quad) \end{aligned}$$

in $\mathcal{C}_{\gamma_0(z^0)}(\mathcal{C}(s \mapsto \gamma_{1-s}(z_+^{1-s}))\mathcal{M}_1(z_+^1), \mathcal{M}_0(z_+^0))$. Note that Lemma 1.6 implies that they are both isomorphisms since the 2-morphisms $[\gamma_s(z_-^t), s \leq t]$ and $[\gamma_{1-t}(z_+^{1-s}), s \leq t]$ are invertible in $\mathcal{P}_2^+(S)$.

Now define $\mathcal{F}^{z^0}(\varphi)$ to be

$$\begin{aligned} &\mathcal{C}(z^1; s \mapsto \gamma_s(z_-^s) \quad , z^0 \quad , s \mapsto \gamma_{1-s}(z_+^{1-s}) \quad ; [(s, t) \mapsto z^s(t)]) \\ &\quad (\mathcal{M}(s \mapsto \gamma_s(z_-^s)) \quad , \varphi \quad , \mathcal{M}(s \mapsto \gamma_{1-s}(z_+^{1-s})) \quad). \end{aligned}$$

Note that $\sigma_{z^0} = [(s, t) \mapsto z^s(t)]$ is thin (because either it does not depend on t or it is contained in ∂S), so in fact $[(s \mapsto \gamma_s(z_-^s))z^0(s \mapsto \gamma_{1-s}(z_+^{1-s}))] = [z^1]$ and

$$(4.9) \quad \mathcal{F}^{z^0}(\varphi) = \mathcal{M}(s \mapsto \gamma_s(z_-^s)) \cdot \mathcal{C}(s \mapsto \gamma_s(z_-^s))\varphi \cdot \mathcal{C}((s \mapsto \gamma_s(z_-^s))z^0)\mathcal{M}(s \mapsto \gamma_{1-s}(z_+^{1-s})).$$

Proposition 4.8. $(\psi, \mathcal{F}): \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ defined above is a naïve isomorphism.

Remark 4.9. The isomorphism in Proposition 4.8 does not necessarily preserve the action, i.e. if the double points or Reeb chords $x^0, y^0 \in Z_0$ correspond to $x^1, y^1 \in Z_1$, then it is possible to have $A(x^0) < A(y^0)$, but $A(x^1) > A(y^1)$.

Proof. ψ is a bijection by construction, and it preserves degree by Lemmas 3.26 and 3.27. As $\mathcal{M}(s \mapsto \gamma_s(z_-^s))$ and $\mathcal{M}(s \mapsto \gamma_{1-s}(z_+^{1-s}))$ are isomorphisms, (4.9) implies that $\mathcal{F}^z$ is an isomorphism of dg-modules. Therefore, by Lemma 4.7 it remains to prove that the higher products given by (4.6) are preserved.

Fix $z_0^0, \dots, z_n^0 \in Z_0$ and note that the sign before the summation in (4.6) is the same since ψ and $\mathcal{F}^z$ preserve the degree.

Recall that by Lemma 3.19 we have smooth families of maps $\Phi^s: S \rightarrow S$, $\psi^s: I \rightarrow I$, $s \in [0, 1]$ such that $\Phi^0 = \text{id}$, $\psi^0 = \text{id}$, $\Phi^s \circ \gamma_0 = \gamma_s \circ \psi^s$ for $s \in [0, 1]$, $\Phi^s \in \text{Diff}(S)$, $\psi^s \in \text{Diff}(I)$ for $s \in [0, 1]$, and $\Phi^1|_{S \setminus (\Phi^1)^{-1}(z^1)}$, $\psi^1|_{I \setminus (\psi^1)^{-1}\{z_+^1, z_-^1\}}$ are diffeomorphisms onto their images.

Claim. Φ^1 establishes a bijection

$$\Delta^0(L_0; z_0^0; z_n^0, \dots, z_1^0) \rightarrow \Delta^0(L_1; z_0^1; z_n^1, \dots, z_1^1), \mathbf{u}^0 \mapsto \mathbf{u}^1$$

Proof of Claim. If none of $z_0^1, \dots, z_n^1$ is the point-chord z^1 , then there is nothing to prove. If we are in Case 1 above, i.e. the family of Reeb chord y_0^s collapsing to z^1 have positive action, then there is also nothing to prove since we treat the point-chord as a Reeb chord. Therefore suppose we are in Case 2, and let x_0^s be the family of double points collapsing to z^1 .

If $\mathbf{u}^1$ is a degenerate polygon, i.e. $z_0^1 = z_2^1 z_1^1$, then, since z^1 has no non-trivial subchords, either $z_1^1 = z^1$ or $z_2^1 = z^1$. In either case, $\mathbf{u}^0$ is the triangle with vertices x_0^0 , $\gamma_0(y_0^0)$, $\gamma_0(y_0^0)$, see Figs. 12 and 13.

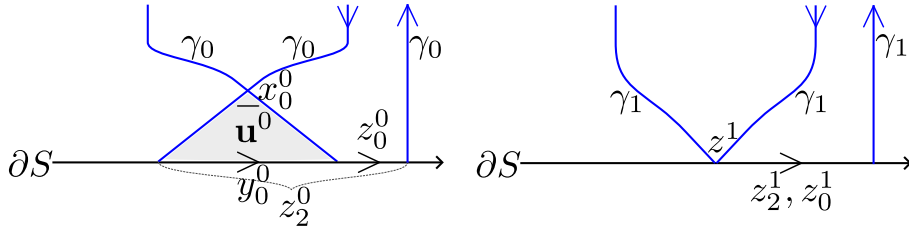


FIGURE 12. $\Delta(L_0; z_0^0; z_2^0, x_0^0) \rightarrow \Delta(L_1; z_0^1; z_2^1, z^1)$.

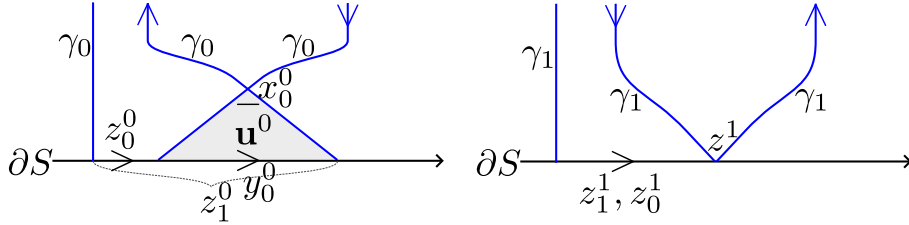


FIGURE 13. $\Delta(L_0; z_0^0; x_0^0, z_1^0) \rightarrow \Delta(L_1; z_0^1; z^1, z_1^1)$.

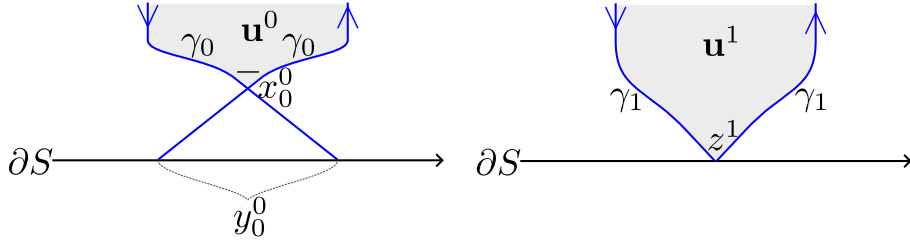
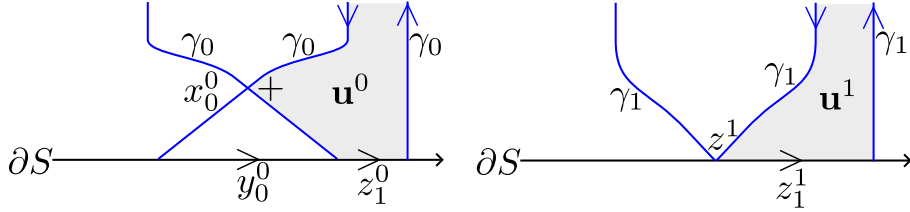
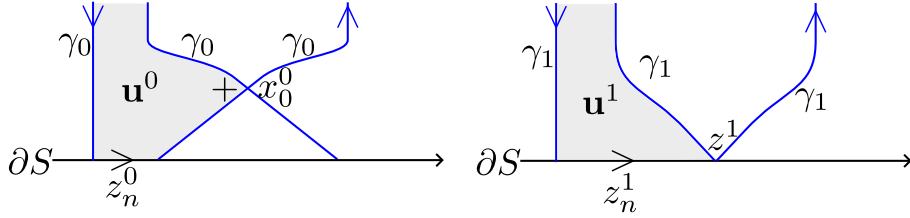
Now suppose $\mathbf{u}^1$ is a non-degenerate polygon. If $z^1 = z_i^1$, then $z_i^0 = x_0^0$ and the polygons $\mathbf{u}^0$ and $\mathbf{u}^1$ look like in Fig. 14.

Finally, if $z^1 = z_0^1$, then, depending on whether we are in the case (2a) or case (2b) of Definition 3.21, the polygons $\mathbf{u}^0$ and $\mathbf{u}^1$ look like in Figs. 15 and 16.

We organised the cases above depending on what happens at $s = 1$. Observe that at $s = 0$ the cases depend on what corner of x_0^0 is covered by $\mathbf{u}^0$, and we covered all possibilities. This observation finishes the proof of the claim. $\square$

We therefore have a bijection between the summands in (4.6) for $\mathcal{L}_0$ and $\mathcal{L}_1$. Fix $\mathbf{u}^0 \in \Delta^0(L_0; z_0^0; z_n^0, \dots, z_1^0)$. For the signs we have

$$\prod_{i=0}^n (\varepsilon_i^{\mathbf{u}^0})^{|z_i^0|} = \prod_{i=0}^n (\varepsilon_i^{\mathbf{u}^1})^{|z_i^1|}$$

FIGURE 14. $\Delta(L_0; z_0^0; \dots, x_0^0, \dots) \rightarrow \Delta(L_1; z_1^1; \dots, z^1, \dots)$.FIGURE 15. $\Delta(L_0; x_0^0; \dots, z_1^0) \rightarrow \Delta(L_1; z_1^1; \dots, z_1^1)$.FIGURE 16. $\Delta(L_0; x_0^0; z_n^0, \dots) \rightarrow \Delta(L_1; z_1^1; z_n^1, \dots)$.

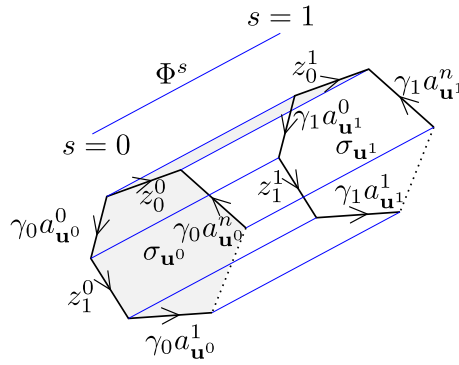
since Φ^1 is orientation-preserving and in Figs. 12 to 16 the orientations from Definition 4.4 are the same except for Fig. 13, for which we have

$$\prod_{i=0}^2 \left(\frac{\varepsilon_i^{\mathbf{u}^1}}{\varepsilon_i^{\mathbf{u}^0}} \right)^{|z_i^0|} = (-1)^{|z_0^0|} (-1)^{|z_1^0|} (-1)^{|z_2^0|} = (-1)^{|z_1^0|+|z_2^0|} (-1)^{|z_1^0|} (-1)^{|z_2^0|} = 1.$$

It therefore remains to check that

$$(4.10) \quad \mu^{\mathbf{u}^1}(\mathcal{F}^{z_0^0}(\varphi^n), \dots, \mathcal{F}^{z_1^0}(\varphi^1)) = \mathcal{F}^{z_0^0} \circ \mu^{\mathbf{u}^0}(\varphi^n, \dots, \varphi^1).$$

This follows by Lemma 1.6 from the isotopy given by Φ^s in Fig. 17.

FIGURE 17. Isotopy of 2-morphism from Φ^s .

Namely, the right-hand side of (4.10) is given by

$$(4.11) \quad \mathcal{C}\left(z_0^1; [s \mapsto \gamma_s(z_{0-}^s)], \quad \gamma_0(a_{\mathbf{u}^0}^n), \dots, \quad \gamma_0(a_{\mathbf{u}^0}^1), z_1^0, \quad \gamma_0(a_{\mathbf{u}^0}^0), [s \mapsto \gamma_{1-s}(z_{0+}^{1-s})]; \sigma_{z_0^0} \cup \sigma_{\mathbf{u}^0}\right) \\ \left(\mathcal{M}(s \mapsto \gamma_s(z_{0-}^s)), \quad \mathcal{M}_0(a_{\mathbf{u}^0}^n), \dots, \quad \mathcal{M}_0(a_{\mathbf{u}^0}^1), \varphi^1, \quad \mathcal{M}_0(a_{\mathbf{u}^0}^0), \mathcal{M}(s \mapsto \gamma_{1-s}(z_{0+}^{1-s}))\right),$$

and the left-hand side of (4.10) is given by

$$(4.12) \quad \mathcal{C}\left(z_0^1; \gamma_1(a_{\mathbf{u}^1}), \quad \dots, [s \mapsto \gamma_s(z_{1-}^s)], \quad z_1^0, [s \mapsto \gamma_{1-s}(z_{1+}^{1-s})], \quad \gamma_1(a_{\mathbf{u}^1}^0); \sigma\right) \\ \left(\mathcal{M}_1(a_{\mathbf{u}^1}^n), \quad \dots, \mathcal{M}(s \mapsto \gamma_s(z_{1-}^s)), \quad \varphi^1, \mathcal{M}(s \mapsto \gamma_{1-s}(z_{1+}^{1-s})), \quad \mathcal{M}_1(a_{\mathbf{u}^1}^0)\right)$$

where $\sigma = \sigma_{\mathbf{u}^1} \cup \sigma_{z_1^0} \cup \dots \cup \sigma_{z_n^0}$. Expanding $\mathcal{M}$ and $\mathcal{M}_1$ by definition we see that (4.11) is equal to the composition along the shaded part of the boundary of the prism in Fig. 17 and (4.12) is equal to the composition along the unshaded part of the boundary of the prism. Thus the solid prism $\Phi^s \circ \sigma_{\mathbf{u}^0}$ provides the required isotopy. $\square$

4.3. A_∞ -relations. Let $\mathcal{L} = (L = (I, \gamma, P), \text{gr}, \mathcal{M})$ be a Lagrangian pre-brane in $(\mathcal{S}; \mathcal{C})$. Recall that in §4.1.3 we defined the obstruction algebra $\mathcal{A}_{\mathcal{L}}$ of $\mathcal{L}$.

Proposition 4.10. *The obstruction algebra $\mathcal{A}_{\mathcal{L}}$ is a structured A_∞ -algebra over k .*

The argument is based on the decomposition of polygons with one concave corner by Chekanov [13, Theorem 3.3]. There are versions of this argument that take into account signs or local systems, see e.g. Abouzaid [1, §3] for the case of immersed curves in closed surfaces that do not bound any teardrops and Etnyre–Ng [21, §3] for an overview of various flavours of Chekanov–Eliashberg DGAs.

Proof. The set $Z = Z_L$ of double points of L and Reeb chords between the ends of L of positive action is finite by Lemma 3.11. The sums defining the structure maps μ^n in (4.6) are finite by Lemma 3.24(2). The degree of any μ^n , $n \geq 0$ is equal to $2 - n$ by (3.18). The action property from Definition 4.2 is satisfied by Lemma 3.24(1). It remains to check the A_∞ -relations (4.1). By Lemma 3.18 and Proposition 4.8 we can assume L is sufficiently self-wrapped.

Let $C_0, \dots, C_n$ be (not necessarily distinct) connected components of I and $x^0, \dots, x^n \in X_L = Z_L$ be such that

$$x_+^0 \in C_0, x_-^0 \in C_n \text{ and } x_+^i \in C_{i-1}, x_-^i \in C_i \text{ for } i = 1, \dots, n.$$

We show that the entry

$$(4.13) \quad \mathcal{A}_{\mathcal{L}}(x^n) \otimes \dots \otimes \mathcal{A}_{\mathcal{L}}(x^1) \rightarrow \mathcal{A}_{\mathcal{L}}(x^0)$$

of the map in the left hand side of the A_∞ -relation (4.1) vanishes.

We first handle the part of the A_∞ -relations coming from the dg-coefficients, i.e. given by (4.5). $\mu^{x^0; x^0}$ squares to 0 since $d_{\mathcal{A}_{\mathcal{L}}(x^0)}$ is a differential. Other summands of (4.13) involving (4.5) group into

$$\mu^{x^0; x^0} \circ \mu^{\mathbf{u}}(\varphi^n, \dots, \varphi^1) + \sum_{i=1}^n (-1)^{\sum_{j < i} (|x^j| + |\varphi^j| - 1) + \sum_{j > i} (|x^j| - 1)} \mu^{\mathbf{u}}(\varphi^n, \dots, \mu^{x^i; x^i}(\varphi^i), \dots, \varphi^1) = \\ (-1)^{\sum_{i=1}^n (|\varphi^i| + |x^i| - 1)} d(\mathcal{C}(x^0; \gamma_{\mathbf{u}}^n, x^n, \dots, \gamma_{\mathbf{u}}^1, x^1, \gamma_{\mathbf{u}}^0; \sigma_{\mathbf{u}})(\mathcal{M}(\gamma_{\mathbf{u}}^n), \varphi^n, \dots, \mathcal{M}(\gamma_{\mathbf{u}}^1), \varphi^1, \mathcal{M}(\gamma_{\mathbf{u}}^0))) + \\ \sum_{i=1}^n (-1)^{\sum_{j \leq i} |\varphi^j| + \sum_{j=1}^n (|x^j| - 1) + 1} \mathcal{C}(x^0; \dots, \gamma_{\mathbf{u}}^i, x^i, \gamma_{\mathbf{u}}^{i-1}, \dots; \sigma_{\mathbf{u}})(\dots, \mathcal{M}(\gamma_{\mathbf{u}}^i), d\varphi^i, \mathcal{M}(\gamma_{\mathbf{u}}^{i-1}), \dots)$$

where $\mathbf{u} \in \Delta^0(L; x^0; x^n, \dots, x^1)$ and we omitted the common sign. Recall from (1.1) that the composition along a 2-morphism is a certain composition in $\mathcal{C}_{x^0}$. Since all $\mathcal{M}(\gamma_{\mathbf{u}}^i)$ are closed morphisms of degree 0 and $\sigma_{\mathbf{u}}$ is a closed morphism of dg-functors, the above expression vanishes by the Leibniz rule.

Each other summand of the entry (4.13) is given by

$$(4.14) \quad \pm \mu^{\mathbf{u}_1}(\varphi^n, \dots, \varphi^{k+j+1}, \mu^{\mathbf{u}_0}(\varphi^{k+j}, \dots, \varphi^{k+1}), \varphi^k, \dots, \varphi^1)$$

where

- $i, j, k \in \mathbb{Z}_{\geq 0}$ and $i + j + k = n$;
- x is a double point of L ;
- $\mathbf{u}_0 = (\Pi_0, u_0) \in \Delta^0(L; x; x^{k+j}, \dots, x^{k+1})$;
- $\mathbf{u}_1 = (\Pi_1, u_1) \in \Delta^0(L; x^0; x^n, \dots, x^{k+j+1}, x, x^k, \dots, x^1)$.

Denote by T the set of the tuples $\mathbf{t} = ((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1)$ as above.

By [13, §7] there is an involution ι on T and a ι -invariant double cover $\theta: T \rightarrow \Delta^1(L; x^0; x^n, \dots, x^1)$ such that

- if $\theta((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1) = \mathbf{u} = (\Pi, u)$, then the pullback of $\mathbf{u}_0$ and $\mathbf{u}_1$ to Π via u is diffeomorphic to one of the pictures in Figs. 18 to 22 where Π has vertices at each point marked by x^i for $i = 0, \dots, n$, the corners which are not depicted are all negative;
- the involution ι swaps the pictures in Fig. 18 with that in Fig. 19, the two pictures in Fig. 20, and the pictures in Fig. 21 with that in Fig. 22.

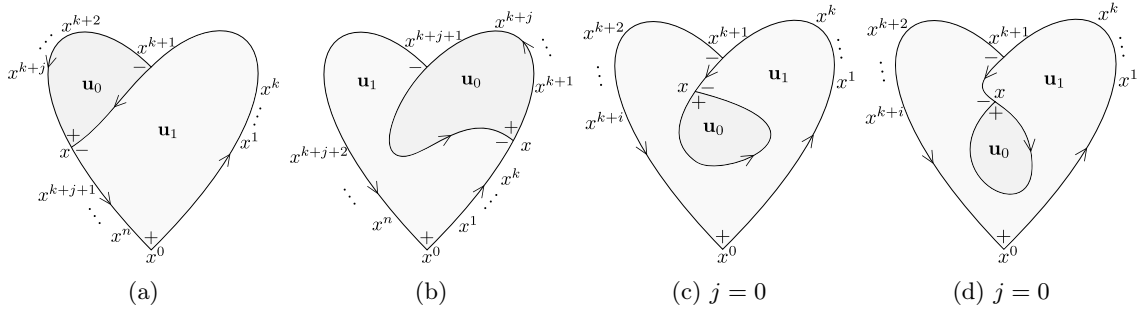


FIGURE 18. Extension of the incoming side of a negative concave corner.

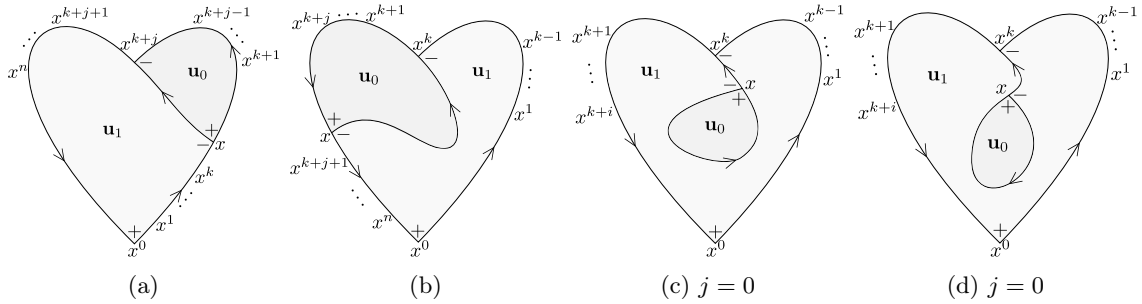


FIGURE 19. Extension of the outgoing side of a negative concave corner.

It remains to prove that for each $\mathbf{t} = ((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1) \in T$ the summands given by $\mathbf{t}$ and $\iota(\mathbf{t})$ sum to zero. Let $\mathbf{u} = (\Pi, u) = \theta(\mathbf{t}) \in \Delta^1(L; x^0; x^n, \dots, x^1)$ and $\varphi = \mu^{\mathbf{u}_0}(\varphi^{k+j}, \dots, \varphi^{k+1})$. By Lemma 1.6 we have

$$\begin{aligned} & \mu^{\mathbf{u}_1}(\varphi^n, \dots, \varphi^{k+j+1}, \mu^{\mathbf{u}_0}(\varphi^{k+j}, \dots, \varphi^{k+1}), \varphi^k, \dots, \varphi^1) = \\ & \mathcal{C}(x^0; \gamma_{\mathbf{u}}^n, x^n, \dots, \gamma_{\mathbf{u}}^1, x^1, \gamma_{\mathbf{u}}^0; \sigma_{\mathbf{u}})(\mathcal{M}(\gamma_{\mathbf{u}}^n), \varphi^n, \dots, \mathcal{M}(\gamma_{\mathbf{u}}^1), \varphi^1, \mathcal{M}(\gamma_{\mathbf{u}}^0)) \end{aligned}$$

where $\gamma_{\mathbf{u}}^i$ is the part of the boundary of $\mathbf{u}$ going from x^i to x^{i+1} (with $x^{n+1} = x^0$). That is, up to sign, the contributions of $\mathbf{t}$ and $\iota(\mathbf{t})$ are the same. It remains to compute the sign in (4.14).

The sign comes from the following.

- (1) From A_{∞} -relation (4.1):

$$(-1)^{\sum_{0 < \alpha \leq k} (|\varphi^{\alpha}| + |x^{\alpha}| - 1)}.$$

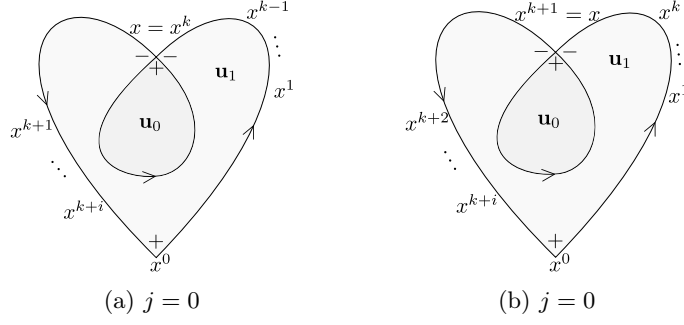


FIGURE 20. Decompositions of a polygon with one negative concave corner arising when extensions of the incoming and outgoing sides coincide.

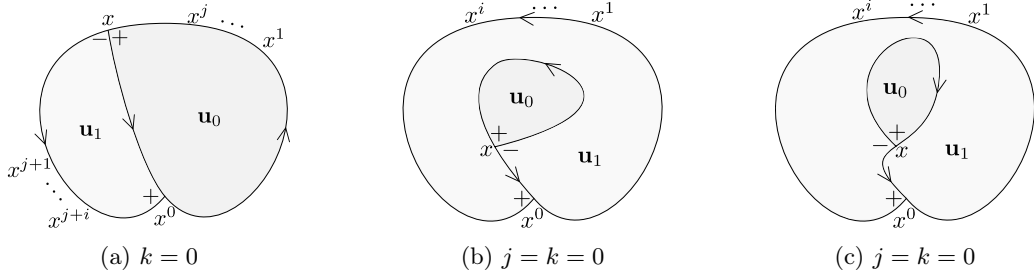


FIGURE 21. Extension of the outgoing side of a positive concave corner.

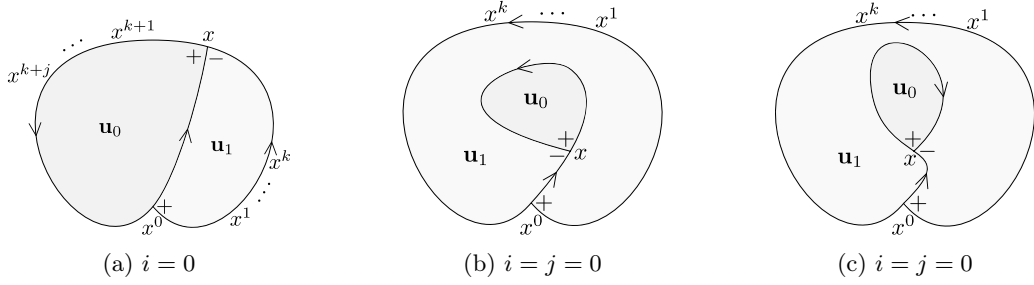


FIGURE 22. Extension of the incoming side of a positive concave corner.

(2) From the sign before the sum in (4.6) for $\mathbf{u}_0$:

$$(-1)^{\sum_{k < \alpha < \beta \leq k+j} |\varphi^\alpha| (|x^\beta| - 1)}.$$

(3) From the sign before the sum in (4.6) for $\mathbf{u}_1$:

$$(-1)^{\sum_{\alpha < \beta \leq k} |\varphi^\alpha| \left(\sum_{\alpha < \beta \leq k} (|x^\beta| - 1) + |x| - 1 + \sum_{k+j < \beta \leq n} (|x^\beta| - 1) \right) + |\varphi| \sum_{k+j < \beta \leq n} (|x^\beta| - 1) + \sum_{k+j < \alpha < \beta \leq n} |\varphi^\alpha| (|x^\beta| - 1)}.$$

(4) From the signs $\varepsilon_{\mathbf{u}_0}$ and $\varepsilon_{\mathbf{u}_1}$ in (4.6). For $\alpha = 1, \dots, n$, let ε_α (resp. ε_0) be 1 if the orientation of C_α (resp. C_n) coincides with the orientation induced by $\partial\Pi$ and -1 otherwise. The difference of the sign from

$$\prod_{0 \leq \alpha \leq n} \varepsilon_\alpha^{|x^\alpha|}$$

is

- for Figs. 18(a), 18(c), and 21: 1;
- for Figs. 18(b), 18(d), and 20(b): $(-1)^{|x|}$;
- for Fig. 19(a): $(-1)^{|x^{k+j}| + |x|}$;

- for Figs. 19(b) to 19(d) and 20(a): $(-1)^{|x^k|}$;
- for Fig. 22: $(-1)^{|x^0|+|x|}$,

Now multiply all up and use that by (3.18) and (1.1) we have

$$\begin{aligned} |x| &= |x^{k+j}| + \dots + |x^{k+1}| + 2 - j; \\ |\varphi| &= |\varphi^{k+j}| + \dots + |\varphi^{k+1}|; \\ |x^0| &= |x^n| + \dots + |x^1| + 3 - n. \end{aligned}$$

Let l be the number of vertices of Π on the open interval of $\partial\Pi$ from x^0 to the concave corner, i.e. for Figs. 18 to 20 the concave corner is x^{l+1} and $l = 0$ for Figs. 21 and 22. We obtain that the overall sign is equal to

$$(-1)^{\sum_{0 < \alpha \leq l} (|x^\alpha| - 1) + \sum_{0 < \alpha < \beta \leq n} |\varphi^\alpha| (|x^\beta| - 1)} \prod_{0 \leq \alpha \leq n} \varepsilon_\alpha^{|x^\alpha|};$$

for Figs. 18, 20(b), and 21 and is opposite to that for Figs. 19, 20(a), and 22. Now it is clear that the summands corresponding to $\mathbf{t}$ and $\iota(\mathbf{t})$ cancel out for each $\mathbf{t} \in T$. $\square$

4.4. Lagrangian branes. Before defining Lagrangian branes in §4.4.2 we recall what bounding cochains in curved A_∞ -algebras are in §4.4.1.

4.4.1. Bounding cochains. Let $\mathcal{A}$ be a nilpotent A_∞ -algebra. Given any elements $\delta_0, \dots, \delta_n \in \mathcal{A}^1$ we can define a map $\mu_{\delta_n, \dots, \delta_0}^n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ by

$$(4.15) \quad \mu_{\delta_n, \dots, \delta_0}^n(a^n, \dots, a^1) = \sum_{k_0, \dots, k_n \geq 0} \mu^{n+k_n+\dots+k_0}(\underbrace{\delta_n, \dots, \delta_n}_{k_n}, a^n, \dots, a^2, \underbrace{\delta_1, \dots, \delta_1}_{k_1}, a^1, \underbrace{\delta_0, \dots, \delta_0}_{k_0}).$$

The sum in (4.15) is finite since $\mathcal{A}$ is nilpotent. These maps satisfy a version of the A_∞ -relation:

$$\sum_{n \geq j \geq i \geq 0} (-1)^{\dagger} \mu_{\delta_n, \dots, \delta_j, \delta_i, \dots, \delta_0}^{n-j+1+i} \left(a^n, \dots, a^{j+1}, \mu_{\delta_j, \dots, \delta_i}^{j-i} (a^j, \dots, a^{i+1}), a^i, \dots, a^1 \right) = 0.$$

Specialising to $\delta_0, \dots, \delta_n = \delta$ we get a curved A_∞ -algebra structure

$$(4.16) \quad \mu_\delta = \{\mu_{\delta, \dots, \delta}^n\}_{n=0}^\infty \text{ on } \mathcal{A}.$$

Note that if $(\mathcal{A}, \mu)$ is structured, then $(\mathcal{A}, \mu_\delta)$ is also structured, with the same Z , action, and degree.

A *bounding cochain* in $\mathcal{A}$ is an element $\delta \in \mathcal{A}^1$ such that $(\mathcal{A}, \mu_\delta)$ is uncurved, i.e. such that the *Maurer–Cartan equation*

$$(4.17) \quad \mu_\delta^0() = \sum_{n=0}^\infty \mu^n(\underbrace{\delta, \dots, \delta}_n) = 0$$

holds.

4.4.2. Lagrangian branes.

Definition 4.11. A *Lagrangian brane* in the weighted Liouville surface $(\mathcal{S}; \mathcal{C})$ is the data of

- a Lagrangian pre-brane $\mathcal{L}$ in $(\mathcal{S}; \mathcal{C})$;
- a bounding cochain δ in the obstruction algebra $\mathcal{A}_\mathcal{L}$.

Lagrangian branes will be the objects of the Fukaya category. For a Lagrangian brane $\mathbb{L} = (\mathcal{L}, \delta)$ denote by $(\mathcal{A}_\mathbb{L}, \mu_\mathbb{L})$ the uncurved A_∞ -algebra $(\mathcal{A}_\mathcal{L}, \mu_\delta)$.

Remark 4.12. Classically (see Akaho–Joyce [4]), given a Lagrangian pre-brane $\mathcal{L}$, one looks for bounding cochains in a much larger algebra, the curved A_∞ -endomorphism algebra of $\mathcal{L}$, which we will not define in our setting, but only explain what the underlying k -module is. It is the direct sum of the following:

- (1) for each circle component C of the domain I the cohomology space $H^*(C; \mathcal{M}|_C)$;

- (2) for each self-intersection point x of $\mathcal{L}$ the spaces $\mathcal{C}_x(\mathcal{M}(x_+), \mathcal{M}(x_-))[-|x|]$ and $\mathcal{C}_x(\mathcal{M}(x_-), \mathcal{M}(x_+))[-|x| - 1]$;
- (3) for each Reeb chord y between the ends of $\mathcal{L}$ (including the constant Reeb chords) the space $\mathcal{C}_{\gamma(y_-)}(\mathcal{C}(y)\mathcal{M}(y_+), \mathcal{M}(y_-))[-|y|]$.

It is clear that the obstruction algebra is a submodule of the above, and it can be shown that it is a curved A_∞ -subalgebra, see Haiden [26, Proposition 3.1] for the proof in a related setting. The benefit of working with the obstruction algebra instead of the curved endomorphism algebra is not only that the former is smaller, but also that one does not need to perturb the Lagrangian to define the A_∞ -operations.

Remark 4.13. Recall the relation between obstruction algebra and Chekanov–Eliashberg algebra from Remark 4.5. Assume for concreteness that $k = \mathbb{F}_2$, $\mathcal{C}$ is the trivial 2-functor with fibre k . Consider a Lagrangian pre-brane $\mathcal{L}$ with closed domain and local system of rank 1. Then $\mathcal{A}_{\mathcal{L}}^1$ can be identified with the degree 1 maps from $\mathbb{F}_2$ to $\mathcal{A}_{\mathcal{L}}$ and taking the dual of such map gives a degree 0 map from $B\mathcal{A}_{\mathcal{L}}^\vee$ to $\mathbb{F}_2$. We therefore get a map

$$\mathcal{A}_{\mathcal{L}}^1 \rightarrow \text{Hom}_{\mathbb{F}_2}(B\mathcal{A}_{\mathcal{L}}^\vee, \mathbb{F}_2), \quad \delta \mapsto \epsilon_\delta$$

that establishes a bijection between bounding cochains in $\mathcal{A}_{\mathcal{L}}$ and augmentations of $B\mathcal{A}_{\mathcal{L}}^\vee$, i.e. maps of dg-algebras $(B\mathcal{A}_{\mathcal{L}}^\vee, \partial) \rightarrow (\mathbb{F}_2, 0)$.

The linearised contact cohomology $LCH_{\varepsilon_\delta}^*(\mathcal{L})$ is then identified with the cohomology of $(\mathcal{A}_{\mathcal{L}}, \mu_\delta)$, see Etnyre–Ng [21, §4] for details.

5. A_∞ -MACHINERY

In this section we study A_∞ -pre-categories with enough homotopy equivalences and develop the algebraic machinery that allows us to pass from the geometric Fukaya A_∞ -pre-category to a strictly unital A_∞ -category over an arbitrary commutative ring k . The section is purely algebraic and is independent of the geometric material in §§3 and 4.

In §5.1 we define homotopy equivalences in A_∞ -pre-categories and A_∞ -pre-categories with enough homotopy equivalences, and establish basic properties of homotopy equivalences in Lemmas 5.3 to 5.6. In §5.2 we define the triangulated envelope of an A_∞ -pre-category and strongly triangulated A_∞ -pre-categories, and prove that homotopy equivalences behave well under passing to the triangulated envelope in Proposition 5.8. In §5.3 we construct a strictly unital A_∞ -category spanned by a collection of objects admitting a system of strict units in Proposition 5.12 and give a sufficient condition for this construction to be canonical up to homotopy equivalence in Proposition 5.14.

5.1. Homotopy equivalence in A_∞ -pre-categories. In §5.1.1 we recall A_∞ -pre-categories and in §5.1.2 we define homotopy equivalences in them and A_∞ -pre-categories with enough homotopy equivalences. In §5.1.3 we prove that homotopy equivalences are closed under composition and satisfy the 2-out-of-6 property, and deduce that transverse homotopy equivalent objects are connected by a single homotopy equivalence. In §5.1.4 we characterise exact closed morphisms of degree zero in terms of pre- and post-composition. In §5.1.5 we prove that homotopy equivalence can be tested pointwise. In §5.1.6 we characterise homotopy equivalences in strictly unital A_∞ -categories in terms of strict units.

5.1.1. A_∞ -pre-categories. We recall the definition of A_∞ -pre-category by Kontsevich–Soibelman [33, §4.3]. Let k be a unital commutative ring. A (small $\mathbb{Z}$ -graded k -linear) A_∞ -pre-category $\mathcal{W}$ is the following data.

- A set of objects $\text{Ob } \mathcal{W}$.
- For each $n \geq 1$ a subset $\text{Ob}_{tr}^n \mathcal{W}$ of $(\text{Ob } \mathcal{W})^n$, called the set of *transverse sequences*, such that $\text{Ob}_{tr}^1 \mathcal{W} = \text{Ob } \mathcal{W}$ and subsequences of transverse sequences are transverse.
- For each pair $(X_0, X_1) \in \text{Ob}_{tr}^2 \mathcal{W}$ a $\mathbb{Z}$ -graded k -module of morphisms $\mathcal{W}(X_0, X_1)$.
- For each $n \geq 1$ and a transverse sequence $(X_0, \dots, X_n)$ a k -linear map

$$\mu^n: \mathcal{W}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{W}(X_0, X_1) \rightarrow \mathcal{W}(X_0, X_n)$$

of degree $2 - n$ satisfying the A_∞ -relation (4.1) for all chains of morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$.

Note that, unlike A_∞ -algebras, all A_∞ -pre-categories are uncurved, i.e. $\mu^0 = 0$.

An A_∞ -category is an A_∞ -pre-category such that any finite sequence of objects is transverse. Given a transverse sequence $\mathbf{X} = (X_0, \dots, X_n)$ in an A_∞ -pre-category $\mathcal{W}$, we define the *directed* A_∞ -category $\mathcal{W}|_{\mathbf{X}}$ by setting $\text{Ob } \mathcal{W}|_{\mathbf{X}} = \{X_0, \dots, X_n\}$,

$$\mathcal{W}|_{\mathbf{X}}(X_i, X_j) = \begin{cases} \mathcal{W}(X_i, X_j), & \text{if } i < j; \\ 0, & \text{otherwise.} \end{cases}$$

and the A_∞ -operations given by that in $\mathcal{W}$.

Let $\mathcal{W}$ be an A_∞ -pre-category, I a set and $\mathbf{X}_I = \{X_i\}_{i \in I}$ a collection of objects in $\mathcal{W}$. We say that $\mathbf{X}_I$ is a *transverse set* (this is a slight abuse of language since objects of $\mathcal{W}$ can be repeated in $\mathbf{X}_I$) if for any finite sequence of pairwise distinct elements $i_0, \dots, i_n \in I$ the sequence $(X_{i_0}, \dots, X_{i_n})$ is transverse. Unless stated otherwise, our standing assumption is that transverse sequences in A_∞ -pre-categories *form* transverse sets, that is, any permutation of a transverse sequence is a transverse sequence.

5.1.2. Homotopy equivalences in A_∞ -pre-category. Let $\mathcal{W}$ be an A_∞ -pre-category. The structure maps μ^1 make morphism spaces into cochain complexes by the A_1 -relation. The A_2, A_3 -relations imply that $a \circ b = (-1)^{|b|} \mu^2(a, b)$ satisfies the graded Leibniz rule with respect to the differential $da = (-1)^{|a|} \mu^1(a)$ and descends to an associative (partially defined) composition on cohomology. We will use the associated homological algebra language (closed, cohomologous, etc.) for morphisms in $\mathcal{W}$. Denote the degree k component of $\mathcal{W}(X_0, X_1)$ by $\mathcal{W}^k(X_0, X_1)$.

Recall our standing assumption that transverse sequences in $\mathcal{W}$ form transverse sets. Recall from [33, Definition 9] that a closed morphism $q^1 \in \mathcal{W}^0(X_0, X_1)$ is called a *quasi-isomorphism* if for any transverse set $\{T, X_0, X_1\}$ the maps

$$(\mathcal{W}(T, X_0), d) \xrightarrow{q^1 \circ -} (\mathcal{W}(T, X_1), d) \text{ and } (\mathcal{W}(X_1, T), d) \xrightarrow{- \circ q^1} (\mathcal{W}(X_0, T), d)$$

are quasi-isomorphisms. Since we are working with arbitrary modules over an arbitrary commutative ring, quasi-isomorphisms do not necessarily have quasi-inverses. We therefore introduce a more convenient notion of a homotopy equivalence in A_∞ -pre-category.

Definition 5.1. Let $\{X_0, X_1\}$ be a transverse set in $\mathcal{W}$ and $q^1 \in \mathcal{W}^0(X_0, X_1)$, $q^0 \in \mathcal{W}^0(X_1, X_0)$ two closed morphisms. We say that q^1 and q^0 are *homotopy inverse* if for any transverse set $\{T, X_0, X_1\}$ and $i = 0, 1$ the map $\mu^2(q^i, \mu^2(q^{1-i}, -)) = q^i \circ (q^{1-i} \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(T, X_i)}$ and the map $\mu^2(\mu^2(-, q^i), q^{1-i}) = (- \circ q^i) \circ q^{1-i}$ is homotopic to $\text{id}_{\mathcal{W}(X_i, T)}$.

A degree zero closed morphism is called a *homotopy equivalence* if it admits a homotopy inverse. Two objects $X, X' \in \text{Ob } \mathcal{W}$ are called *homotopy equivalent* if for some $n \geq 0$ there exist a chain of homotopy equivalences

$$X = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n = X'.$$

Note that, in the notation of Definition 5.1, the morphisms $\mu^2(q^0, q^1)$ and $\mu^2(q^1, q^0)$ are usually not defined since (X_0, X_1, X_0) and (X_1, X_0, X_1) are usually not transverse sequences. Standard homological algebra implies that homotopy equivalences are quasi-isomorphisms.

The following definition is a natural adaptation of [33, Definition 10] from quasi-isomorphisms to homotopy equivalences.

Definition 5.2. We say that the A_∞ -pre-category $\mathcal{W}$ *has enough homotopy equivalences* if transverse sequences in $\mathcal{W}$ form transverse sets and for any object $X \in \text{Ob } \mathcal{W}$ and a finite set of finite transverse sets $\{S_i\}_{i \in I}$ there exists an object X' and a homotopy equivalence $q: X \rightarrow X'$ such that the set $S_i \sqcup \{X'\}$ is transverse for each $i \in I$.

Having enough homotopy equivalences is a substitute for being unital for A_∞ -pre-categories and makes homotopy equivalences satisfy the expected properties, see §5.1.

Note that having enough homotopy equivalences is a property of an A_∞ -pre-category rather than an additional structure. Also observe that, in contrast with the homotopy unit framework of Fukaya–Oh–Ohta–Ono [23, §3.3] we do not require any higher coherences from homotopy equivalences. This works well in our setting since our A_∞ -pre-categories are triangulated by construction.

5.1.3. We prove that composition of homotopy equivalences is a homotopy equivalence whenever it is defined and that homotopy equivalences satisfy the 2-out-of-6 property. Note that one does not follow from the other since we do not have identity morphisms, but the 2-out-of-3 property follows from these.

Lemma 5.3. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences. Then*

- (1) *Composition of homotopy equivalences is a homotopy equivalence: if $\{X_0, X_1, X_2\}$ is a transverse set and $q^1 \in \mathcal{W}(X_0, X_1)$, $q^2 \in \mathcal{W}(X_1, X_2)$ are homotopy equivalences, then $q^2 \circ q^1$ is a homotopy equivalence.*
- (2) *Homotopy equivalences satisfy the 2-out-of-6 property: for a transverse set $\{X_0, X_1, X_2, X_3\}$ and closed morphisms $q^1 \in \mathcal{W}^0(X_0, X_1)$, $q^2 \in \mathcal{W}^0(X_1, X_2)$, $q^3 \in \mathcal{W}^0(X_2, X_3)$ if $q^2 \circ q^1$ and $q^3 \circ q^2$ are homotopy equivalences, then so are q^1, q^2, q^3 .*

As can be seen from the proof below, having enough homotopy equivalences is crucial for any homotopy argument in A_∞ -pre-categories.

Proof. We prove (1). Denote $q^0 = q^2 \circ q^1$. First note that $|q^0| = |q^1| + |q^2| + (2 - 2) = 0$ and $\mu^1(q^0) = 0$ by the A_2 -relation. Let $p^1 \in \mathcal{W}(X_1, X_0)$ be a homotopy inverse to q^1 and $p^2 \in \mathcal{W}(X_2, X_1)$ a homotopy inverse to q^2 . We show that $p^0 = p^1 \circ p^2$ is homotopy inverse to q^0 . Let $T \in \text{Ob } \mathcal{W}$ be such that $\{T, X_0, X_2\}$ is a transverse set. Since $\mathcal{W}$ has enough homotopy equivalences, there is an object T' and homotopy inverse morphisms $e \in \mathcal{W}(T', T)$, $e' \in \mathcal{W}(T, T')$ such that the sets $\{T', T, X_0, X_2\}$ and $\{T', X_0, X_1, X_2\}$ are transverse.

The A_3 -relation implies that the following diagram is homotopy commutative.

$$\begin{array}{ccccc}
 \mathcal{W}(T, X_0) & \xrightarrow{q^0 \circ -} & \mathcal{W}(T, X_2) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T, X_0) \\
 \downarrow - \circ e & & \downarrow - \circ e & \uparrow - \circ e' & \uparrow - \circ e' \\
 \mathcal{W}(T', X_0) & \xrightarrow{q^0 \circ -} & \mathcal{W}(T', X_2) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T', X_0) \\
 \searrow q^1 \circ - & & \nearrow q^2 \circ - & \searrow p^2 \circ - & \nearrow p^1 \circ - \\
 & \mathcal{W}(T', X_1) & & \mathcal{W}(T', X_1) &
 \end{array}$$

Using that $(- \circ e) \circ e'$, $p^2 \circ (q^2 \circ -)$, $p^1 \circ (q^1 \circ -)$ are all homotopic to identity we conclude that so is $p^0 \circ (q^0 \circ -)$. The other three maps from Definition 5.1 are homotopic to identity for similar reasons.

We prove (2). Let $p^{20} \in \mathcal{W}(X_2, X_0)$ be a homotopy inverse to $q^{02} = q^2 \circ q^1$ and $p^{31} \in \mathcal{W}(X_3, X_1)$ a homotopy inverse to $q^{13} = q^3 \circ q^2$. We show that $p_0^2 = q^1 \circ p^{20}$ and $p_3^2 = p^{31} \circ q^3$ are homotopy inverse to q^2 . Let $T \in \text{Ob } \mathcal{W}$ be such that $\{T, X_1, X_2\}$ is a transverse set. Since $\mathcal{W}$ has enough homotopy equivalences, there is an object T' and homotopy inverse morphisms $e \in \mathcal{W}(T', T)$, $e' \in \mathcal{W}(T, T')$ such that the sets $\{T', T, X_1, X_2\}$ and $\{T', X_0, X_1, X_2, X_3\}$ are transverse.

The A_3 -relation implies that the following diagram is homotopy commutative.

$$\begin{array}{ccccc}
 \mathcal{W}(X_2, T) & \xrightarrow{- \circ q^2} & \mathcal{W}(X_1, T) & \xrightarrow{- \circ p_0^2} & \mathcal{W}(X_2, T) \\
 \downarrow e' \circ - & & \downarrow e' \circ - & \uparrow e \circ - & \uparrow e \circ - \\
 \mathcal{W}(X_2, T') & \xrightarrow{- \circ q^2} & \mathcal{W}(X_1, T') & \xrightarrow{- \circ p_0^2} & \mathcal{W}(X_2, T') \\
 \searrow - \circ q^{02} & & \downarrow - \circ q^1 & \nearrow - \circ p^{20} & \\
 & \mathcal{W}(X_0, T') & & &
 \end{array}$$

Therefore, $(- \circ q^2) \circ p_0^2$ is homotopic to $\text{id}_{\mathcal{W}(X_2, T)}$. A similar diagram implies that $(- \circ p_3^2) \circ q^2$ is homotopic to $\text{id}_{\mathcal{W}(X_1, T)}$. Hence $- \circ p_0^2$ and $- \circ p_3^2$ are homotopic as morphisms $\mathcal{W}(X_1, T) \rightarrow \mathcal{W}(X_2, T)$, $(- \circ q^2) \circ p_3^2$ is homotopic to $\text{id}_{\mathcal{W}(X_2, T)}$, and $(- \circ p_0^2) \circ q^2$ is homotopic to $\text{id}_{\mathcal{W}(X_1, T)}$. The other four maps from Definition 5.1 applied to (q^2, p_0^2) and (q^2, p_3^2) are homotopic to identity for similar reasons hence p_0^2 and p_3^2 are homotopy inverse to q^2 .

A similar argument shows that $p^{20} \circ q^2$ is homotopy inverse to q^1 and $q^2 \circ p^{31}$ is homotopy inverse to q^3 . $\square$

Lemma 5.4. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences. Then for any pair of homotopy equivalent objects X_0, X_1 such that $\{X_0, X_1\}$ is a transverse set there exists a homotopy equivalence $q: X_0 \rightarrow X_1$.*

Note that Lemma 5.4 implies that for any homotopy equivalent objects X_0, X_1 there exists an object X_2 and homotopy equivalences $X_0 \rightarrow X_2$ and $X_2 \rightarrow X_1$.

Proof. Since X_0 and X_1 are homotopy equivalent, there exist a sequence $X_0 = Y_0, Y_1, \dots, Y_n = X_1$ of objects in $\mathcal{W}$ and homotopy equivalences $q^i \in \mathcal{W}(Y_{i-1}, Y_i)$ for $i = 1, \dots, n$. We need to construct a homotopy equivalence $q \in \mathcal{W}(Y_0, Y_n)$ provided that Y_0 and Y_n are transverse. As $\mathcal{W}$ has enough homotopy equivalences, we can take an object Y' and homotopy inverse morphisms $q'_0 \in \mathcal{W}(Y', Y_0)$ and $p'_0 \in \mathcal{W}(Y_0, Y')$ such that the sets $\{Y', Y_0, Y_n\}$ and $\{Y', Y_{i-1}, Y_i\}$, $i = 1, \dots, n$ are transverse. Using Lemma 5.3(1) we obtain homotopy equivalences $q'_i \in \mathcal{W}(Y', Y_i)$ by $q'_i = q_i \circ q'_{i-1}$ for $i = 1, \dots, n$. Setting $q = q'_n \circ p'_0$ gives the desired homotopy equivalence from $Y_0 = X_0$ to $Y_n = X_1$. $\square$

5.1.4. We give criteria for a degree zero closed morphism to be exact.

Lemma 5.5. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences and $q \in \mathcal{W}^0(X_0, X_1)$ a closed morphism. Then the following are equivalent.*

- (1) q is exact;
- (2) for any transverse set $\{T, X_0, X_1\}$ the map $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ is null-homotopic;
- (3) there exists a homotopy equivalence $e_0 \in \mathcal{W}(X'_0, X_0)$ such that the set $\{X'_0, X_0, X_1\}$ is transverse and $q \circ e_0 \in \mathcal{W}(X'_0, X_1)$ is exact;
- (4) for any transverse set $\{X_0, X_1, T\}$ the map $\mathcal{W}(X_1, T) \xrightarrow{- \circ q} \mathcal{W}(X_0, T)$ is null-homotopic;
- (5) there exists a homotopy equivalence $e'_1 \in \mathcal{W}(X_1, X'_1)$ such that the set $\{X_0, X_1, X'_1\}$ is transverse and $e'_1 \circ q \in \mathcal{W}(X_0, X'_1)$ is exact.

Proof. We show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. The argument for $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ is similar.

Assume $q = \mu^1(a)$ for some $a \in \mathcal{W}^{-1}(X_0, X_1)$. Let $\{T, X_0, X_1\}$ be a transverse set. The A_2 -relation then implies that

$$q \circ b = \mu^1(\mu^2(a, b)) + \mu^2(a, \mu^1(b)) \text{ for any homogeneous } b \in \mathcal{W}(T, X_0).$$

Hence $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ is indeed null-homotopic.

That (2) implies (3) is clear since we can take X'_0 for T and $\mu^1(e_0) = 0$.

Now assume (3). As e_0 is a homotopy equivalence, $\mathcal{W}(X_0, X_1) \xrightarrow{- \circ e_0} \mathcal{W}(X'_0, X_1)$ is a homotopy equivalence. Since $q \circ e_0 \in \mathcal{W}(X'_0, X_1)$ is exact, so is $q \in \mathcal{W}(X_0, X_1)$. $\square$

5.1.5. We prove that homotopy equivalence can be tested pointwise.

Lemma 5.6. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences, and $X_0 \xrightarrow{q} X_1$ a closed morphism of degree zero in $\mathcal{W}$. Suppose that for any transverse set $\{T, X_0, X_1\}$ the maps $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ and $\mathcal{W}(X_1, T) \xrightarrow{- \circ q} \mathcal{W}(X_0, T)$ are homotopy equivalences.*

Then q is a homotopy equivalence in $\mathcal{W}$.

Proof. It suffices to construct two degree zero closed morphisms $p^0, p^1 \in \mathcal{W}(X_1, X_0)$ such that for any transverse set $\{T, X_0, X_1\}$ the maps $p^0 \circ (q \circ -)$, $(- \circ p^0) \circ q$, $q \circ (p^1 \circ -)$, and $(- \circ q) \circ p^1$ are homotopic to identity. Then the maps $p^0 \circ -$ and $p^1 \circ -$ are homotopic hence by Lemma 5.5 p^0 is cohomologous to p^1 and they are both homotopy inverse to q .

To construct p^0 pick homotopy inverse morphisms $e \in \mathcal{W}(X'_0, X_0)$ and $e' \in \mathcal{W}(X_0, X'_0)$ such that $\{X'_0, X_0, X_1\}$ is a transverse set. Let $\mathcal{W}(X_0, X'_0) \xrightarrow{\alpha} \mathcal{W}(X_1, X'_0)$ be a homotopy inverse to $- \circ q$. Set $p^0 = e \circ \alpha(e') \in \mathcal{W}(X_1, X_0)$. Clearly, p^0 is closed of degree zero. Let $\{T, X_0, X_1\}$ be a transverse set. As in the proof of Lemma 5.3 we can assume that $\{T, X_0, X_1, X'_0\}$ is transverse. Note that $\alpha(e') \circ q$ is cohomologous to e' since α is homotopy inverse to $- \circ q$. The A_3 -relation and Lemma 5.5 now imply that the diagram

$$\begin{array}{ccccc} \mathcal{W}(T, X_0) & \xrightarrow{q \circ -} & \mathcal{W}(T, X_1) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T, X_0) \\ & & \searrow \alpha(e') \circ - & & \nearrow e \circ - \\ & & \mathcal{W}(T, X'_0) & & \end{array}$$

$e' \circ -$

is homotopy commutative. As e is homotopy inverse to e' we conclude that $p^0 \circ (q \circ -)$ is homotopic to identity. A similar argument shows that $(- \circ p^0) \circ q$ is homotopic to identity.

To construct p^1 pick homotopy inverse morphisms $e \in \mathcal{W}(X'_1, X_1)$ and $e' \in \mathcal{W}(X_1, X'_1)$ such that $\{X'_1, X_0, X_1\}$ is a transverse set. Let $\mathcal{W}(X'_1, X_1) \xrightarrow{\alpha} \mathcal{W}(X'_1, X_0)$ be a homotopy inverse to $q \circ -$. Set $p^1 = \alpha(e) \circ e' \in \mathcal{W}(X_1, X_0)$. Clearly, p^1 is closed of degree zero. Let $\{T, X_0, X_1\}$ be a transverse set. As in the proof of Lemma 5.3 we can assume that $\{T, X_0, X_1, X'_1\}$ is transverse. Note that $q \circ \alpha(e)$ is cohomologous to e since α is homotopy inverse to $q \circ -$. The A_3 -relation and Lemma 5.5 now imply that the diagram

$$\begin{array}{ccccc} \mathcal{W}(T, X_1) & \xrightarrow{p^1 \circ -} & \mathcal{W}(T, X_0) & \xrightarrow{q \circ -} & \mathcal{W}(T, X_1) \\ & & \nearrow \alpha(e) \circ - & & \searrow e \circ - \\ & & \mathcal{W}(T, X'_1) & & \end{array}$$

$e' \circ -$

is homotopy commutative. As e is homotopy inverse to e' we conclude that $q \circ (p^1 \circ -)$ is homotopic to identity. A similar argument shows that $(- \circ q) \circ p^1$ is homotopic to identity.

We have therefore constructed the desired $p^0, p^1 \in \mathcal{W}(X_1, X_0)$. $\square$

5.1.6. Recall that an A_∞ -category $\mathcal{C}$ is called *strictly unital* if for each $X \in \text{Ob } \mathcal{C}$ there exists a closed morphism $1_X \in \mathcal{C}^0(X, X)$ such that $1_X \circ a = a$ for any $a \in \mathcal{C}(Y, X)$, $a \circ 1_X = a$ for any $a \in \mathcal{C}(X, Y)$, and $\mu^n(\dots, 1_X, \dots) = 0$ for $n \geq 3$. Note that strict units 1_X , $X \in \text{Ob } \mathcal{C}$ are unique if they exist.

Observe that strict units in $\mathcal{C}$ are homotopy equivalences and that they certify that $\mathcal{C}$ has enough homotopy equivalences. The following characterisation of homotopy equivalences in strictly unital A_∞ -categories indicates that homotopy equivalences are the right notion of invertible morphism in an A_∞ -pre-category.

Lemma 5.7. *Let $\mathcal{C}$ be a strictly unital A_∞ -category. Let $q \in \mathcal{C}^0(X_0, X_1)$ be a closed morphism.*

- (1) *If $p \in \mathcal{C}^0(X_1, X_0)$ is homotopy inverse to q , then $p \circ q$ is cohomologous to 1_{X_0} and $q \circ p$ is cohomologous to 1_{X_1} .*
- (2) *If there exists a closed morphism $p^1 \in \mathcal{C}^0(X_1, X_0)$ such that $q \circ p^1$ is cohomologous to 1_{X_1} and a closed morphism $p^0 \in \mathcal{C}^0(X_1, X_0)$ such that $p^0 \circ q$ is cohomologous to 1_{X_0} , then q is a homotopy equivalence.*

Proof. (1) follows immediately from Lemma 5.5.

To prove (2) note that by Lemma 5.5 p^0, q, p^1 satisfy the assumption of Lemma 5.3(2) hence q is a homotopy equivalence. $\square$

5.2. Triangulated A_∞ -pre-categories. In §5.2.1 we define the triangulated envelope $\text{Tw } \mathcal{W}$ of an A_∞ -pre-category $\mathcal{W}$. In §5.2.2 we recall A_∞ -functors between A_∞ -pre-categories. In §5.2.3 we define strongly triangulated A_∞ -pre-categories, i.e. those admitting a strict totalisation functor $\text{Tw } \mathcal{W} \rightarrow \mathcal{W}$, and prove in Proposition 5.8 that for such $\mathcal{W}$ with enough homotopy equivalences homotopy equivalences behave well under passing to $\text{Tw } \mathcal{W}$; in particular, any twisted complex is homotopy equivalent to an object of $\mathcal{W}$. In §5.2.4 we define shifts and cones in $\text{Tw } \mathcal{W}$ and

in strongly triangulated A_∞ -pre-categories and prove that upper triangular morphisms of twisted complexes with homotopy equivalences on the diagonal are homotopy equivalences in Lemma 5.11.

5.2.1. Triangulated envelope. The triangulated envelope of an A_∞ -pre-category is defined similarly to the case of A_∞ -categories, and the construction (in the case of dg-categories) goes back to Bondal–Kapranov [9]. We follow Efimov [17, §4] with slight modifications taking into account that transverse sequences form transverse sets in our A_∞ -pre-categories.

Let $\mathcal{W}$ be a $\mathbb{Z}$ -graded k -linear A_∞ -pre-category where transverse sequences form transverse sets. The *triangulated envelope* $\text{Tw } \mathcal{W}$ of $\mathcal{W}$ is the $\mathbb{Z}$ -graded k -linear A_∞ -pre-category defined as follows.

Objects. An object of $\text{Tw } \mathcal{W}$ is a twisted complex, i.e. a pair (X, δ) where

- $X = \bigoplus_{i \in I} X_i[m_i]$ is a formal direct sum of formal shifts of objects in $\mathcal{W}$ such that I is finite, $\{X_i\}_{i \in I}$ is a transverse set in $\mathcal{W}$, and $m_i \in \mathbb{Z}$ for $i \in I$;
- δ is an element of degree 1 in the graded k -module

$$\bigoplus_{i_0, i_1 \in I: i_0 \neq i_1} \mathcal{W}(X_{i_0}, X_{i_1})[m_{i_1} - m_{i_0}]$$

and there exists a partial order $<$ on I such that

$$\delta \in \mathcal{A}_X^< = \bigoplus_{i_0, i_1 \in I: i_0 < i_1} \mathcal{W}(X_{i_0}, X_{i_1})[m_{i_1} - m_{i_0}]$$

is a bounding cochain in the nilpotent (uncurved) A_∞ -algebra $\mathcal{A}_X^<$ where the A_∞ -operations in $\mathcal{A}_X^<$ are given by that in $\mathcal{W}$ with a Koszul sign rule,

$$(5.1) \quad \mu^n(a_n[i_n], \dots, a_1[i_1]) = (-1)^{\sum_{1 \leq \alpha < \beta \leq n} i_\alpha(|a_\beta| - 1)} \mu^n(a_n, \dots, a_1)[i_1 + \dots + i_n]$$

for homogeneous morphisms $a_1, \dots, a_n$ in $\mathcal{W}$ and $i_1, \dots, i_n \in \mathbb{Z}$.

We have

$$\delta = \sum_{i_0, i_1 \in I} \delta_{i_0 i_1}[m_{i_1} - m_{i_0}] \text{ where } \delta_{i_0 i_1} \in \mathcal{W}^{m_{i_1} - m_{i_0} + 1}(X_{i_0}, X_{i_1}),$$

the morphisms $\delta_{i_0 i_1}$ are called *components* of δ . Note that if $\delta_{i_0 i_1} \neq 0$, then $i_0 < i_1$. The partial order on I is not part of the data, but we can always consider the weakest partial order such that $i_0 < i_1$ whenever $\delta_{i_0 i_1} \neq 0$.

Transverse sequences. A sequence of twisted complexes $((X^0, \delta^0), \dots, (X^n, \delta^n))$ with $X^j = \bigoplus_{i \in I^j} X_i^j[m_i^j]$ for $j = 0, \dots, n$ is transverse if and only if

$$\bigsqcup_{j=0}^n \bigsqcup_{i \in I^j} \{X_i^j\}$$

is a transverse set in $\mathcal{W}$.

Morphisms. For a transverse pair of twisted complexes $((X^0, \delta^0), (X^1, \delta^1))$ with $X^j = \bigoplus_{i \in I^j} X_i^j[m_i^j]$ for $j = 0, 1$ the morphism space is the graded k -module

$$(5.2) \quad \text{Tw } \mathcal{W}((X^0, \delta^0), (X^1, \delta^1)) = \bigoplus_{i^0 \in I^0, i^1 \in I^1} \mathcal{W}(X_{i^0}^0, X_{i^1}^1)[m_{i^1}^1 - m_{i^0}^0].$$

For a morphism $a \in \text{Tw } \mathcal{W}((X^0, \delta^0), (X^1, \delta^1))$ of degree d we have

$$a = \sum_{i^0 \in I^0, i^1 \in I^1} a_{i^0 i^1}[m_{i^1}^1 - m_{i^0}^0] \text{ where } a_{i^0 i^1} \in \mathcal{W}^{m_{i^1}^1 - m_{i^0}^0 + d}(X_{i^0}^0, X_{i^1}^1),$$

the morphisms $a_{i^0 i^1}$ are called *components* of a .

Structure maps. For a transverse sequence $((X^0, \delta^0), \dots, (X^n, \delta^n))$ with $X^j = \bigoplus_{i \in I^j} X_i^j[m_i^j]$ for $j = 0, \dots, n$, the A_∞ -operation μ^n is given by μ_δ from (4.16) in the A_∞ -algebra $\mathcal{A}_X^\leq$ where $\delta = \delta^0 + \dots + \delta^n$ is the bounding cochain in $\mathcal{A}_X^\leq$ with $X = X^0 \oplus \dots \oplus X^n$ and a partial order $<$ on $I^0 \sqcup \dots \sqcup I^n$ that includes partial orders on each I^j and satisfies $i^j < i^{j'}$ for any $j < j'$ and $i^j \in I^j, i^{j'} \in I^{j'}$.

It is instructive to think of $X[m]$ for $X \in \mathcal{W}$ and $m \in \mathbb{Z}$ as a formal tensor product $k[m] \otimes X$. The Koszul sign in (5.1) comes from this point of view and the fact that μ^n acts from the right.

Note that $\mathcal{W}$ is naturally a full subcategory of $\text{Tw } \mathcal{W}$ by identifying X and the twisted complex $(X[0], 0)$ for each $X \in \text{Ob } \mathcal{W}$. Observe that transverse sequences in $\text{Tw } \mathcal{W}$ form transverse sets.

5.2.2. A_∞ -functors. Let $\mathcal{W}$ and $\mathcal{V}$ be two A_∞ -pre-categories. An A_∞ -functor $\mathcal{F}: \mathcal{W} \rightarrow \mathcal{V}$ is the following data.

- A map $\mathcal{F}^0: \text{Ob } \mathcal{W} \rightarrow \text{Ob } \mathcal{V}$ such that for any transverse sequence $(X_0, \dots, X_n)$ in $\mathcal{W}$ the sequence $(\mathcal{F}^0(X_0), \dots, \mathcal{F}^0(X_n))$ is transverse in $\mathcal{V}$.
- For each $n \geq 1$ and a transverse sequence $(X_0, \dots, X_n)$ in $\mathcal{W}$ a k -linear map

$$\mathcal{F}^n: \mathcal{W}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{W}(X_0, X_1) \rightarrow \mathcal{V}(\mathcal{F}^0(X_0), \mathcal{F}^0(X_n))$$

of degree $1 - n$ satisfying the A_n -functor-relation

$$(5.3) \quad \sum_{n \geq j > i \geq 0} (-1)^\dagger \mathcal{F}^{n-j+1+i} \left(a^n, \dots, a^{j+1}, \mu_{\mathcal{W}}^{j-i}(a^j, \dots, a^{i+1}), a^i, \dots, a^1 \right) = \sum_{r > 0} \sum_{n=i_r > \dots > i_0=0} \mu_{\mathcal{V}}^r \left(\mathcal{F}^{n-i_r-1}(a^n, \dots, a^{i_r-1+1}), \dots, \mathcal{F}^{i_1}(a^{i_1}, \dots, a^1) \right).$$

where $\dagger = |a^i| + \dots + |a^1| - i$ for any $n \geq 1$ and any chain of homogeneous morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$ in $\mathcal{W}$.

The A_∞ -functor $\mathcal{F}$ is called *strict* if $\mathcal{F}^n = 0$ for $n \geq 2$. For strict A_∞ -functors the A_∞ -relations (5.3) boils down to

$$\mathcal{F}^1(\mu_{\mathcal{W}}^n(a^n, \dots, a^1)) = \mu_{\mathcal{V}}^n(\mathcal{F}^1(a^n), \dots, \mathcal{F}^1(a^1))$$

for any $n \geq 1$ and any chain of morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$ in $\mathcal{W}$.

5.2.3. Triangulated A_∞ -pre-category. We call an A_∞ -pre-category $\mathcal{W}$ *strongly triangulated* (cf. Drinfeld [16, §2.4]) if there exists a strict A_∞ -functor $\mathcal{T}: \text{Tw } \mathcal{W} \rightarrow \mathcal{W}$, called the *totalisation functor*, such that

- (1) A sequence $(\mathbf{X}^0, \dots, \mathbf{X}^n)$ of objects in $\text{Tw } \mathcal{W}$ is transverse if $(\mathcal{T}^0 \mathbf{X}^0, \dots, \mathcal{T}^0 \mathbf{X}^n)$ is a transverse sequence in $\mathcal{W}$;
- (2) $\mathcal{T}^1: \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathcal{T}^0(\mathbf{X}^0), \mathcal{T}^0(\mathbf{X}^1))$ is an isomorphism of graded modules for each transverse pair $(\mathbf{X}^0, \mathbf{X}^1)$ in $\text{Tw } \mathcal{W}$;
- (3) $\mathcal{T}^0(X) = X$ for each $X \in \text{Ob } \mathcal{W}$;
- (4) $\mathcal{T}^1$ is identity on $\text{Tw } \mathcal{W}(X^0, X^1) = \mathcal{W}(X^0, X^1)$ for each transverse pair (X^0, X^1) in $\mathcal{W}$.

It is not hard to show that $\text{Tw } \mathcal{W}$ is strongly triangulated for any A_∞ -pre-category $\mathcal{W}$.

If $\mathcal{W}$ has enough homotopy equivalences, then we can define $\mathcal{W}$ to be *triangulated* if any twisted complex over $\mathcal{W}$ is homotopy equivalent to an object of $\mathcal{W}$. The next statement shows that this alternative definition is implied by ours.

Proposition 5.8. *Let $\mathcal{W}$ be a strongly triangulated A_∞ -pre-category with enough homotopy equivalences. Then two twisted complexes $\mathbf{X}^0, \mathbf{X}^1 \in \text{Ob } \text{Tw } \mathcal{W}$ are homotopy equivalent in $\text{Tw } \mathcal{W}$ if and only if $X^0 = \mathcal{T}^0(\mathbf{X}^0)$ is homotopy equivalent to $X^1 = \mathcal{T}^0(\mathbf{X}^1)$ in $\mathcal{W}$. In particular,*

- (1) *homotopy equivalences between objects of $\mathcal{W}$ remain homotopy equivalences in $\text{Tw } \mathcal{W}$;*
- (2) *$\text{Tw } \mathcal{W}$ has enough homotopy equivalences;*
- (3) *any $\mathbf{X}^0 \in \text{Ob } \text{Tw } \mathcal{W}$ is homotopy equivalent to $\mathcal{T}^0(\mathbf{X}^0) \in \text{Ob } \mathcal{W}$; and*
- (4) *homotopy equivalence of twisted complexes can be tested only on objects of $\mathcal{W}$.*

Remark 5.9. If $\mathcal{W}$ has enough homotopy equivalences, but not strongly triangulated, then we do not know how to prove that homotopy inverse morphisms in $\mathcal{W}$ remain homotopy inverse in $\mathrm{Tw} \mathcal{W}$ or that $\mathrm{Tw} \mathcal{W}$ has enough homotopy equivalences. The issue is that we did not impose any conditions on $\mu^n(\dots, q, \dots)$, $n \geq 3$ for homotopy equivalences q . This can probably be fixed in a way similar to the rather involved homotopy unit framework of [23, §3.3], but we do not need it since the Fukaya categories that we construct geometrically are strongly triangulated.

Proof of Proposition 5.8. Since homotopy equivalence is defined as having a chain of homotopy equivalences between objects, it is enough to show that $\mathcal{T}^1$ and $(\mathcal{T}^1)^{-1}$ map homotopy inverse morphisms to homotopy inverse morphisms. That a strict surjective A_∞ -functor which is isomorphism on morphism spaces preserves homotopy inverses is clear.

It remains to show that if $e^1 \in \mathcal{W}(X^0, X^1)$ and $e^0 \in \mathcal{W}(X^1, X^0)$ are homotopy inverse morphisms in $\mathcal{W}$, then $\mathbf{e}^1 = \mathcal{T}_{\mathbf{X}^0, \mathbf{X}^1}^{-1}(e^1) \in \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1)$ and $\mathbf{e}^0 = \mathcal{T}_{\mathbf{X}^1, \mathbf{X}^0}^{-1}(e^0) \in \mathrm{Tw} \mathcal{W}(\mathbf{X}^1, \mathbf{X}^0)$ are homotopy inverse morphisms in $\mathrm{Tw} \mathcal{W}$. To see that this is indeed the case let $\{\mathbf{T}, \mathbf{X}^0, \mathbf{X}^1\}$ be a transverse set in $\mathrm{Tw} \mathcal{W}$. Denote $T = \mathcal{T}^0(\mathbf{T})$. If H is a homotopy of $\mathcal{W}(T, X^0)$ certifying that $e^0 \circ (e^1 \circ -)$ is homotopic to $\mathrm{id}_{\mathcal{W}(T, X^0)}$, then $\mathcal{T}_{\mathbf{T}, \mathbf{X}^0}^{-1} H \mathcal{T}_{\mathbf{T}, \mathbf{X}^0}$ certifies that $\mathbf{e}^0 \circ (\mathbf{e}^1 \circ -)$ is homotopic to $\mathrm{id}_{\mathrm{Tw} \mathcal{W}(\mathbf{T}, \mathbf{X}^0)}$. The argument for the other three homotopies is analogous. $\square$

Now suppose that an A_∞ -pre-category $\mathcal{W}$ with enough homotopy equivalences has two strict totalisation functors $\mathcal{T}_0, \mathcal{T}_1: \mathrm{Tw} \mathcal{W} \rightarrow \mathcal{W}$. We make the following observation.

Lemma 5.10. *Let $\mathbf{X}^0$ and $\mathbf{X}^1$ be two twisted complexes over $\mathcal{W}$. Then $\mathbf{X}^0$ is homotopy equivalent to $\mathbf{X}^1$ in $\mathrm{Tw} \mathcal{W}$ if and only if $\mathcal{T}_0^0(\mathbf{X}^0)$ is homotopy equivalent to $\mathcal{T}_1^0(\mathbf{X}^1)$ in $\mathcal{W}$.*

Proof. Applying Proposition 5.8 repeatedly we obtain two chains of homotopy equivalences

$$\mathcal{T}_1^0(\mathbf{X}^0) \cong \mathbf{X}^0 \cong \mathcal{T}_0^0(\mathbf{X}^0)$$

and

$$\mathcal{T}_1^0(\mathbf{X}^1) \cong \mathbf{X}^1 \cong \mathcal{T}_0^0(\mathbf{X}^1),$$

from which the claim clearly follows. $\square$

5.2.4. Shifts and cones. Let $\mathcal{W}$ be an A_∞ -pre-category where transverse sequences form transverse sets. For a twisted complex $\mathbf{X} = (\bigoplus_{i \in I} X_i[m_i], \delta)$ over $\mathcal{W}$ and $k \in \mathbb{Z}$ we define the *shift*

$$\mathbf{X}[k] = \left(\bigoplus_{i \in I} X_i[m_i + k], \delta \right)$$

where δ has the same components as before. This is indeed a twisted complex since the Koszul sign in (5.1) only depends on the differences $m_{i_1} - m_{i_0}$, so $\mathcal{A}_{\mathbf{X}[k]}^< = \mathcal{A}_{\mathbf{X}}^<$. The same observation applied to (5.2) and to the definition of μ in $\mathrm{Tw} \mathcal{W}$ gives that $[k]$ with identity on morphisms is a strict autoequivalence of $\mathrm{Tw} \mathcal{W}$.

Now let $\mathbf{X}^j = (\bigoplus_{i \in I^j} X_i^j[m_i^j], \delta^j)$, $j = 0, 1$ be a transverse pair of twisted complexes over $\mathcal{W}$ and $a \in \mathrm{Tw} \mathcal{W}^0(\mathbf{X}^0, \mathbf{X}^1)$ a closed morphism. We define the *cone* of a to be the twisted complex

$$\mathrm{Cone}(a) = \left(\bigoplus_{i \in I^0} X_i^0[m_i^0 + 1] \oplus \bigoplus_{i \in I^1} X_i^1[m_i^1], \delta^a \right) \text{ where } \delta_{i^0 i^1}^a = \delta_{i^0 i^1}^0 + \delta_{i^0 i^1}^1 + (-1)^{m_{i^1}^1} a_{i^0 i^1}.$$

Extending the orders on I^0 and I^1 to $I^0 \sqcup I^1$ so that $I^0 < I^1$ we see that $\delta^a \in \mathcal{A}_{\mathrm{Cone}(a)}^<$. A straightforward computation then shows that δ^a is a bounding cochain. Note that if $\mathbf{X}^1$ is an object of $\mathcal{W}$, then the sign in δ^a is trivial.

If $\mathcal{W}$ is strongly triangulated with a totalisation functor $\mathcal{T}$, then we define the *shift* $X[k] = \mathcal{T}^0(X[k])$ of $X \in \mathrm{Ob} \mathcal{W}$ and the *cone* $\mathrm{Cone}(a) = \mathcal{T}^0(\mathrm{Cone}(a))$ of a closed morphism $a \in \mathcal{W}^0(X^0, X^1)$, where on the right-hand sides X and a are viewed as an object and a morphism in $\mathrm{Tw} \mathcal{W}$. In particular, $\mathcal{W}$ is closed under shifts and cones, i.e. triangulated in the sense of the Introduction. If moreover $\mathcal{W}$ has enough homotopy equivalences, then by Proposition 5.8 the shift and the cone in $\mathcal{W}$ are homotopy equivalent in $\mathrm{Tw} \mathcal{W}$ to the corresponding twisted complexes.

We finish with a criterion for a morphism of twisted complexes to be a homotopy equivalence.

Lemma 5.11. *Let $\mathcal{W}$ be a strongly triangulated A_∞ -pre-category with enough homotopy equivalences and $\mathbf{X} = (\bigoplus_{i \in I} X_i[m_i], \delta^{\mathbf{X}})$ and $\mathbf{Y} = (\bigoplus_{i \in I} Y_i[m_i], \delta^{\mathbf{Y}})$ a transverse pair of twisted complexes over $\mathcal{W}$ with respect to the same partial order $<$ on I . Let $q \in \text{Tw } \mathcal{W}^0(\mathbf{X}, \mathbf{Y})$ be a closed morphism which is upper triangular, i.e. $q_{i_0 i_1} \neq 0$ implies $i_0 \leq i_1$, and such that $q_{ii} \in \mathcal{W}^0(X_i, Y_i)$ is a homotopy equivalence in $\mathcal{W}$ for each $i \in I$. Then q is a homotopy equivalence in $\text{Tw } \mathcal{W}$.*

Proof. By Proposition 5.8 $\text{Tw } \mathcal{W}$ has enough homotopy equivalences, so by Lemma 5.6 it suffices to show that for any transverse set $\{\mathbf{T}, \mathbf{X}, \mathbf{Y}\}$ in $\text{Tw } \mathcal{W}$ the maps $\text{Tw } \mathcal{W}(\mathbf{T}, \mathbf{X}) \xrightarrow{q \circ -} \text{Tw } \mathcal{W}(\mathbf{T}, \mathbf{Y})$ and $\text{Tw } \mathcal{W}(\mathbf{Y}, \mathbf{T}) \xrightarrow{- \circ q} \text{Tw } \mathcal{W}(\mathbf{X}, \mathbf{T})$ are homotopy equivalences of complexes. We treat $q \circ -$, the argument for $- \circ q$ is analogous with the roles of sources and targets exchanged.

Identify I with $\{1, \dots, n\}$ with the order on the latter extending $<$. For $r = 1, \dots, n+1$ let $F^r \text{Tw } \mathcal{W}(\mathbf{T}, \mathbf{X})$ be the sum of the summands of (5.2) with target X_s for $s \geq r$, and similarly for $\mathbf{Y}$. Since the components of $\delta^{\mathbf{X}}$, $\delta^{\mathbf{Y}}$, and q do not decrease the index, the definition of μ in $\text{Tw } \mathcal{W}$ shows that F^r are subcomplexes, which are direct summands as graded modules, and that $q \circ -$ preserves them. Moreover, F^r/F^{r+1} is the complex $\text{Tw } \mathcal{W}(\mathbf{T}, X_r[m_r])$ and $q \circ -$ induces

$$\text{Tw } \mathcal{W}(\mathbf{T}, (X_r[m_r], 0)) \xrightarrow{q_{rr} \circ -} \text{Tw } \mathcal{W}(\mathbf{T}, (Y_r[m_r], 0))$$

on it. Since the shift is a strict autoequivalence of $\text{Tw } \mathcal{W}$, this map coincides with $\text{Tw } \mathcal{W}(\mathbf{T}[-m_r], X_r) \xrightarrow{q_{rr} \circ -} \text{Tw } \mathcal{W}(\mathbf{T}[-m_r], Y_r)$, which is a homotopy equivalence since q_{rr} is a homotopy equivalence in $\text{Tw } \mathcal{W}$ by Proposition 5.8.

It remains to recall that a map of complexes of k -modules is a homotopy equivalence if and only if its cone is contractible, and that a complex with a finite filtration by subcomplexes which are direct summands as graded modules is contractible if all the associated graded pieces are. Applying this to the cone of $q \circ -$ with the induced filtration we conclude that $q \circ -$ is a homotopy equivalence. $\square$

5.3. From A_∞ -pre-categories to A_∞ -categories. Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences. We construct strictly unital A_∞ -categories spanned by collections of objects in $\mathcal{W}$. The construction requires a collection of objects $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ to admit a system of strict units $\mathbf{X}^\bullet$, which we define in §5.3.1. In §5.3.2 we construct the A_∞ -category $\mathcal{W}_{\mathbf{X}^\bullet}$ with $\text{Ob } \mathcal{W}_{\mathbf{X}^\bullet} = \mathbf{X}^0$, and in §5.3.3 we prove that it is strictly unital and that two objects of $\mathbf{X}^0$ are homotopy equivalent in $\mathcal{W}_{\mathbf{X}^\bullet}$ if and only if they are homotopy equivalent in $\mathcal{W}$, see Proposition 5.12. In §5.3.4 we introduce saturated collections $\mathcal{SU}$ of systems of strict units and prove that they give a strictly unital A_∞ -category $\mathcal{W}_{\mathcal{SU}}$, well-defined up to homotopy equivalence, see Proposition 5.14. In §5.3.5 we prove that $\mathcal{W}_{\mathcal{SU}}$ is triangulated if $\mathcal{W}$ is strongly triangulated, see Lemma 5.15.

5.3.1. System of strict units. A system of strict units on a collection $\{X_i^0\}_{i \in I}$ of objects in $\mathcal{W}$ is the data of

- a transverse set $\{X_i^j\}_{j \in \mathbb{Z}_{\geq 0}}$ in $\mathcal{W}$ for each $i \in I$;
- homotopy equivalences $e_i^{j_0, j_1} \in \mathcal{W}(X_i^{j_0}, X_i^{j_1})$ for each $i \in I$ and $j_0 \neq j_1$

such that

- (1) $e_i^{j_1, j_2} \circ e_i^{j_0, j_1} = e_i^{j_0, j_2}$ for any $i \in I$ and pairwise distinct $j_0, j_1, j_2 \geq 0$;
- (2) for any $i_0, i_1 \in I$ there exists $N > 0$ such that for any $j_0, j_1, j'_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \min\{j_1, j'_1\} < \max\{j_1, j'_1\}$ the set $\{X_{i_0}^{j_0}, X_{i_1}^{j_1}, X_{i_1}^{j'_1}\}$ is transverse and the map

$$e_{i_1}^{j'_1, j_1} \circ (e_{i_1}^{j_1, j'_1} \circ -) \text{ is the identity on } \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1});$$

- (3) for any $i_0, i_1 \in I$ there exists $N > 0$ such that for any $j_0, j'_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $\min\{j_0, j'_0\} < \max\{j_0, j'_0\} < j_1$ the set $\{X_{i_0}^{j_0}, X_{i_0}^{j'_0}, X_{i_1}^{j_1}\}$ is transverse and the map

$$(- \circ e_{i_0}^{j'_0, j_0}) \circ e_{i_0}^{j_0, j'_0} \text{ is the identity on } \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1});$$

- (4) for any $n \geq 3$, $0 \leq l \leq n$, and $i_0, \dots, i_{l-2}, i_{l-1} = i = i_l, i_{l+1}, \dots, i_n \in I$ there exists $N > 0$ such that for any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_{l-2} < \min\{j_{l-1}, j_l\} < \max\{j_{l-1}, j_l\} < j_{l+1} < \dots < j_n$ the set $\{X_{i_0}^{j_0}, \dots, X_{i_n}^{j_n}\}$ is transverse and the map

$$\begin{array}{c} \mathcal{W}(X_{i_{n-1}}^{j_{n-1}}, X_{i_n}^{j_n}) \dots \mathcal{W}(X_{i_l}^{j_l}, X_{i_{l+1}}^{j_{l+1}}) \otimes \mathcal{W}(X_{i_{l-2}}^{j_{l-2}}, X_{i_l}^{j_{l-1}}) \dots \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1}) \\ \downarrow \mu^n(\dots, e_i^{j_{l-1}, j_l}, \dots) \\ \mathcal{W}(X_{i_0}^{j_0}, X_{i_n}^{j_n}) \end{array}$$

vanishes.

5.3.2. A_∞ -category spanned by a collection of objects. Let

$$\mathbf{X}^\bullet = \left\{ \left(\{X_i^j\}^{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$$

be a system of strict units on a collection $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ of objects in $\mathcal{W}$. We define an A_∞ -category $\mathcal{W}_{\mathbf{X}^\bullet}$ spanned by $\mathbf{X}^0$ as follows.

Objects. $\text{Ob } \mathcal{W}_{\mathbf{X}^\bullet} = \mathbf{X}^0$.

Morphisms. Fix $X_{i_0}^0, X_{i_1}^0 \in \mathbf{X}^0$. The morphism space is given by

$$\mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) = \text{colim } \mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1})$$

where the colimit is taken with respect to the directed system

$$e_{i_1}^{j_1, j'_1} \circ - : \mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1}) \rightarrow \mathcal{W}(X_{i_0}^0, X_{i_1}^{j'_1})$$

for $0 \ll j_1 < j'_1$. Properties (1) and (4) in §5.3.1 and the A_3 -relation imply that the above is indeed a directed system. Property (2) implies that the natural map $\mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1}) \rightarrow \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0)$ is an isomorphism of graded k -modules for $j_1 \gg 0$. Combining this with Property (3), for some $N > 0$ we obtain isomorphisms

$$\varphi_{i_0, i_1}^{j_0, j_1} : \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) \rightarrow \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1})$$

for any $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$.

A_∞ -operations. For $X_{i_0}^0, \dots, X_{i_n}^0 \in \mathbf{X}^0$ the A_∞ -operation

$$\mu_{\mathcal{W}_{\mathbf{X}^\bullet}}^n : \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_{n-1}}^0, X_{i_n}^0) \otimes \dots \otimes \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) \rightarrow \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_n}^0)$$

is determined by

$$\varphi_{i_0, i_n}^{j_0, j_n} \circ \mu_{\mathcal{W}_{\mathbf{X}^\bullet}}^n = \mu_{\mathcal{W}}^n \circ \left(\varphi_{i_{n-1}, i_n}^{j_{n-1}, j_n} \otimes \dots \otimes \varphi_{i_0, i_1}^{j_0, j_1} \right)$$

for sufficiently large $N > 0$ and any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$. We check that this definition makes sense in Proposition 5.12.

5.3.3. We check that the definition of an A_∞ -category spanned by a collection of objects in §5.3.2 makes sense and prove that homotopy equivalence in $\mathcal{W}$ and $\mathcal{W}_{\mathbf{X}^\bullet}$ is the same.

Proposition 5.12. *Let $\mathbf{X}^\bullet = \left\{ \left(\{X_i^j\}^{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$ be a system of strict units on a collection of objects $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ in an A_∞ -pre-category $\mathcal{W}$ with enough homotopy equivalences.*

Then $\mathcal{W}_{\mathbf{X}^\bullet}$ is a strictly unital A_∞ -category with the strict unit $e_i \in \mathcal{W}_{\mathbf{X}^\bullet}(X_i^0, X_i^0)$ defined by $\varphi_{i, i}^{j_0, j_1}(e_i) = e_i^{j_0, j_1}$ for sufficiently large $N > 0$ and $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$.

Moreover, $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\mathcal{W}_{\mathbf{X}^\bullet}$.

Proof. That $\mathcal{W}_{\mathbf{X}^\bullet}$ is a strictly unital A_∞ -category follows from the statement below.

Claim. *Let $n \geq 1$ and $i_0, \dots, i_n \in I$. Then there exists $N > 0$ such that for any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$ the following holds.*

(1) if $j'_0 \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_0 < j'_0 < j_1$, then

$$\mu_{\mathcal{W}}^n(a_n, \dots, a_2, a_1 \circ e_{i_0}^{j'_0, j_0}) = \mu_{\mathcal{W}}^n(a_n, \dots, a_1) \circ e_{i_0}^{j'_0, j_0}$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;

(2) if $j'_l \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_l < j'_l < j_{l+1}$, then

$$\mu_{\mathcal{W}}^n(a_n, \dots, a_{l+2}, a_{l+1} \circ e_{i_l}^{j'_l, j_l}, e_{i_l}^{j_l, j'_l} \circ a_l, a_{l-1}, \dots, a_1) = \mu_{\mathcal{W}}^n(a_n, \dots, a_1)$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;

(3) if $j'_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_n < j'_n$, then

$$\mu_{\mathcal{W}}^n(e_{i_n}^{j_n, j'_n} \circ a_n, a_{n-1}, \dots, a_1) = e_{i_n}^{j_n, j'_n} \circ \mu_{\mathcal{W}}^n(a_n, \dots, a_1)$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;

The Claim implies that for $n \geq 1$ and $i_0, \dots, i_n \in I$ there exists $N > 0$ such that

$$\left(\varphi_{i_0, i_n}^{j_0, j_n} \right)^{-1} \circ \mu_{\mathcal{W}}^n \circ \left(\varphi_{i_{n-1}, i_n}^{j_{n-1}, j_n} \otimes \dots \otimes \varphi_{i_0, i_1}^{j_0, j_1} \right)$$

does not depend on the choice of $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$. Indeed, if we have two such choices $j'_0 < \dots < j'_n$ and $j''_0 < \dots < j''_n$, then they give the same result as $j_0 < \dots < j_n$ for $j_0 > \max\{j'_n, j''_n, N\}$. Therefore, $\mu_{\mathcal{W}_{\mathbf{X}^\bullet}}$ is well-defined and satisfies the A_∞ -relations.

Property (1) in §5.3.1 and the definition of φ imply that $\left(\varphi_{i, i}^{j_0, j_1} \right)^{-1} \left(e_i^{j_0, j_1} \right)$ does not depend on the choice of $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$ for sufficiently large $N > 0$. Since $e_i^{j_0, j_1}$ is a homotopy equivalence, e_i is a closed morphism of degree zero. Properties (2) and (3) imply that e_i satisfies the part of the definition of strict unit involving $\circ$, and Property (4) implies that e_i satisfies the remaining part of the definition, involving μ^n for $n \geq 3$.

Suppose $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$. Then by Lemma 5.4 there exists a homotopy equivalence $X_{i_0}^{j_0} \xrightarrow{q^{j_0, j_1}} X_{i_1}^{j_1}$ in $\mathcal{W}$ for $0 \ll j_0 < j_1$ hence $q = \left(\varphi_{i_0, i_1}^{j_0, j_1} \right)^{-1} (q^{j_0, j_1})$ is a homotopy equivalence in $\mathcal{W}_{\mathbf{X}^\bullet}$ by Lemma 5.6 since $q \circ -$ (resp. $- \circ q$) on $\mathcal{W}_{\mathbf{X}^\bullet}(X_{i_2}^0, X_{i_0}^0)$ (resp. on $\mathcal{W}_{\mathbf{X}^\bullet}(X_{i_1}^0, X_{i_2}^0)$) can be computed as $\varphi_{i_0, i_1}^{j_0, j_1}(q) \circ -$ (resp. $- \circ \varphi_{i_0, i_1}^{j_0, j_1}(q)$) on $\mathcal{W}(X_{i_2}^{j_2}, X_{i_0}^{j_0})$ (resp. $\mathcal{W}(X_{i_1}^{j_1}, X_{i_2}^{j_2})$) for $0 \ll j_2 < j_0 < j_1$ (resp. $0 \ll j_0 < j_1 < j_2$).

Now suppose that $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}_{\mathbf{X}^\bullet}$. Then by Lemma 5.7 there exist $j_0 < j_1 < j'_0 < j'_1$ and closed morphisms of degree zero $X_{i_0}^{j_0} \xrightarrow{q^{j_0, j_1}} X_{i_1}^{j_1} \xrightarrow{q^{j_1, j'_0}} X_{i_0}^{j'_0} \xrightarrow{q^{j'_0, j'_1}} X_{i_1}^{j'_1}$ such that $q^{j_1, j'_0} \circ q^{j_0, j_1}$ is cohomologous to $e_{i_0}^{j'_0, j'_0}$ and $q^{j'_0, j'_1} \circ q^{j_1, j'_0}$ is cohomologous to $e_{i_1}^{j'_1, j'_1}$. By Lemma 5.5 (1) and Lemma 5.3 q^{j_0, j_1} is a homotopy equivalence and hence $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$. $\square$

Proof of Claim. Take $N > 0$ such that Properties (2) and (3) in §5.3.1 are satisfied for any $i, i' \in \{i_0, \dots, i_n\}$ and Property (4) is satisfied for any tuple $(i'_0, \dots, i'_{n+2})$ where $i'_0, \dots, i'_{n+2} \in \{i_0, \dots, i_n\}$.

(1) of the Claim is equivalent to

$$\mu^n(a_n, \dots, a_2, \mu^2(a_1, e_{i_0}^{j'_0, j_0})) + (-1)^{0-1} \mu^2(\mu^n(a_n, \dots, a_1), e_{i_0}^{j'_0, j_0}) = 0,$$

which is the A_{n+1} -relation since $\mu^1(e_{i_0}^{j'_0, j_0}) = 0$ and all other terms vanish by Property (4).

(3) of the Claim is equivalent to

$$\begin{aligned} & (-1)^{|a_n|} \mu^n \left(\mu^2(e_{i_n}^{j_n, j'_n}, a_n), a_{n-1}, \dots, a_1 \right) + \\ & (-1)^{|a_n| + \dots + |a_1| - n - 1} \mu^2(e_{i_n}^{j_n, j'_n}, \mu^n(a_n, \dots, a_1)) = 0, \end{aligned}$$

which is a multiple of the A_{n+1} -relation since $\mu^1(e_{i_n}^{j'_n, j_n}) = 0$ and all other terms vanish by Property (4).

To prove (2) of the Claim consider the A_{n+1} -relation for the sequence of morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_l} X_{i_l}^{j_l} \xrightarrow{e_{i_l}^{j_l, j'_l}} X_{i_l}^{j'_l} \xrightarrow{\mu^2(a_{l+1}, e_{i_l}^{j'_l, j_l})} X_{i_{l+1}}^{j_{l+1}} \xrightarrow{a_{l+2}} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ and use that $e_{i_l}^{j_l, j'_l}$ is closed and Property (4) to get

$$\begin{aligned} & (-1)^{|a_{l-1}|+\dots+|a_1|+l} \mu^n \left(a_n, \dots, a_{l+2}, \mu^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), \mu^2(e_{i_l}^{j_l, j'_l}, a_l), a_{l-1}, \dots, a_1 \right) + \\ & (-1)^{|a_l|+\dots+|a_1|+l-1} \mu^n \left(a_n, \dots, a_{l+2}, \mu^2 \left(\mu^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), e_{i_l}^{j_l, j'_l} \right), a_l, \dots, a_1 \right) = 0. \end{aligned}$$

Using Property (3) we obtain

$$\begin{aligned} & \mu_{\mathcal{W}}^n \left(a_n, \dots, a_{l+2}, a_{l+1} \circ e_{i_l}^{j'_l, j_l}, e_{i_l}^{j_l, j'_l} \circ a_l, a_{l-1}, \dots, a_1 \right) = \\ & (-1)^{|a_l|} \mu^n \left(a_n, \dots, a_{l+2}, \mu^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), \mu^2(e_{i_l}^{j_l, j'_l}, a_l), a_{l-1}, \dots, a_1 \right) = \\ & \mu^n \left(a_n, \dots, a_{l+2}, \mu^2 \left(\mu^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), e_{i_l}^{j_l, j'_l} \right), a_l, \dots, a_1 \right) = \mu_{\mathcal{W}}^n(a_n, \dots, a_1) \end{aligned}$$

as required. $\square$

5.3.4. Invariance. A strict A_∞ -functor $\mathcal{F}$ is called a *strict equivalence* if $\mathcal{F}^0$ is a bijection and $\mathcal{F}^1$ is an isomorphism of graded k -modules on each morphism space in $\mathcal{W}$. Note that this is equivalent to $\mathcal{F}$ having an inverse *strict* A_∞ -functor. A strict equivalence from $\mathcal{W}$ to itself is called a *strict autoequivalence* of $\mathcal{W}$.

Let

$$\mathbf{X}^\bullet = \left\{ \left(\{X_i^j\}_{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$$

be a system of strict units in $\mathcal{W}$ and $\mathcal{I}$ a strict autoequivalence of $\mathcal{W}$. Then

$$\mathcal{I}_* \mathbf{X}^\bullet = \left\{ \left(\{\mathcal{I}^0 X_i^j\}_{j=0, \dots, \infty}, \{\mathcal{I}^1 e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$$

is also a system of strict units in $\mathcal{W}$ and $\mathcal{I}$ induces a strict equivalence of strictly unital A_∞ -categories $\mathcal{W}_{\mathbf{X}^\bullet} \rightarrow \mathcal{W}_{\mathcal{I}_* \mathbf{X}^\bullet}$.

We call a system of strict units $\mathbf{X}^\bullet$ *full* if any $X \in \text{Ob } \mathcal{W}$ is homotopy equivalent in $\mathcal{W}$ to some $X^0 \in \mathbf{X}^0$.

Definition 5.13. A collection $\mathcal{SU}$ of systems of strict units in $\mathcal{W}$ is called *saturated* if

- (1) $\mathcal{SU}$ contains a full system of strict units;
- (2) for any two systems of strict units $\mathbf{X}_0^\bullet, \mathbf{X}_1^\bullet \in \mathcal{SU}$ there exists a strict autoequivalence $\mathcal{I}$ of $\mathcal{W}$ such that $\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet \in \mathcal{SU}$.

Proposition 5.14. Let $\mathcal{SU}$ be a saturated collection of systems of strict units in $\mathcal{W}$. Then the strictly unital A_∞ -categories $\mathcal{W}_{\mathbf{X}^\bullet}$ for full $\mathbf{X}^\bullet \in \mathcal{SU}$ are all homotopy equivalent.

We denote (the homotopy equivalence class of) any of the $\mathcal{W}_{\mathbf{X}^\bullet}$ with $\mathbf{X}^\bullet \in \mathcal{SU}$ full by $\mathcal{W}_{\mathcal{SU}}$; such $\mathbf{X}^\bullet$ exists by Definition 5.13 (1). Since $\mathbf{X}^\bullet$ is full, by Proposition 5.12 every object of $\mathcal{W}$ is homotopy equivalent in $\mathcal{W}$ to an object of $\mathcal{W}_{\mathcal{SU}}$, and two objects of $\mathcal{W}_{\mathcal{SU}}$ are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\mathcal{W}_{\mathcal{SU}}$. Moreover, for any $\mathbf{X}^\bullet \in \mathcal{SU}$, applying Definition 5.13 (2) to $\mathbf{X}^\bullet$ and a full $\mathbf{Y}^\bullet \in \mathcal{SU}$ we get a strict autoequivalence $\mathcal{I}$ of $\mathcal{W}$ such that $\mathcal{I}_* \mathbf{X}^\bullet \sqcup \mathbf{Y}^\bullet \in \mathcal{SU}$. The latter system of strict units is full, so $\mathcal{I}$ induces a strict full embedding $\mathcal{W}_{\mathbf{X}^\bullet} \hookrightarrow \mathcal{W}_{\mathcal{I}_* \mathbf{X}^\bullet \sqcup \mathbf{Y}^\bullet} = \mathcal{W}_{\mathcal{SU}}$.

The homotopy equivalence of strictly unital A_∞ -categories in Proposition 5.14 is defined as follows. If $\mathcal{C}$ and $\mathcal{D}$ are two strictly unital A_∞ -categories, then an A_∞ -functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is called *strictly unital* if $\mathcal{F}^1(1_X) = 1_{\mathcal{F}^0(X)}$ and $\mathcal{F}^n(\dots, 1_X, \dots) = 0$ for $n \geq 2$ for any $X \in \text{Ob } \mathcal{C}$. A strictly unital A_∞ -functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is called a *homotopy equivalence* if

- For each $X_0, X_1 \in \text{Ob } \mathcal{C}$ the map $\mathcal{F}^1: \mathcal{C}(X_0, X_1) \rightarrow \mathcal{D}(\mathcal{F}^0(X_0), \mathcal{F}^0(X_1))$ is a homotopy equivalence; and
- each $Y \in \text{Ob } \mathcal{D}$ is homotopy equivalent to $\mathcal{F}^0(X)$ for some $X \in \text{Ob } \mathcal{C}$.

Two strictly unital A_∞ -categories $\mathcal{C}$ and $\mathcal{D}$ are called *homotopy equivalent* if there exists a finite sequence of strictly unital A_∞ -categories $\mathcal{C}_0, \dots, \mathcal{C}_n$, $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_n = \mathcal{D}$ such that for any $i = 1, \dots, n$ there exists a homotopy equivalence from $\mathcal{C}_{i-1}$ to $\mathcal{C}_i$ or vice versa.

Proof of Proposition 5.14. Consider two full systems of strict units $\mathbf{X}_0^\bullet, \mathbf{X}_1^\bullet \in SU$. Pick a strict autoequivalence $\mathcal{I}$ of $\mathcal{W}$ such that $\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet \in SU$ and set $\mathcal{V} = \mathcal{W}_{\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet}$. Then the inclusion $\mathcal{W}_{\mathbf{X}_1^\bullet} \hookrightarrow \mathcal{V}$ and the functor $\mathcal{W}_{\mathbf{X}_0^\bullet} \rightarrow \mathcal{V}$ induced by $\mathcal{I}$ are strict full embeddings. Let X be an object of $\mathcal{V}$. Since $\mathbf{X}_1^\bullet$ and $\mathbf{X}_0^\bullet$ are full (and $\mathcal{I}$ is a strict autoequivalence of $\mathcal{W}$), we see that X is homotopy equivalent in $\mathcal{W}$ to an object of $\mathbf{X}_1^0$ and to an object of $\mathcal{I}^0 \mathbf{X}_0^0$, hence also in $\mathcal{V}$ by Proposition 5.12. Thus the inclusions above are homotopy equivalences of strictly unital A_∞ -categories hence $\mathcal{W}_{\mathbf{X}_0^\bullet}$ and $\mathcal{W}_{\mathbf{X}_1^\bullet}$ are homotopy equivalent. $\square$

5.3.5. *Induced triangulated structure.* We show that triangulated structure on $\mathcal{W}$ induces a triangulated structure on $\mathcal{W}_{SU}$.

Lemma 5.15. *Let $\mathcal{W}$ be a strongly triangulated A_∞ -pre-category with enough homotopy equivalences and SU a saturated collection of systems of strict units on $\mathcal{W}$. Then $\mathcal{W}_{SU}$ is triangulated.*

Recall that a strictly unital A_∞ -category $\mathcal{C}$ is triangulated if any twisted complex over $\mathcal{C}$ is homotopy equivalent to an object of $\mathcal{C}$. It is clear that being triangulated is invariant under homotopy equivalence of strictly unital A_∞ -categories.

Proof. Fix a full system of strict units $\mathbf{X}^\bullet \in SU$. Let $\mathbf{Y} = (\bigoplus_{\alpha \in A} X_{i_\alpha}^0[m_\alpha], \delta)$ be a twisted complex over $\mathcal{W}_{\mathbf{X}^\bullet}$. We need to show $\mathbf{Y}$ is homotopy equivalent to an object of $\mathcal{W}_{\mathbf{X}^\bullet}$. Choose $j_\alpha \gg 0$ for each $\alpha \in A$ such that $j_{\alpha_0} < j_{\alpha_1}$ and $\varphi_{i_{\alpha_0}, i_{\alpha_1}}^{j_{\alpha_0}, j_{\alpha_1}}$ is an isomorphism whenever $\alpha_0 < \alpha_1$, and all $\mu_{\mathcal{W}_{\mathbf{X}^\bullet}}$ on components of δ can be computed in

$$\bigoplus_{\alpha_0 < \alpha_1} \mathcal{W}(X_{i_{\alpha_0}}^{j_{\alpha_0}}, X_{i_{\alpha_1}}^{j_{\alpha_1}})[m_{\alpha_1} - m_{\alpha_0}]$$

via $\varphi_{i_{\alpha_0}, i_{\alpha_1}}^{j_{\alpha_0}, j_{\alpha_1}}$ and Koszul sign. We then obtain a twisted complex $\mathbf{Y}' = (\bigoplus_{\alpha \in A} X_{i_\alpha}^{j_\alpha}[m_\alpha], \delta')$ over $\mathcal{W}$ where $\delta'_{\alpha_0 \alpha_1} = \varphi_{i_{\alpha_0}, i_{\alpha_1}}^{j_{\alpha_0}, j_{\alpha_1}}(\delta_{\alpha_0 \alpha_1})$. Let $X_k^0 \in \text{Ob } \mathcal{W}_{\mathbf{X}^\bullet}$ be an object homotopy equivalent to $\mathcal{T}\mathbf{Y}'$ in $\mathcal{W}$, which exists since $\mathbf{X}^\bullet$ is full. We prove that $\mathbf{Y}$ is homotopy equivalent to X_k^0 in $\text{Tw } \mathcal{W}_{\mathbf{X}^\bullet}$.

By (4) in §5.3.1 we can assume that $\{X_{i_\alpha}^{j_\alpha}\}_{\alpha \in A} \sqcup \{X_k^0, X_k^j\}$ is a transverse set for some $j \gg \max\{j_\alpha\}_{\alpha \in A}$. Define a twisted complex $\mathbf{Y}'' = (\bigoplus_{\alpha \in A} X_{i_\alpha}^{j'_\alpha}[m_\alpha], \delta'')$ analogously to $\mathbf{Y}'$, but now with all $j'_\alpha \gg j$. Again by (4) in §5.3.1 we can assume that $\{X_{i_\alpha}^{j'_\alpha}\}_{\alpha \in A} \sqcup \{X_k^0, X_k^j\} \sqcup \{X_{i_\alpha}^{j'_\alpha}\}_{\alpha \in A}$ is a transverse set. By Lemma 5.4 and Proposition 5.8 we obtain a homotopy equivalence $q \in \text{Tw } \mathcal{W}^0(\mathbf{Y}', X_k^j)$. Since q is a homotopy equivalence, there is a morphism $p \in \text{Tw } \mathcal{W}^0(X_k^0, \mathbf{Y}')$ such that $q \circ p$ is cohomologous to e_k^{0j} and a morphism $p' \in \text{Tw } \mathcal{W}^0(X_k^j, \mathbf{Y}'')$ such that $p' \circ q$ is cohomologous to $e_{\mathbf{Y}', \mathbf{Y}''} = \sum_{\alpha \in A} e_{i_\alpha}^{j_\alpha j'_\alpha}$. All of these morphisms define the corresponding morphisms in $\text{Tw } \mathcal{W}_{\mathbf{X}^\bullet}$ via φ^{-1} , which is assembled from the inverses to $\varphi_{i_0 i_1}^{j_0 j_1}$. This yields $\varphi^{-1}(q) \circ \varphi^{-1}(p)$ cohomologous to $1_{X_k^0}$ and $\varphi^{-1}(p') \circ \varphi^{-1}(q)$ cohomologous to $1_{\mathbf{Y}}$. Therefore, $\varphi^{-1}(q) \in \text{Tw } \mathcal{W}_{\mathbf{X}^\bullet}^0(\mathbf{Y}, X_k^0)$ is a homotopy equivalence by Lemma 5.3. $\square$

6. GEOMETRIC FUKAYA CATEGORY

In this section we define the geometric Fukaya A_∞ -pre-category, show that it satisfies the assumptions of the machinery of §5, and prove Theorems 1 and 3 to 5.

In §6.1 we define the geometric Fukaya A_∞ -pre-category $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$, whose A_∞ -operations are computed in obstruction algebras of composite Lagrangian branes, and prove its invariance properties. In §6.2 we perturb morphisms of local systems on a Lagrangian brane to morphisms between the brane and its transverse perturbation and compute A_∞ -operations involving them. Using these perturbed morphisms, we prove that $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ has enough homotopy equivalences in §6.3, construct a saturated collection of systems of strict units on it in §6.4, and prove that it is strongly triangulated in §6.5, see Propositions 6.17, 6.20, and 6.22. In §6.6 we identify the

A_∞ -category of an arc brane system with the A_∞ -category of the corresponding tame collection of Lagrangian branes and prove that full arc brane systems generate the same subcategory. In §6.7 we deduce Theorems 1 and 3 to 5.

6.1. Fukaya A_∞ -pre-category. Fix a weighted Liouville surface $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C})$.

In §6.1.1 we define transverse tuples of Lagrangians and their composite Lagrangians, obtained by shifting the potentials vertically, and show that a suitable shift makes the actions of any given finitely many intersection points and Reeb chords positive in Lemma 6.1. In §6.1.2 we define the geometric Fukaya A_∞ -pre-category $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$, whose A_∞ -operations are computed in the obstruction algebras of composite Lagrangian branes. In §6.1.3 we prove that these operations do not depend on the choice of the shift and that $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ is an A_∞ -pre-category, see Proposition 6.4. In §6.1.4 we prove that graded exact symplectomorphisms induce strict equivalences of geometric Fukaya A_∞ -pre-categories, and deduce that $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ is invariant under isotopies of η and $\mathcal{C}$ and does not depend on the choice of λ up to strict equivalence.

6.1.1. Transverse Lagrangians. Let $L_i = (I_i, \gamma_i, P_i)$, $i = 0, \dots, n$ be Lagrangians in $\mathcal{S}$ such that

$$\gamma = \bigsqcup_{i=0}^n \gamma_i: I = \bigsqcup_{i=0}^n I_i \rightarrow S$$

has at worst transverse double points and $\gamma|_{\partial I}$ is an embedding. In this case we say that $L_0, \dots, L_n$ are *transverse*. Note that γ is an immersion satisfying conditions (1)–(4) from Definition 3.9.

Given a vector $h = (h_0, \dots, h_n) \in \mathbb{R}^{n+1}$ define

$$P_h: I \rightarrow \mathbb{R}, P_h(t) = P_i(t) + h_i \text{ if } t \in I_i.$$

The data $L_h = L_h(L_0, \dots, L_n) = (I, \gamma, P_h)$ satisfies all conditions on being a Lagrangian in Definition 3.9 except that P_h can fail to satisfy conditions (6) and (7). Call the vector $h \in \mathbb{R}^{n+1}$ *admissible* for $L_0, \dots, L_n$ if L_h is a Lagrangian and let $\mathcal{H}(L_0, \dots, L_n)$ be the set of admissible shift vectors. For $h \in \mathcal{H}(L_0, \dots, L_n)$ we call the Lagrangian L_h the *composite Lagrangian* of $L_0, \dots, L_n$ with *shift vector* h .

For $i, j \in \{0, \dots, n\}$, $i \neq j$ let $Z(L_i, L_j)$ be the union of the set $X(L_i, L_j)$ of intersection points of γ_i and γ_j and the set $Y(L_i, L_j)$ of Reeb chords from an end of γ_i to an end of γ_j . Note that $Z(L_i, L_j)$ is infinite if L_i and L_j both have ends on a component C of ∂S with $C \cap \Lambda = \emptyset$. The set $Z(L_i, L_j)$ will give rise to the morphism space in the Fukaya category between Lagrangian branes supported on L_i and L_j .

For $z \in X(L_i, L_j)$ let $z_+ \in I_i$ and $z_- \in I_j$ be such that $\gamma_i(z_+) = z = \gamma_j(z_-)$, and for $z \in Y(L_i, L_j)$ let $z_+ \in I_i$ and $z_- \in I_j$ be such that z goes from $\gamma_i(z_+)$ to $\gamma_j(z_-)$.

Given a (not necessarily admissible) shift vector h and $z \in Z(L_i, L_j)$ for $i \neq j$, we define the action

$$(6.1) \quad A_h(z) = \begin{cases} P_h(z_+) - P_h(z_-), & \text{if } z \text{ is an intersection point;} \\ P_h(z_+) - \int_z \lambda - P_h(z_-), & \text{if } z \text{ is a Reeb chord.} \end{cases}$$

Note that h being admissible is equivalent to the non-vanishing of A_h for all z , but, unlike the action in (3.8), $A_h(z)$ can be negative for intersection points. Observe that

$$A_h(z) - A_0(z) = h_i - h_j \text{ for any } z \in Z(L_i, L_j).$$

We therefore have a characterisation of admissible shift vectors,

$$\mathcal{H}(L_0, \dots, L_n) = \left\{ h \in \mathbb{R}^{n+1}: h_j - h_i \neq A_0(z) \text{ for all } \begin{array}{l} i, j \in \{0, \dots, n\}, i \neq j \\ \text{and } z \in Z(L_i, L_j) \end{array} \right\}.$$

That is, $\mathcal{H}(L_0, \dots, L_n)$ is the complement of a hyperplane arrangement in $\mathbb{R}^{n+1}$. The number of hyperplanes in the arrangement can be infinite, but the arrangement is discrete since $\{z \in Z(L_i, L_j): A_0(z) > B\}$ is finite for all $i, j \in \{0, \dots, n\}$, $i \neq j$ and $B \in \mathbb{R}$ by Lemma 3.11.

Let L_h be a composite Lagrangian of $L_0, \dots, L_n$. Fix $i, j \in \{0, \dots, n\}, i \neq j$. Then A_h and A_{L_h} are two action functions defined on $Z(L_i, L_j)$. Note that

$$A_{L_h}(z) = \begin{cases} |A_h(z)|, & \text{if } z \in X(L_i, L_j); \\ A_h(z), & \text{if } z \in Y(L_i, L_j). \end{cases}$$

We finish this discussion with the following easy but useful lemma.

Lemma 6.1. *Given finite sets $Z_{ij} \subset Z(L_i, L_j)$, $0 \leq i < j \leq n$ there exists an admissible shift vector $h \in \mathbb{R}^{n+1}$ such that for any $0 \leq i < j \leq n$ we have*

$$(6.2) \quad A_h(z) > 0 \text{ for any } z \in Z_{ij} \text{ and } A_h(z) < 0 \text{ for any } z \in Z(L_j, L_i)$$

Proof. Recall that $A_h(z) = A_0(z) + h_i - h_j$ for $z \in Z(L_i, L_j)$ and that $\{z \in Z(L_i, L_j) : A_0(z) > 0\}$ is finite for all $i, j \in \{0, \dots, n\}, i \neq j$. Therefore, for each $0 \leq i < j \leq n$ there exists

$$M_{ij} = \min(\{0\} \cup \{A_0(z) \mid z \in Z_{ij}\} \cup \{-A_0(z) \mid z \in Z(L_j, L_i)\}) \in \mathbb{R}.$$

Observe that if $h_j - h_i < M_{ij}$ for each $0 \leq i < j \leq n$, then (6.2) holds. Since the left-hand sides of these inequalities are given by linear combinations with non-negative coefficients of n linearly independent functionals $h \mapsto h_i - h_{i-1}, i = 1, \dots, n$, these inequalities define a non-empty open subset of $\mathbb{R}^{n+1}$. The lemma follows since $\mathcal{H}(L_0, \dots, L_n)$ is dense in $\mathbb{R}^{n+1}$. $\square$

If we are given a grading gr_i for each L_i , then we get a grading gr on γ and hence can define the degree $|z|$ as in §3.4.2, but with the new definition of $z_{\pm}$.

Let $\mathcal{L}_i = (I_i, \gamma_i, P_i, \text{gr}_i, \mathcal{M}_i), i = 0, \dots, n$ be Lagrangian pre-branes such that their underlying Lagrangians are transverse and $h \in \mathbb{R}^{n+1}$ an admissible shift vector, i.e. $L = L_h(L_0, \dots, L_n)$ is a Lagrangian. Then $\mathcal{L} = (L, \text{gr}, \sqcup \mathcal{M}_i) = \mathcal{L}_h(\mathcal{L}_0, \dots, \mathcal{L}_n)$ is a Lagrangian pre-brane, so the obstruction algebra $(\mathcal{A}_{\mathcal{L}}, \mu_{\mathcal{L}})$ is defined. For any $i = 0, \dots, n$ and $z \in Z_{L_i}$ we have $\mathcal{A}_{\mathcal{L}}(z) = \mathcal{A}_{\mathcal{L}_i}(z)$, and if $z^0, \dots, z^k \in Z_L$ are such that $z^j \in Z_{L_{i_j}}$ for each $j = 0, \dots, k$, then

$$\mu_{\mathcal{L}}^{z^0, z^k, \dots, z^1} = \begin{cases} \mu_i^{z^0, z^k, \dots, z^1}, & \text{if } i_0 = \dots = i_k = i; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\bigoplus_{i=0}^n (\mathcal{A}_{\mathcal{L}_i}, \mu_i)$ is an A_{∞} -subalgebra of $(\mathcal{A}_{\mathcal{L}}, \mu_{\mathcal{L}})$.

If $\delta_i \in \mathcal{A}_{\mathcal{L}_i}, i = 0, \dots, n$ are bounding cochains, i.e. $\mathbb{L}_i = (\mathcal{L}_i, \delta_i)$ are Lagrangian branes, then we obtain a bounding cochain $\delta = \delta_0 + \dots + \delta_n \in \mathcal{A}_{\mathcal{L}}$. Indeed, δ lies in the A_{∞} -subalgebra $\bigoplus_{i=0}^n (\mathcal{A}_{\mathcal{L}_i}, \mu_i)$ and hence $\mu_{\delta}^0 = \mu_{\delta_0}^0 + \dots + \mu_{\delta_n}^0 = 0$. We call the resulting Lagrangian brane $\mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n) = (\mathcal{L}, \delta)$ the *composite* Lagrangian brane of $\mathbb{L}_0, \dots, \mathbb{L}_n$ with the shift vector h . The A_{∞} -operations in the Fukaya category will be computed in A_{∞} -algebras $(\mathcal{A}_{\mathbb{L}}, \mu_{\mathbb{L}})$ for composite Lagrangian branes $\mathbb{L}$.

6.1.2. Definition of Fukaya A_{∞} -pre-category. We define the geometric Fukaya A_{∞} -pre-category $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ of a graded Liouville surface $\mathcal{S} = (S, Q, \eta, \lambda)$ with coefficients in the graded system of dg-categories $\mathcal{C} : \mathcal{P}_2^+(S, Q) \rightarrow \text{dgc}at_k$ as a $\mathbb{Z}$ -graded k -linear A_{∞} -pre-category.

Objects. An object of $\mathcal{W}$ is a Lagrangian brane $\mathbb{L} = (I, \gamma, P, \text{gr}, \mathcal{M}, \delta)$ in $(\mathcal{S}; \mathcal{C})$.

Transverse sequences. A sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n) \in (\text{Ob } \mathcal{W})^{n+1}$ is transverse if the underlying Lagrangians $L_0, \dots, L_n$ are transverse.

Morphisms. For a transverse pair of objects $(\mathbb{L}_0, \mathbb{L}_1)$ let $Z = Z(\mathbb{L}_0, \mathbb{L}_1)$ be the set of intersection points between γ_0 and γ_1 and Reeb chords from an end of γ_0 to an end of γ_1 . The morphism space is the graded k -module

$$(6.3) \quad \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1) = \bigoplus_{z \in Z} \mathcal{W}(z)[-|z|]$$

where

$$(6.4) \quad \mathcal{W}(z) = \mathcal{C}_{\gamma_1(z_-)}(\mathcal{C}(z)\mathcal{M}_0(z_+), \mathcal{M}_1(z_-))$$

A_∞ -operations. For $n \geq 1$, a transverse sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n)$, and intersection points or Reeb chords

$$z^0 \in Z(\mathbb{L}_0, \mathbb{L}_n), z^i \in Z(\mathbb{L}_{i-1}, \mathbb{L}_i), i = 1, \dots, n$$

the entry

$$(6.5) \quad \mu^{z^0; z^n, \dots, z^1} : \mathcal{W}(z^n) \otimes \dots \otimes \mathcal{W}(z^1) \rightarrow \mathcal{W}(z^0)$$

of the structure map μ^n is defined as follows. Pick an admissible shift vector $h \in \mathcal{H}(L_0, \dots, L_n)$ such that $A_h(z^0), \dots, A_h(z^n) > 0$ and define

$$(6.6) \quad \mu^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1}$$

where $\mathbb{L} = \mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n)$ is the composite Lagrangian brane of $\mathbb{L}_0, \dots, \mathbb{L}_n$ with the shift vector h .

See Proposition 6.4 for the proof that the choices required to define the structure maps can be made and do not affect the result and that we get a $\mathbb{Z}$ -graded k -linear A_∞ -pre-category where transverse sequences form transverse sets.

Remark 6.2. A related approach to defining Fukaya category of surfaces is outlined by Haiden [26]. The setup there is different in that k is a field, $D = \emptyset$, the system of dg-coefficients is constant, and the Lagrangians are only allowed to end in a fixed finite set of points on ∂S , so Reeb chords on the boundary do not enter the construction. The resulting category is called an infinitesimal Fukaya category of a surface.

The idea of using a version of Chekanov–Eliashberg algebra as an obstruction algebra also appears in Chantraine–Dimitroglou Rizell–Ghiggini–Golovko [12].

Remark 6.3. If S is a disc, $Q = \emptyset$, $\mathcal{C}$ is a constant system equal to a field k , and L is a graded Lagrangian whose domain is closed, then the positive augmentation A_∞ -category $\text{Aug}_+(L)$ of Ng–Rutherford–Shende–Sivek–Zaslow [40] is defined. Modulo sign conventions, the A_∞ -operation μ^n on a transverse sequence of Lagrangian branes $\mathbb{L}_0, \dots, \mathbb{L}_n$ such that each $\mathbb{L}_i$ is supported on a perturbation of L can be computed in the augmentation category $\text{Aug}_+(L)$.

6.1.3. *Well-definedness.* In this section we demonstrate the following.

Proposition 6.4. *The geometric Fukaya A_∞ -pre-category $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ defined in §6.1.2 is a small $\mathbb{Z}$ -graded k -linear A_∞ -pre-category where transverse sequences form transverse sets.*

We defined the A_∞ -operations μ in $\mathcal{W}$ by computing them in certain composite Lagrangian branes. We first check that this computation does not depend on the choice of a composite Lagrangian brane.

Lemma 6.5. *Let $\mathbb{L}_i = (I_i, \gamma_i, P_i, \text{gr}_i, \mathcal{M}_i, \delta_i)$, $i = 0, \dots, n$ be Lagrangian branes, $h, h' \in \mathbb{R}^{n+1}$ two admissible shift vectors, and $\mathbb{L} = \mathbb{L}_h = (I, \gamma, P, \text{gr}, \mathcal{M}, \delta)$, $\mathbb{L}' = \mathbb{L}_{h'} = (I, \gamma, P', \text{gr}, \mathcal{M}, \delta)$ the corresponding composite Lagrangian branes. Then*

- (1) *For any $z \in Z(\mathbb{L}_i, \mathbb{L}_j)$ if the actions $A_h(z)$ and $A_{h'}(z)$ are positive, then $A_L(z) = A_h(z)$, $A_{L'}(z) = A_{h'}(z)$, and $\mathcal{A}_{\mathbb{L}}(z) = \mathcal{A}_{\mathbb{L}'}(z)$.*
- (2) *Let $z^0 \in Z(\mathbb{L}_0, \mathbb{L}_n)$ and $z^i \in Z(\mathbb{L}_{i-1}, \mathbb{L}_i)$, $i = 1, \dots, n$ be such that $A_h(z^i) > 0$ and $A_{h'}(z^i) > 0$ for all $i = 1, \dots, n$. Suppose that*

$$\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} : \mathcal{A}_{\mathbb{L}}(z^n) \otimes \dots \otimes \mathcal{A}_{\mathbb{L}}(z^1) \rightarrow \mathcal{A}_{\mathbb{L}}(z^0)$$

is non-zero. Then $A_h(z^0) > 0$, $A_{h'}(z^0) > 0$ and $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}'}^{z^0; z^n, \dots, z^1}$.

Proof. To show (1) first note that $A_h(z) \neq 0$ and $A_{h'}(z) \neq 0$ since h and h' are admissible. Positivity of these actions implies that the definitions of z_+ and z_- are the same for $\mathbb{L}$, $\mathbb{L}'$ (these are given in §3.2.2) and for $(L_0, \dots, L_n)$ (given in §6.1.1). Namely, we have $z_+ \in I_i$ and $z_- \in I_j$. This implies $A_L(z) = A_h(z)$, $A_{L'}(z) = A_{h'}(z)$ by comparing definitions in (3.8), (3.10), and (6.1), and $\mathcal{A}_{\mathbb{L}}(z) = \mathcal{A}_{\mathbb{L}'}(z)$ by (4.3).

We prove (2). Observe that

$$A_h(z^i) = A_L(z^i) > 0 \text{ and } A_{h'}(z^i) = A_{L'}(z^i) > 0 \text{ for } i = 1, \dots, n$$

by (1) of this lemma and

$$A_{L_i}(z) = A_h(z) = A_{h'}(z) = A_L(z) \text{ for any } z \in Z_{L_i}, i = 0, \dots, n$$

by definition. Non-vanishing of $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1}$ implies that there exists

$$\mathbf{u} \in \Delta^0(L; z^0; z_{j_n}^n, \dots, z_1^n, z^n, z_{j_{n-1}}^{n-1}, \dots, z_1^1, z^1, z_{j_0}^0, \dots, z_1^0)$$

for some $j_0, \dots, j_n \geq 0$ and $z_j^i \in Z_{L_i}$.

Note that $z_+^0 \in I_0$ and $z_-^0 \in I_n$ by any definition. For L it follows from the existence of $\mathbf{u}$, and for L' this follows by considering $\mathbf{u}$ as an element in $\Delta^0(L'; z^0; \dots)$ and applying Lemma 3.24(2). Therefore, we have $A_h(z^0) = A_L(z^0) > 0$ and $A_{h'}(z^0) = A_{L'}(z^0) > 0$. Since any such $\mathbf{u}$ can be viewed as an element of $\Delta^0(L'; z^0; \dots)$ and vice versa, and as the definition of $\mu_{\mathbb{L}}$ and $\mu_{\mathbb{L}'}$ does not depend on the values of the actions, but only on their signs, we conclude $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}'}^{z^0; z^n, \dots, z^1}$. $\square$

Proof of Proposition 6.4. That subsets of transverse sets of Lagrangians are transverse is clear from the definition in §6.1.1 hence transverse sequences form transverse sets in $\mathcal{W}$. $\mathcal{W}$ is k -linear by construction. It is small since we chose a set of domains of curves in (3.6).

That the choice of an admissible shift vector required in the definition of A_∞ -operations in $\mathcal{W}$ from §6.1.2 can be made follows from Lemma 6.1. By Lemma 6.5 the operation $\mu^{z^0; z^n, \dots, z^1}$ does not depend on this choice, and has the correct degree since the A_∞ -algebra $\mathcal{A}_{\mathbb{L}}$ is $\mathbb{Z}$ -graded.

To prove that the A_∞ -relations hold in $\mathcal{W}$ consider a transverse sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n)$ of Lagrangian branes and a finite set of morphisms $\{a^k \in \mathcal{W}(\mathbb{L}_{i_k}, \mathbb{L}_{j_k})\}_{k \in K}$ such that $i_k < j_k$ for all $k \in K$. By Lemma 6.1 there exists a composite Lagrangian brane $\mathbb{L} = \mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n)$ such that $A_h(z) > 0$ whenever the $\mathcal{W}(z)$ -component of one of the morphisms a^k is non-zero. We call such composite Lagrangian brane $\mathbb{L}$ *adapted* to $\{a^k\}_{k \in K}$. Note that if $\mathbb{L}$ is adapted to $\{a^k\}_{k \in K}$, then for any $l \geq 1$, $k_1, \dots, k_l \in K$ we have

$$(6.7) \quad \mu_{\mathbb{L}}^l(a^{k_l}, \dots, a^{k_1}) = \begin{cases} \mu_{\mathcal{W}}^l(a^{k_l}, \dots, a^{k_1}), & \text{if } \mu_{\mathcal{W}}^l(a^{k_l}, \dots, a^{k_1}) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases}$$

That is, the A_∞ -relation on a given sequence $\mathbb{L}_0 \xrightarrow{a^1} \dots \xrightarrow{a^n} \mathbb{L}_n$ of morphisms in $\mathcal{W}$ boils down to an A_∞ -relation in a composite Lagrangian brane $\mathbb{L}$ and holds since $\mathcal{A}_{\mathbb{L}}$ is an uncurved A_∞ -algebra. $\square$

6.1.4. Invariance. We prove two invariance statements for the geometric Fukaya A_∞ -pre-categories. One is that a graded exact symplectomorphism induces a strict equivalence of the Fukaya categories, see Lemma 6.6. Another is that deforming the line field or the weighted system of dg-categories (Lemma 6.7) or changing the Liouville form (Corollary 6.8) gives a strictly equivalent Fukaya category.

Let $\Phi: (S_0, Q_0) \rightarrow (S_1, Q_1)$ be a diffeomorphism of marked surfaces (i.e. Φ is orientation preserving and $\Phi(Q_0) = Q_1$). We have an induced 2-functor $\Phi_*: \mathcal{P}_2^+(S_0, Q_0) \rightarrow \mathcal{P}_2^+(S_1, Q_1)$. Therefore, given a weighted system of dg-categories $\mathcal{C}: \mathcal{P}_2^+(S_1, Q_1) \rightarrow \text{dgc}at_k$ we obtain a weighted system of dg-categories $\Phi^*\mathcal{C}: \mathcal{P}_2^+(S_0, Q_0) \rightarrow \text{dgc}at_k$ by pre-composing $\mathcal{C}$ with Φ_* .

Now let $\mathcal{S}_i = (S_i, Q_i, \eta_i, \lambda_i)$, $i = 0, 1$ be two graded Liouville marked surfaces, $\Phi = (\Phi, \tilde{\Phi}): \mathcal{S}_0 \rightarrow \mathcal{S}_1$ a graded exact symplectomorphism, and $\mathcal{C}_i: \mathcal{P}_2^+(\mathcal{S}_i) \rightarrow \text{dgc}at_k$ two weighted systems of dg-categories such that $\Phi^*\mathcal{C}_1 = \mathcal{C}_0$. We construct a strict equivalence of A_∞ -pre-categories $\mathcal{I}_\Phi: \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_0; \mathcal{C}_0) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_1; \mathcal{C}_1)$. First note that we have a bijection between Lagrangian pre-branes in $(\mathcal{S}_0; \mathcal{C}_0)$ and $(\mathcal{S}_1; \mathcal{C}_1)$ given by

$$\Phi_*\mathcal{L} = (I, \Phi \circ \gamma, P - g \circ \gamma, \Phi_*(\text{gr}) \cdot (\Phi\gamma)^*(\tilde{\Phi})^{-1}, \mathcal{M}) \text{ for } \mathcal{L} = (I, \gamma, P, \text{gr}, \mathcal{M})$$

where $g \in C^\infty(S_0, \mathbb{R})$ is such that $\Phi^*\lambda_1 - \lambda_0 = dg$ and $g(q_0) = 0$ for some fixed basepoint $q_0 \in S_0$. Secondly, observe that Φ induces a naïve isomorphism $\mathcal{F}_\Phi: \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\Phi_*\mathcal{L}}$ for any Lagrangian pre-brane $\mathcal{L}$ in $(\mathcal{S}_0; \mathcal{C}_0)$. Indeed, for $z \in Z_{\mathcal{L}}$ and $\Phi(z) \in Z_{\Phi_*\mathcal{L}}$ we have $|z| = |\Phi(z)|$ and

(6.8)

$$\mathcal{A}_{\mathcal{L}}(z) = (\mathcal{C}_0)_{\gamma(z_-)}(\mathcal{C}_0(z)\mathcal{M}(z_+), \mathcal{M}(z_-)) = (\mathcal{C}_1)_{\Phi\gamma(z_-)}(\mathcal{C}_1(\Phi(z))\mathcal{M}(z_+), \mathcal{M}(z_-)) = \mathcal{A}_{\Phi_*\mathcal{L}}(\Phi(z)),$$

and Φ gives a bijection between the polygons contributing to $\mathcal{A}_{\mathcal{L}}$ and to $\mathcal{A}_{\Phi_*\mathcal{L}}$ via (4.6).

Therefore, we obtain a bijection

$$\mathcal{I}_{\Phi}^0: \text{Ob } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_0; \mathcal{C}_0) \rightarrow \text{Ob } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_1; \mathcal{C}_1), (\mathcal{L}, \delta) \mapsto (\Phi_*\mathcal{L}, \mathcal{F}_{\Phi}(\delta)).$$

Note that Φ_* preserves transverse sets. Given two transverse Lagrangian branes $\mathbb{L}_0, \mathbb{L}_1$ in $(\mathcal{S}_0; \mathcal{C}_0)$ and $z \in Z(\mathbb{L}_0, \mathbb{L}_1)$ we have

$$\mathcal{I}_{\Phi}^1(z) = \text{id}: \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_0; \mathcal{C}_0)(z) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_1; \mathcal{C}_1)(\Phi(z))$$

via the same computation as in (6.8). Since A_{∞} -operations in the Fukaya categories are computed via A_{∞} -operations in obstruction algebras, we obtain

Lemma 6.6. $\mathcal{I}_{\Phi}: \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_0; \mathcal{C}_0) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}_1; \mathcal{C}_1)$ is a strict equivalence of A_{∞} -pre-categories.

We now pass to the other invariance. Let $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C})$ be a weighted Liouville surface.

Lemma 6.7. Let $\Phi_s \in \text{Diff}(S, Q)$, $s \in [0, 1]$ be a family of diffeomorphisms with $\Phi_0 = \text{id}$. Then

- (1) $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, (\Phi_1)_*\eta, \lambda); \mathcal{C})$ is strictly equivalent to $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, \eta, \lambda); \mathcal{C})$.
- (2) $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Phi_1^*\mathcal{C})$ is strictly equivalent to $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$.

Proof. To show (1) observe that the identity functor works since the line field only enters the constructions to determine the degrees, the degrees are defined as differences in gradings, and a grading with respect to $(\Phi_1)_*\eta$ can be obtained as concatenation of the grading with respect to η and $s \mapsto (\Phi_{1-s})_*\eta$.

The proof of (2) is similar to the proof of Proposition 4.8. Namely, given a Lagrangian pre-brane $\mathcal{L} = (I, \gamma, P, \text{gr}, \mathcal{M})$ in $(\mathcal{S}; \mathcal{C})$ we define the Lagrangian pre-brane $\mathcal{L}_1 = (I, \gamma, P, \text{gr}, \mathcal{M}_1)$ in $(\mathcal{S}; \Phi_1^*\mathcal{C})$ by setting $\mathcal{M}_1(p) = \mathcal{C}(s \mapsto \Phi_s\gamma(p))\mathcal{M}(p)$ for $p \in I$ and for a path $a: [0, 1] \rightarrow I$ the transition map is

$$\mathcal{M}_1(a) = \mathcal{C}(\Phi_1\gamma a; s \mapsto \Phi_s\gamma a(1), \gamma a, s \mapsto \Phi_{1-s}\gamma a(0); \sigma_a)(\text{id}_{\mathcal{M}_1(a(1))}, \mathcal{M}(a), \text{id}_{\mathcal{M}(a(0))})$$

where σ_a is the 2-morphism in $\mathcal{P}_2^+(\mathcal{S})$ given by $\Phi_s\gamma a(t)$.

Then we define a naïve isomorphism $(\text{id}, \mathcal{F}): \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ by

$$(6.9) \quad \begin{aligned} \mathcal{F}^z(\varphi) &= \mathcal{C}(\Phi_1 z; s \mapsto \Phi_s\gamma(z_-), z, s \mapsto \Phi_{1-s}\gamma(z_+); [(s, t) \mapsto \Phi_s z(t)]) \\ &\quad (\text{id}_{\mathcal{M}_1(z_-)}, \varphi, \text{id}_{\mathcal{M}(z_+)}) \end{aligned}$$

for $z \in Z_{\mathcal{L}}$ and $\varphi \in \mathcal{A}_{\mathcal{L}}(z) = \mathcal{C}_{\gamma(z_-)}(\mathcal{C}(z)\mathcal{M}(z_+), \mathcal{M}(z_-))$. The family Φ_s then gives the isotopies of polygons contributing via (4.6) that certify that $(\text{id}, \mathcal{F})$ is a naïve isomorphism.

Therefore, we obtain a bijection

$$\mathcal{I}^0: \text{Ob } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C}) \rightarrow \text{Ob } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Phi_1^*\mathcal{C}), (\mathcal{L}, \delta) \mapsto (\mathcal{L}_1, \mathcal{F}(\delta)).$$

Given two Lagrangian branes $\mathbb{L}_0, \mathbb{L}_1$ in $(\mathcal{S}; \mathcal{C})$ and $z \in Z(\mathbb{L}_0, \mathbb{L}_1)$ we have $\mathcal{I}^1(z): \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})(z) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Phi_1^*\mathcal{C})(z)$ via (6.9). Since A_{∞} -operations in the Fukaya A_{∞} -pre-categories are computed via A_{∞} -operations in obstruction algebras, we conclude that $\mathcal{I}: \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C}) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Phi_1^*\mathcal{C})$ is a strict equivalence of A_{∞} -pre-categories. $\square$

Corollary 6.8. Let λ' be another Liouville form on S compatible with the marking Q . Then $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, \eta, \lambda); \mathcal{C})$ is strictly equivalent to $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, \eta, \lambda'); \mathcal{C})$.

Proof. By Corollary 3.6 there exists a family $\Phi_s \in \text{Diff}(S, Q)$, $s \in [0, 1]$ of diffeomorphisms such that $\Phi_0 = \text{id}$ and Φ_1 is an exact symplectomorphism from (S, Q, λ) to (S, Q, λ') . By Lemma 6.6 $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, \eta, \lambda); \mathcal{C})$ is strictly equivalent to $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, (\Phi_1)_*\eta, \lambda'); (\Phi_1^{-1})^*\mathcal{C})$. By Lemma 6.7 the latter category is strictly equivalent to $\mathcal{W}_{\text{pre}}^{\text{geom}}((S, Q, \eta, \lambda'); \mathcal{C})$. $\square$

6.2. Transverse perturbations. Fix a weighted Liouville surface $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgcat}_k)$. Denote by $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ the geometric Fukaya A_∞ -pre-category.

In §6.2.1 we define transverse perturbations of Lagrangian branes and in §6.2.2 morphisms of local systems compatible with the brane structures. In §§6.2.3 and 6.2.4 we describe the intersection points and Reeb chords between a Lagrangian and its transverse perturbation and the polygons bounded by a composite Lagrangian containing them, see Lemma 6.12. In §6.2.5 we perturb a morphism f of local systems on a Lagrangian brane to a morphism in $\mathcal{W}$ and compute its differential. In §6.2.6 we prove that composition with the perturbation of f is given by f and that all higher products involving it vanish.

6.2.1. Transverse perturbations. Let $L_0 = (I, \gamma_0, P_0)$ be a Lagrangian in $\mathcal{S}$. Recall from Lemma 3.16 that a neighbourhood of L_0 in $\text{Lag}_I(\mathcal{S})$ is parametrised by a neighbourhood of $\{0\} \times \{\text{id}\}$ in $C^\infty(I, \mathbb{R}) \times \text{Diff}(I)$ via $(F, \varphi) \mapsto L(F, \varphi)$.

Definition 6.9. A Lagrangian $L_1 = (I, \gamma_1, P_1)$ is called a *transverse perturbation* of L_0 if $L_1 = L(F, \text{id})$ for some $F \in C^\infty(I, \mathbb{R})$ and $L_s = L(sF, \text{id})$ is transverse to L_0 for any $s \in (0, 1]$.

Note that $L(F, \text{id})$ is transverse to L_0 if and only if F' has only simple zeros and is nowhere zero on ∂I . In particular, it is enough for L_1 to be transverse to L_0 in Definition 6.9. Also note that if L_1 is a transverse perturbation of L_0 , then L_0 is a transverse perturbation of L_1 after choosing an appropriate Weinstein chart for L_1 .

Let L_1 be a transverse perturbation of L_0 . Recall from §4.2.2 that if we have a grading gr_0 or a local system $\mathcal{M}_0$ on L_0 , then we can transport them from L_0 to L_1 along the isotopy L_s from Definition 6.9. We therefore get the notion of *transverse perturbation* for graded Lagrangians and Lagrangian pre-branes.

Let $\mathbb{L}_0 = (\mathcal{L}_0, \delta_0)$ be a Lagrangian brane in $(\mathcal{S}; \mathcal{C})$ and $\mathcal{L}_1$ a transverse perturbation of the Lagrangian pre-brane $\mathcal{L}_0$. Then, by Proposition 4.8 the obstruction algebras $\mathcal{A}_{\mathcal{L}_0}$ and $\mathcal{A}_{\mathcal{L}_1}$ are naively isomorphic. In particular, we get a bounding cochain $\delta_1 \in \mathcal{A}_{\mathcal{L}_1}$ corresponding to the bounding cochain $\delta_0 \in \mathcal{A}_{\mathcal{L}_0}$. We call the resulting Lagrangian brane $\mathbb{L}_1 = (\mathcal{L}_1, \delta_1)$ a *transverse perturbation* of $\mathbb{L}_0$.

6.2.2. Morphisms of local systems. Let $L_0 = (I, \gamma_0, P_0)$ be a Lagrangian in $\mathcal{S}$ and $\mathcal{M}_0^0, \mathcal{M}_0^1$ two local systems of objects over $\gamma_0^* \mathcal{C}$ on I . A collection of morphisms

$$f(t) \in \mathcal{C}_{\gamma_0(t)}(\mathcal{M}_0^0(t), \mathcal{M}_0^1(t)) \text{ for each } t \in I$$

is called a *morphism of local systems* $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ if it commutes with transition maps, i.e.

$$(6.10) \quad \mathcal{M}_0^1(a) \cdot \mathcal{C}(\gamma_0 a) f(a(0)) = f(a(1)) \cdot \mathcal{M}_0^0(a) \text{ for each path } a: [0, 1] \rightarrow I.$$

Note that (6.10) implies that a morphism $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ is determined by $\{f(t_C)\}_{C \in \pi_0 I}$ for any choice of $t_C \in C$. If $C \in \pi_0 I$ is a segment, then (6.10) poses no restriction on $f(t_C)$. If $C \in \pi_0 I$ is a circle, then (6.10) for the component C is equivalent to

$$\mathcal{M}_0^1(a) \cdot \mathcal{C}(\gamma_0 a) f(t_C) = f(t_C) \cdot \mathcal{M}_0^0(a)$$

for one path $a: [0, 1] \rightarrow C$ that parametrises C with $a(0) = a(1) = t_C$.

If $\mathbb{L}_0^i = (L_0, \text{gr}_0, \mathcal{M}_0^i, \delta_0^i)$ are Lagrangian branes, then a morphism of local systems $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ is *compatible with the brane structures* if

$$(6.11) \quad f(z_-) \cdot (\delta_0^0)_z = (-1)^{|f|(1-|z|)} (\delta_0^1)_z \cdot \mathcal{C}(z) f(z_+) \text{ for any } z \in Z_{L_0}$$

where $|f| = m$ if $|f(t)| = m$ for all $t \in I$ and (6.11) is understood to hold for each homogeneous component of f .

6.2.3. Transverse double of a Lagrangian. We introduce some notation for intersection points and Reeb chords between a Lagrangian and its transverse perturbation.

Definition 6.10. A *transverse double* of a Lagrangian $L_0 = (I, \gamma_0, P_0)$ is a composite Lagrangian $L_h = L_h(L_0, L_1)$ such that

- (1) $L_1 = L(F_1, \text{id})$ is a transverse perturbation of L_0 and $h \in \mathbb{R}^2$ is an admissible shift vector for $(L_0, L(sF_1, \text{id}))$ for any $s \in (0, 1]$.

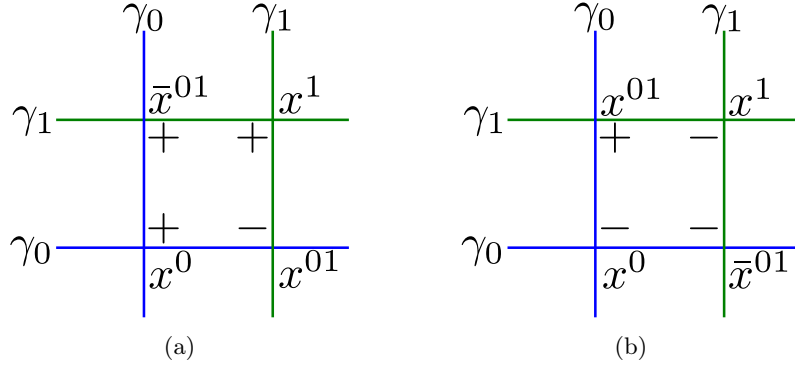


FIGURE 23. Neighbourhood of a perturbed self-intersection

- (2) For each double point x^0 of L_0 , there is a double point $x^1 = x^1(x^0)$ of L_1 , intersection points $x^{01} = x^{01}(x^0)$, $\bar{x}^{01} = \bar{x}^{01}(x^0) \in X(L_0, L_1)$ and a neighbourhood U_{x^0} such that $\{U_{x^0}\}_{x^0 \in X_{L_0}}$ are pairwise disjoint and L_h in U_{x^0} is diffeomorphic to one of the pictures in Fig. 23.
- (3) For each connected component C of I that is a circle, there is an integer $k_C \geq 1$ such that the intersection points in $Z(L_0|_C, L_1|_C) = X(L_0|_C, L_1|_C)$ that are not of the form x^{01} or $\bar{x}^{01}$ from (2) split into two sets, $Q_C = \{q^i\}_{i \in \mathbb{Z}/k_C\mathbb{Z}}$ and $P_C = \{p^i\}_{i \in \mathbb{Z}/k_C\mathbb{Z}}$. These points satisfy $A_h(q^i) > 0$ and $A_h(p^i) > 0$ for each $i \in \mathbb{Z}/k_C\mathbb{Z}$ and

$$|\Delta^0(L_h; p^i; q^j)| = \begin{cases} 1, & \text{if } i = j \text{ or } i + 1 = j \text{ in } \mathbb{Z}/k_C\mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

That is, there is an immersion $i_C: T_C \subset \mathbb{R}^2 \rightarrow S \setminus (\Lambda \cup D)$, where T_C is the shaded area in Fig. 24(a), that extends the immersions given by bigons from $\Delta^0(L_h; p^i; q^j)$ (that the numbering goes counter-clockwise is a condition on the indexing of Q_C and P_C).

- (4) For each $t \in \partial I$ there exists a unique Reeb chord $y_t \in Z(L_0, L_1) \cup Z(L_1, L_0)$ that is isotoped to the constant path at $\gamma_1(t)$ by the isotopy from L_0 to L_1 . $A_h(y_t) > 0$ if $y_t \in Z(L_0, L_1)$ and $A_h(y) < 0$ for any $y \in Z(L_1, L_0)$.
- (5) For each connected component C of I that is a segment, there is an integer $k_C \geq 1$ such that the intersection points in $Z(L_0|_C, L_1|_C)$ that are not of the form x^{01} or $\bar{x}^{01}$ from (2) and the Reeb chords $y_t \in Z(L_0|_C, L_1|_C)$ for $t \in \partial I$ split into two sets, $Q_C = \{q^i\}_{i=1}^{k_C}$ and $P_C = \{p^i\}_{i=1}^{k_C-1}$. These points and chords satisfy $A_h(q^i) > 0$ and $A_h(p^i) > 0$ for each i and

$$|\Delta^0(L_h; p^i; q^j)| = \begin{cases} 1, & \text{if } i = j \text{ or } i + 1 = j; \\ 0, & \text{otherwise.} \end{cases}$$

That is, there is an immersion $i_C: T_C \subset \mathbb{R}^2 \rightarrow S \setminus (\Lambda \cup D)$, where T_C is the closed shaded area in Fig. 24(b), that extends the immersions given by bigons from $\Delta^0(L_h; p^i; q^j)$.

- (6) The set X_{L_h} of self-intersection points of L_h satisfies

$$X_{L_h} \sqcup (\{y_t \mid t \in \partial I\} \cap Z(L_0, L_1)) = \bigsqcup_{x^0 \in X_{L_0}} \{x^0, x^1, x^{01}, \bar{x}^{01}\} \sqcup \bigsqcup_{C \in \pi_0 I} Q_C \sqcup P_C.$$

If L_1 is any transverse perturbation of L_0 , then by Lemma 6.1 we can pick an admissible shift vector $h \in \mathbb{R}^2$ such that $L_h = L_h(L_0, L_1)$ is a transverse double of L_0 . Note that even if L_1 is C^∞ -close to L_0 , we might not be able to choose the admissible vector h close to $0 \in \mathbb{R}^2$. Namely, for $x^0 \in X_{L_0}$ the local model from (2) around this self-intersection implies $A_0(\bar{x}^{01}) + h_0 - h_1 = A_h(\bar{x}^{01}) > 0$, but $A_0(\bar{x}^{01}) = P_0(\bar{x}_+^{01}) - P_1(\bar{x}_-^{01})$ is close to $P_0(x_-^0) - P_0(x_+^0) = -A_{L_0}(x^0)$ which is a negative number that depends only on L_0 , not on the perturbation.

Suppose we have a grading gr_0 on L_0 . Recall from §4.2.2 that then gr_0 transports to a grading gr_1 on L_1 . A straightforward computation using (3.18) gives

- (1) for each double point x^0 of L_0 we have $|x^1| = |x^{01}| = |x^0|$ and $|\bar{x}^{01}| = 1 - |x^0|$;

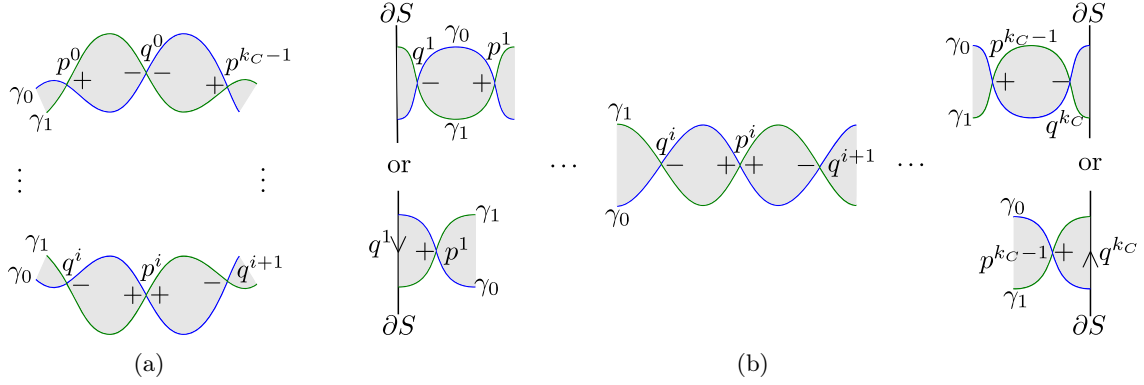


FIGURE 24. Pullbacks of perturbed components of the Lagrangian along bigons

(2) for each connected component C of I we have $|q| = 0$ for any $q \in Q_C$ and $|p| = 1$ for any $p \in P_C$.

6.2.4. *Lagrangians containing a transverse double.* Let $L_0 = (I, \gamma_0, P_0)$ and $L = (J, \gamma, P)$ be two Lagrangians.

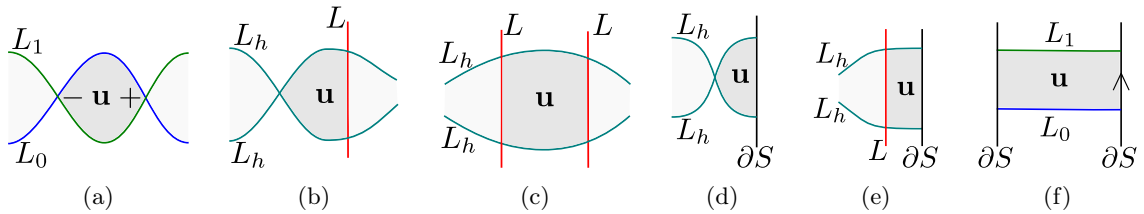
Definition 6.11. We say that L contains a transverse double of L_0 nicely if L is the composite Lagrangian $L_{(h_0, h_1, h_2)}(L_0, L_1, L_2)$ such that

- (1) $L_1 = (I, \gamma_1, P_1) = L(F_1, \text{id})$ is a transverse perturbation of L_0 and $L_{(h_0, h_1)}(L_0, L_1)$ is a transverse double of L_0 .
- (2) For each $s \in (0, 1]$ the Lagrangian $L_{(h_0, h_1)}(L_0, L(sF_1, \text{id}))$ is transverse to L_2 .

Note that a transverse double of L_0 contains itself nicely. Observe that (2) in Definition 6.11 implies that each connected component C of I the closure (in $\mathbb{R}^2$) of the pullback of γ by the immersion i_C to the interior of the shaded region T_C in Fig. 24 is a collection of disjoint segments with one end on γ_0 and another end on γ_1 .

We give a criterion for a polygon with convex corners bounded by L to be contained in the shaded region T_C from Fig. 24.

Lemma 6.12. Let $L = (J, \gamma, P)$ be a Lagrangian containing a transverse double $L_h = L_h(L_0, L_1)$ of $L_0 = (I, \gamma_0, P_0)$ nicely, C a connected component of I , $\mathbf{u} = (\Pi, u)$ a polygon with convex corners bounded by L , and $v, v' \in \Pi$ two vertices of Π such that $v = v'$ or $[v, v']$ is a side of Π . Suppose that $u(v) \in \text{Im } \gamma_0$, $u(v') \in \text{Im } \gamma_1$, and there is an embedding $\varphi: U \rightarrow T_C$ where U is a neighbourhood of $[v, v']$ in Π such that $i_C \circ \varphi = u|_U$. Then φ can be extended to an embedding $\varphi: \Pi \rightarrow T_C$ such that $i_C \circ \varphi = u$. That is, $\mathbf{u}$ is one of the polygons in Fig. 25.

FIGURE 25. Polygons contained in T_C

Note that convex polygons bounded by L that have a negative corner at Q_C or a positive corner at P_C satisfy the assumption of Lemma 6.12. This lemma is similar to [1, Lemma 4.4], our proof is somewhat different.

Proof. Note that i_C induces an orientation on C : it is the counter-clockwise orientation in Fig. 24(a) if C is a circle and the orientation from left to right in Fig. 24(b) if C is a segment. By abuse of notation we will not always distinguish between the objects in S and their pullbacks to $\mathbb{R}^2$ via i_C .

Let the regions $R_1, \dots, R_r \subset S$ be the closures of connected components of $S \setminus \text{Im } \gamma$. Since L contains a transverse double of L_0 nicely, the pullback of γ to $T = T_C$ consists of the boundary of T and a collection of disjoint segments with one end on γ_0 and another end on γ_1 . These segments divide T into regions each of which is a copy of some R_i , $i = 1, \dots, r$ and is diffeomorphic to one of the darker shaded regions in Fig. 25 (with an additional condition that there are no segments of γ inside the region). Let $T_1, \dots, T_l$ be these regions, listed in the direction of the orientation of C . When C is a circle we think of the indices $1, \dots, l$ as elements of $\mathbb{Z}/l\mathbb{Z}$, the indexing as above is not unique in this case, we pick any.

Pulling γ to Π along u we see that Π is glued from copies of the regions R_i , $i = 1, \dots, r$. To prove the lemma it suffices to show that these regions are $T_j, T_{j+1}, \dots, T_{j+m}$ or $T_j, T_{j-1}, \dots, T_{j-m}$ for some $m \geq 0$ and Π is glued from them the same way as T .

Let R_{i_0} be the region that contains a neighbourhood of $[v, v']$ in Π . That $[v, v']$ is contained in one region is clear when $v = v'$ and follows from the existence of φ from the assumption when $[v, v']$ is a side of Π . The existence of φ also implies that R_{i_0} has a copy $T_j \subset T$. Set $\sigma = 1$ if T_j follows $[u(v), u(v')]$ in the direction of C , set $\sigma = -1$ otherwise.

Let v^1 (resp. v^{-1}) be the vertex of Π following (resp. preceding) the (possibly degenerate) segment $[v, v']$ in the counter-clockwise order on $\partial\Pi$. Let w^1 (resp. w^{-1}) be the vertex of R_{i_0} that follows (resp. precedes) the segment $[v, v']$ in the counter-clockwise order on ∂R_{i_0} . If $w^1 \neq v^1$, then define R_{i_1} to be the region that contains the corner at w^1 not contained in R_{i_0} . Observe that this region is the same as the region $T_{j+\sigma}$. Therefore, w^{-1} is a vertex of R_{i_1} and the regions R_{i_0} and R_{i_1} share the side $[w^{-1}, w^1]$. Let w^2 (resp. w^{-2}) be the vertex of R_{i_1} that follows (resp. precedes) w^1 (resp. w^{-1}) in the counter-clockwise order on ∂R_{i_1} . If $w^2 \neq v^1$, then define R_{i_2} to be the region that contains the corner at w^2 not contained in R_{i_1} . Observe that this region is the same as the region $T_{j+2\sigma}$. Therefore, w^{-2} is a vertex of R_{i_2} and the regions R_{i_1} and R_{i_2} share the side $[w^{-2}, w^2]$. Continuing this process while we can we obtain a sequence of regions $R_{i_0}, \dots, R_{i_m} \subset \Pi$ and the corresponding sequence of regions $T_j, \dots, T_{j+m\sigma} \subset T$ for some $m \geq 0$. Since we cannot continue, $w^{m+1} = v^1$.

If $w^{-m-1} = w^{m+1}$, then $R_{i_0}, \dots, R_{i_m}$ cover all Π and we are done. In this case $\mathbf{u}$ is a polygon from Figs. 25(a), 25(b), and 25(d) with $u(w^{m+1}) \in P_C \cup Q_C$. Suppose $w^{-m-1} \neq w^{m+1}$. Then $[w^{-m-1}, w^{m+1}]$ is a side of R_{i_m} since $[u(w^{-m-1}), u(w^{m+1})]$ is a side of $T_{j+m\sigma}$. Moreover, as $v^1 = w^{m+1}$, we have $[w^{-m-1}, w^{m+1}] \subset \partial\Pi$. Therefore, going from v to w^{m+1} to w^{-m-1} to v is a closed curve on $\partial\Pi$ hence $R_{i_0}, \dots, R_{i_m}$ cover all Π and we are done. If $[u(w^{-m-1}), u(w^{m+1})]$ is a sub-segment of γ , then $\mathbf{u}$ is a polygon from Figs. 25(b), 25(c), and 25(e) with $[u(w^{-m-1}), u(w^{m+1})]$ being a red side. If $[u(w^{-m-1}), u(w^{m+1})]$ is a sub-segment of ∂S , then $\mathbf{u}$ is a polygon from Figs. 25(d) to 25(f) with $[u(w^{-m-1}), u(w^{m+1})]$ being a side lying on the boundary. $\square$

6.2.5. Perturbed morphisms. Let $\mathbb{L}_0^i = (L_0 = (I, \gamma_0, P_0), \text{gr}_0, \mathcal{M}_0^i, \delta_0^i)$ be two Lagrangian branes and $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ be a morphism of local systems compatible with the brane structures, i.e. (6.11) holds.

Let $\mathbb{L}_1^1 = (L_1 = (I, \gamma_1, P_1), \text{gr}_1, \mathcal{M}_1^1, \delta_1^1)$ be a transverse perturbation of $\mathbb{L}_0^1$. Recall from §6.2.3 that for each component C of I we have finite sets $Q_C, P_C \subset Z(L_0|_C, L_1|_C)$ with $|z| = 0$ if $z \in Q_C$ and $|z| = 1$ if $z \in P_C$. For any $q \in Q_C$ we have $\mathcal{C}(q)\mathcal{M}_0^1(q_+) = \mathcal{M}_1^1(q_-)$ hence we have a morphism $\mathcal{C}(q)f(q_+) \in \mathcal{W}(q) = \mathcal{C}_{\gamma_1(q_-)}(\mathcal{C}(q)\mathcal{M}_0^0(q_+), \mathcal{M}_1^1(q_-))$. We define the *perturbation of f to $\mathcal{W}(\mathbb{L}_0^0, \mathbb{L}_1^1)$* as

$$(6.12) \quad f_{\mathbb{L}_0^0, \mathbb{L}_1^1} = \sum_{C \in \pi_0 I} \sum_{q \in Q_C} \mathcal{C}(q)f(q_+) \in \mathcal{W}(\mathbb{L}_0^0, \mathbb{L}_1^1).$$

Lemma 6.13. *Let Lagrangian branes $\mathbb{L}_0^0, \mathbb{L}_0^1, \mathbb{L}_1^1$ be as above. Suppose that $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ is a morphism of local systems compatible with the brane structures, homogeneous of degree m . Then*

$$|f_{\mathbb{L}_0^0, \mathbb{L}_1^1}| = m \text{ and}$$

$$(6.13) \quad \mu^1(f_{\mathbb{L}_0^0, \mathbb{L}_1^1}) = (-1)^m (df)_{\mathbb{L}_0^0, \mathbb{L}_1^1}.$$

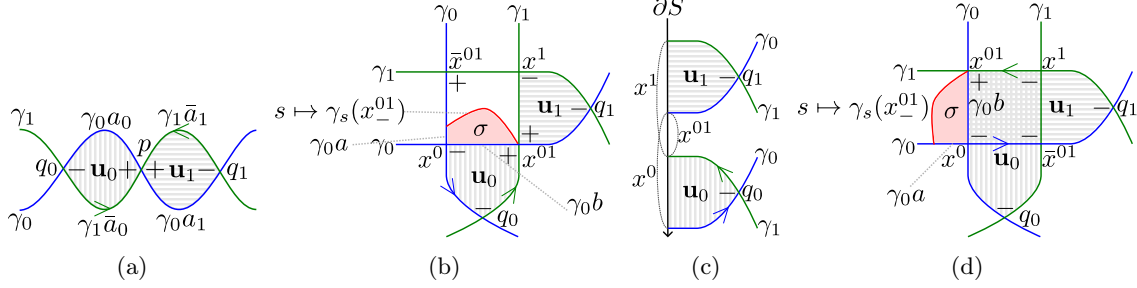


FIGURE 26. Polygons contributing to $\mu^1(f_{01})$

Proof. Denote $f_{01} = f_{\mathbb{L}_0^0, \mathbb{L}_1^1}$. We have $|f_{01}| = m$ by construction since $|q| = 0$ for each $q \in Q_C$, $C \in \pi_0 I$ and $|f(t)| = m$ for $t \in I$. Let $L_h = L_h(L_0, L_1)$ be a transverse double of L_0 , as in §6.2.3. $\mathbb{L}_h = \mathbb{L}_h(\mathbb{L}_0^0, \mathbb{L}_1^1)$ is adapted to f_{01} , so $\mu^1(f_{01}) = \mu_{\mathbb{L}_h}^1(f_{01})$ by definition. By Lemma 3.18 and Proposition 4.8 we can assume that all $q \in Q_C$, $C \in \pi_0 I$ are double points.

The right-hand side of (6.13) is the part of $\mu_{\mathbb{L}_h}^1(f_{01})$ coming from (4.5). Lemma 6.12 applied to L_h implies that the polygons contributing to $\mu_{\mathbb{L}_h}^1(f_{01})$ via (4.6) come in pairs $(\mathbf{u}_0, \mathbf{u}_1)$ as depicted in Fig. 26. It therefore remains to prove the contributions from $\mathbf{u}_0$ and $\mathbf{u}_1$ cancel in each case.

Fig. 26(a). As $|p| = 1$ and $|q_0| = |q_1| = 0$, the contribution of $\mathbf{u}_0$ and $\mathbf{u}_1$ to $\mu_{\mathbb{L}_h}^1(f_{01})$ have opposite sign in (4.6), so we need to prove $\mu^{\mathbf{u}_0}(f(q_{0+})) = \mu^{\mathbf{u}_1}(f(q_{1+}))$.

For $j = 0, 1$ let a_j be the path in I from p_+ to q_{j-} and $\bar{a}_j$ its reverse, as in Fig. 26(a). Then

$$\mu^{\mathbf{u}_j}(f(q_{j+})) = \mathcal{C}(p; \gamma_1 \bar{a}_j, q_j, \gamma_0 a_j; \mathbf{u}_j)(\mathcal{M}_1^1(\bar{a}_j), f(q_{j+}), \mathcal{M}_0^0(a_j))$$

and, by (4.8),

$$\mathcal{M}_1^1(\bar{a}_j) = \mathcal{C}(\gamma_1 \bar{a}_j; p, \gamma_0 \bar{a}_j, q_j; \mathbf{u}_j)(\text{id}, \mathcal{M}_0^0(\bar{a}_j), \text{id}).$$

Since the orientations of $\mathbf{u}_j$ in the above two expressions are opposite, Lemma 1.6 and the fact that f commutes with the transition maps of the local systems imply

$$\mu^{\mathbf{u}_j}(f(q_{j+})) = \mathcal{M}_0^0(\bar{a}_j) \cdot \mathcal{C}(\gamma_0 \bar{a}_j) f(q_{j+}) \cdot \mathcal{C}(\gamma_0 \bar{a}_j) \mathcal{M}_0^0(a_j) = f(p_+) \cdot \mathcal{M}_0^0(\bar{a}_j) \cdot \mathcal{C}(\gamma_0 \bar{a}_j) \mathcal{M}_0^0(a_j) = f(p_+)$$

for $j = 0, 1$.

Figs. 26(b) and 26(c). We have $\mathbf{u}_0 \in \Delta^0(L_h; x^{01}; q_0, x^0)$ and $\mathbf{u}_1 \in \Delta^0(L_h; x^{01}; x^1, q_1)$ for some $x^0 \in X_{L_0}$, $q_0 \in Q_{C_0}$, $q_1 \in Q_{C_1}$, and $C_0, C_1 \in \pi_0 I$. By Lemma 3.18 and Proposition 4.8 we can reduce Fig. 26(c) to Fig. 26(b), so we only consider the latter case.

Note that by the degree computations in §6.2.3 the components $(\delta_0^0)_{x^0}$ and $(\delta_1^1)_{x^1}$ both have degree $1 - |x^0| = 1 - |x^1|$. Comparing the signs in (4.6) for $\mathbf{u}_0$ and $\mathbf{u}_1$, we see that we need to prove

$$(6.14) \quad \mu^{\mathbf{u}_0}(f(q_{0+}), (\delta_0^0)_{x^0}) = (-1)^{m(1-|x^0|)} \mu^{\mathbf{u}_1}((\delta_1^1)_{x^1}, f(q_{1+})).$$

Expanding the definitions for $\mu^{\mathbf{u}_i}$ from (4.7), $\mathcal{M}_1^1$ from (4.8) and $(\delta_1^1)_{x^1} = \mathcal{F}^{x^0}((\delta_0^0)_{x^0})$ from (4.9), using that f commutes with the transition maps of the local systems, the compatibility (6.11) of f with the brane structure at $z = x^0$, and Lemma 1.6, we obtain that both sides of (6.14) are equal to

$$(6.15) \quad \mathcal{C}(x^{01}; s \mapsto \gamma_s(x_-^{01}), \gamma_0 a, x^0, \gamma_0 b; \sigma)(\mathcal{C}(s \mapsto \gamma_s(x_-^{01})) f(x_-^{01}), \mathcal{M}_0^0(a), (\delta_0^0)_{x^0}, \mathcal{M}_0^0(b))$$

in the notation of Fig. 26(b) where b is the path from x_+^{01} to x_+^0 in I and a is the path from x_-^0 to x_-^{01} in I .

Fig. 26(d). We have $\mathbf{u}_0 \in \Delta^0(L_h; x^{01}; x^1, q_0)$ and $\mathbf{u}_1 \in \Delta^0(L_h; x^{01}; q_1, x^0)$ for some $x^0 \in X_{L_0}$, $q_0 \in Q_{C_0}$, $q_1 \in Q_{C_1}$, and $C_0, C_1 \in \pi_0 I$. The signs in (4.6) for $\mathbf{u}_0$ and $\mathbf{u}_1$ imply that we need to prove

$$\mu^{\mathbf{u}_0}((\delta_1^1)_{x^1}, f(q_{0+})) = (-1)^{m(1-|x^0|)} \mu^{\mathbf{u}_1}(f(q_{1+}), (\delta_0^0)_{x^0}).$$

Arguing similarly to the case of Fig. 26(b), we conclude that both sides are equal to $(-1)^{m(1-|x^0|)}$ times the expression in (6.15) in the notation of Fig. 26(d). $\square$

6.2.6. Composition with perturbed morphisms. Let L_0 and T be Lagrangians in $\mathcal{S}$, L_1 a transverse perturbation of L_0 and $L = L_{(h_0, h_1, h_2)}(L_0, L_1, T)$ a composite Lagrangian containing the transverse double $L_{(h_0, h_1)}(L_0, L_1)$ of L_0 nicely (see Definition 6.11). Let $\mathbb{L}_0^i = (L_0, \text{gr}_0, \mathcal{M}_0^i, \delta_0^i)$, $i = 0, 1$ and $\mathbb{T} = (T, \text{gr}_T, \mathcal{M}_T, \delta_T)$ be Lagrangian branes in $(\mathcal{S}; \mathcal{C})$, $\mathbb{L}_1^1$ the transverse perturbation of $\mathbb{L}_0^1$ supported on L_1 , and $\mathbb{L} = \mathbb{L}_{h_0, h_1, h_2}(\mathbb{L}_0^0, \mathbb{L}_1^1, \mathbb{T})$ the corresponding composite Lagrangian brane. Finally, let $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$ be a morphism of local systems compatible with brane structures, homogeneous of degree m .

Note that $\mathbb{L}$ is adapted to the perturbation $f_{01} = f_{\mathbb{L}_0^0, \mathbb{L}_1^1} \in \mathcal{W}(\mathbb{L}_0^0, \mathbb{L}_1^1)$. We describe higher compositions μ that can be computed as $\mu_{\mathbb{L}}$ and have f_{01} as one of the arguments. The following is an immediate corollary of Lemma 6.12.

Lemma 6.14. *Let $\mathbb{L}$ and f_{01} be as above. Then for any $n \geq 3$*

$$\mu_{\mathbb{L}}^n(\dots, f_{01}, \dots) = 0.$$

Recall that $\mu_{\mathbb{L}}^1(f_{01})$ is computed in Lemma 6.13. It remains to describe $\mu_{\mathbb{L}}^2$ when one of the arguments is f_{01} .

Given $z^0 \in Z(\mathbb{T}, \mathbb{L}_0^0)$ (resp. $z^1 \in Z(\mathbb{L}_1^1, \mathbb{T})$) we can pick h_2 above in such a way that $A_{(h_0, h_2)}(z^0) > 0$ (resp. $A_{(h_1, h_2)}(z^1) > 0$). Tracking z^0 (resp. z^1) along the isotopy between L_0 and L_1 we obtain $z^1 \in Z(\mathbb{T}, \mathbb{L}_1^1)$ (resp. $z^0 \in Z(\mathbb{L}_0^0, \mathbb{T})$). The isotopy from L_0 to L_1 induces an isomorphism

$$\mathcal{F}^{z^0}: \mathcal{C}_{\gamma_0(z_-^0)}(\mathcal{C}(z^0)\mathcal{M}_T(z_+^0), \mathcal{M}_0^1(z_-^0)) \rightarrow \mathcal{C}_{\gamma_1(z_-^1)}(\mathcal{C}(z^1)\mathcal{M}_T(z_+^1), \mathcal{M}_1^1(z_-^1)) = \mathcal{W}(z^1)$$

$$\left(\text{resp. } \mathcal{F}^{z^1}: \mathcal{W}(z^1) = \mathcal{C}_{\gamma_T(z_-^1)}(\mathcal{C}(z^1)\mathcal{M}_1^1(z_+^1), \mathcal{M}_T(z_-^1)) \rightarrow \mathcal{C}_{\gamma_T(z_-^0)}(\mathcal{C}(z^0)\mathcal{M}_0^1(z_+^0), \mathcal{M}_T(z_-^0)) \right)$$

as in (4.9). We set

$$f_*(\varphi) = \mathcal{F}^{z^0}(f(z_-^0) \cdot \varphi) \text{ for } \varphi \in \mathcal{W}(z^0)$$

$$\left(\text{resp. } f^*(\varphi) = (-1)^{m|z^1|} \mathcal{F}^{z^1}(\varphi) \cdot \mathcal{C}(z^0)f(z_+^0) \text{ for } \varphi \in \mathcal{W}(z^1) \right).$$

Combining these maps for all $z^0 \in Z(\mathbb{T}, \mathbb{L}_0^0)$ (resp. $z^1 \in Z(\mathbb{L}_1^1, \mathbb{T})$) we obtain morphisms of graded modules of degree m

$$f_* = f_{*}^{\mathbb{L}_0^0, \mathbb{L}_1^1}: \mathcal{W}(\mathbb{T}, \mathbb{L}_0^0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_1^1) \text{ (resp. } f^* = f_{*}^{\mathbb{L}_0^0, \mathbb{L}_1^1}: \mathcal{W}(\mathbb{L}_1^1, \mathbb{T}) \rightarrow \mathcal{W}(\mathbb{L}_0^0, \mathbb{T})).$$

If f is the identity, these are the isomorphisms induced by the isotopy from L_0 to L_1 .

Lemma 6.15. *Let $\mathbb{L}_0^0$, $\mathbb{L}_1^1$, and $\mathbb{T}$ be Lagrangian branes as above. Then*

- (1) $f_{01} \circ - = f_*: \mathcal{W}(\mathbb{T}, \mathbb{L}_0^0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_1^1)$;
- (2) $- \circ f_{01} = f^*: \mathcal{W}(\mathbb{L}_1^1, \mathbb{T}) \rightarrow \mathcal{W}(\mathbb{L}_0^0, \mathbb{T})$.

Proof. We prove (1). Fix $z^0 \in Z(\mathbb{T}, \mathbb{L}_0^0)$ and $\varphi \in \mathcal{W}(z^0)$, and pick h_2 such that $A_{(h_0, h_2)}(z^0) > 0$ and $A_{(h_1, h_2)}(z^s) > 0$ where z^s is the trace of z^0 along the isotopy from L_0 to L_1 . Let C be the connected component of I containing z_-^0 . By Lemma 6.12 the only polygons contributing to $\mu_{\mathbb{L}}^2(f_{01}, \varphi)$ are the polygons $\mathbf{u} \in \Delta^0(L; z^1; q, z^0)$ with $q \in Q_C$. By Lemma 3.18 and Proposition 4.8 we can assume that q is a double point, so $\mathbf{u}$ is a polygon as in Figs. 25(b) and 25(d).

We compute the sign of the contribution of $\mathbf{u}$ in (4.6). The sign in front of the sum is $(-1)^{|\varphi|}$. We have $|z^1| = |z^0|$ and the sides of $\mathbf{u}$ lying on L_0 and on its parallel copy L_1 are traversed in opposite directions, so the product of the signs $\varepsilon_i^{\mathbf{u}}$ is $(-1)^{|z^0|}$. Multiplying all together, we obtain

$$f_{01} \circ \varphi = (-1)^{|\varphi|+|z^0|} \mu_{\mathbb{L}}^2(f_{01}, \varphi) = \mu^{\mathbf{u}}(\mathcal{C}(q)f(q_+), \varphi).$$

Expanding the definition of $\mu^{\mathbf{u}}$ from (4.7), using that f commutes with the transition maps of the local systems to move f along the side of $\mathbf{u}$ lying on L_0 from q_+ to z_-^0 , and applying Lemma 1.6 to the strip of $\mathbf{u}$ between L_0 and L_1 , we conclude

$$\mu^{\mathbf{u}}(\mathcal{C}(q)f(q_+), \varphi) = \mathcal{F}^{z^0}(f(z_-^0) \cdot \varphi) = f_*(\varphi).$$

The proof of (2) is analogous. For $z^1 \in Z(\mathbb{L}_1^1, \mathbb{T})$ and $\varphi \in \mathcal{W}(z^1)$ the only contributing polygons are $\mathbf{u} \in \Delta^0(L; z^0; z^1, q)$ with $q \in Q_C$. The sign in front of the sum in (4.6) is $(-1)^{(|z^1|-1)m}$, and the product of the signs $\varepsilon_i^{\mathbf{u}}$ is 1 since $\varepsilon_0^{\mathbf{u}} = \varepsilon_2^{\mathbf{u}}$ and $|z^0| = |z^1|$. Hence

$$\varphi \circ f_{01} = (-1)^m \mu_{\mathbb{L}}^2(\varphi, f_{01}) = (-1)^{m|z^1|} \mu^{\mathbf{u}}(\varphi, \mathcal{C}(q)f(q_+)) = (-1)^{m|z^1|} \mathcal{F}^{z^1}(\varphi) \cdot \mathcal{C}(z^0)f(z_+^0) = f^*(\varphi). \quad \square$$

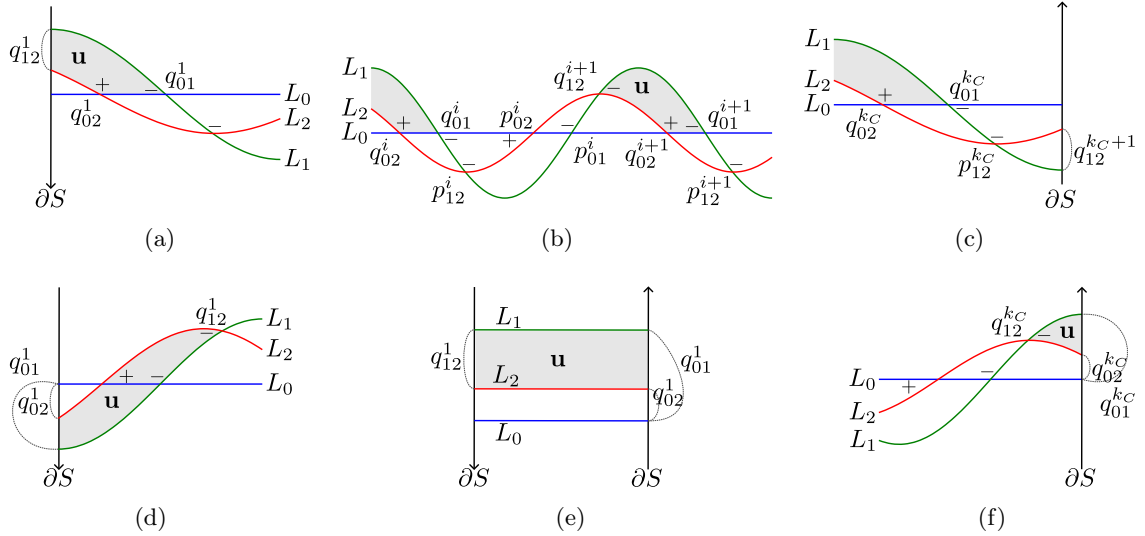


FIGURE 27. L_2 such that L contains $L_{(h_0, h_2)}(L_0, L_2)$ and $L_{(h_1, h_2)}(L_1, L_2)$ nicely.

We finish this section with a useful corollary of Lemma 6.15. Let L_0 be a Lagrangian and $L_1 = L(F_1, \text{id})$ be its transverse perturbation. Then we can find a Lagrangian $L_2 = L(F_2, \text{id})$ such that the composite Lagrangian $L = L_{(h_0, h_1, h_2)}(L_0, L_1, L_2)$ contains the transverse doubles $L_{(h_0, h_2)}(L_0, L_2)$ of L_0 and $L_{(h_1, h_2)}(L_1, L_2)$ of L_1 nicely as in Fig. 27. Note that L_2 can be chosen arbitrarily close in $\text{Lag}_I(\mathcal{S})$ to either L_0 or L_1 .

Now let $\mathbb{L}_0^i = (L_0, \text{gr}_0, \mathcal{M}_0^i, \delta_0^i)$, $i = 0, 1, 2$ be three Lagrangian branes and $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$, $g: \mathcal{M}_0^1 \rightarrow \mathcal{M}_0^2$ two morphisms of local systems compatible with the brane structures. Let $\mathbb{L}_1^1$ be the transverse perturbation of $\mathbb{L}_0^1$ supported on L_1 and $\mathbb{L}_2^2$ the transverse perturbation of $\mathbb{L}_0^2$ supported on L_2 .

Lemma 6.15 together with the fact that the local systems $\mathcal{M}_1$ and $\mathcal{M}_2$ are both obtained from $\mathcal{M}_0$ by transporting along the cotangent fibres gives the following.

Corollary 6.16. *Let $\mathbb{L}_0^0$, $\mathbb{L}_1^1$, $\mathbb{L}_2^2$, f , and g be as above. Then*

$$g_{\mathbb{L}_1^1, \mathbb{L}_2^2} \circ f_{\mathbb{L}_0^0, \mathbb{L}_1^1} = (g \cdot f)_{\mathbb{L}_0^0, \mathbb{L}_2^2}.$$

6.3. Homotopy equivalences. Keep the notation $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ from before. In this subsection we prove the following proposition and then show in Lemma 6.19 that homotopy equivalences of local systems induce homotopy equivalences of Lagrangian branes.

Proposition 6.17. *$\mathcal{W}$ has enough homotopy equivalences (see Definition 5.2).*

Moreover, if $\mathbb{L}_0 = (L_0, \text{gr}_0, \mathcal{M}_0, \delta_0)$ is a Lagrangian brane in $(\mathcal{S}; \mathcal{C})$ and L_1 a Lagrangian in $\mathcal{S}$ isotopic to L_0 , then there exists a Lagrangian brane $\mathbb{L}_1$ supported on L_1 such that $\mathbb{L}_0$ and $\mathbb{L}_1$ are homotopy equivalent as objects of $\mathcal{W}$ (see Definition 5.1).

Let $\mathbb{L}_0 = (L_0, \text{gr}_0, \mathcal{M}_0, \delta_0)$ be a Lagrangian brane in $(\mathcal{S}; \mathcal{C})$ and $\mathbb{L}_1$ be a transverse perturbation of $\mathbb{L}_0$ (see §6.2.1). The morphism of local systems $\text{id}: \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is clearly compatible with the brane structure hence we get a morphism $\text{id}_{\mathbb{L}_0, \mathbb{L}_1} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$ by (6.12). Recall that we can view the reverse of the isotopy from Definition 6.9 as defining $\mathbb{L}_0$ as a transverse perturbation of $\mathbb{L}_1$, hence we get $\text{id}_{\mathbb{L}_1, \mathbb{L}_0} \in \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0)$.

Lemma 6.18. *In the notation above $\text{id}_{\mathbb{L}_0, \mathbb{L}_1}$ and $\text{id}_{\mathbb{L}_1, \mathbb{L}_0}$ are homotopy inverse morphisms in $\mathcal{W}$ (see Definition 5.1).*

Proof. Denote $e_{10} = \text{id}_{\mathbb{L}_1, \mathbb{L}_0}$ and $e_{01} = \text{id}_{\mathbb{L}_0, \mathbb{L}_1}$. By Lemma 6.13 e_{10} and e_{01} are closed morphisms of degree 0. It is enough to prove that for any transverse set $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_1\}$ in $\mathcal{W}$ the map $e_{10} \circ (e_{01} \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_0)}$ and the map $(- \circ e_{10}) \circ e_{01}$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_1)}$ since we can swap $\mathbb{L}_0$ and $\mathbb{L}_1$. We prove that $e_{10} \circ (e_{01} \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_0)}$, the other homotopy is similar.

Fix the transverse set $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_1\}$. Applying the discussion before Corollary 6.16 to L_1 as a transverse perturbation of L_0 and to L_0 as a transverse perturbation of L_1 , we can choose a Lagrangian L_2 arbitrarily close to L_0 which is a transverse perturbation of both L_0 and L_1 such that Corollary 6.16 applies to both triples $(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2)$ and $(\mathbb{L}_1, \mathbb{L}_0, \mathbb{L}_2)$, where $\mathbb{L}_2$ is the transverse perturbation of $\mathbb{L}_0$ (equivalently, of $\mathbb{L}_1$) supported on L_2 . Taking L_2 close enough to L_0 , by Lemma 6.15 and Corollary 6.16 we can moreover assume that $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2\}$ is a transverse set and

$$\begin{aligned} e_{02} \circ - &= \text{id}_*: \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_2) \text{ is an isomorphism;} \\ e_{12} \circ e_{01} &= e_{02} \text{ and } e_{02} \circ e_{10} = e_{12} \end{aligned}$$

where $e_{12} = \text{id}_{\mathbb{L}_1, \mathbb{L}_2}$, and $e_{02} = \text{id}_{\mathbb{L}_0, \mathbb{L}_2}$ are defined by (6.12). The A_∞ -relations imply that

- (1) $e_{02} \circ - = (e_{12} \circ e_{01}) \circ -$ is homotopic to $e_{12} \circ (e_{01} \circ -)$ as maps $\mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_2)$;
- (2) $e_{12} \circ - = (e_{02} \circ e_{10}) \circ -$ is homotopic to $e_{02} \circ (e_{10} \circ -)$ as maps $\mathcal{W}(\mathbb{T}, \mathbb{L}_1) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_2)$.

Combining the two homotopies above we obtain that $e_{02} \circ -$ is homotopic to $e_{02} \circ (e_{10} \circ (e_{01} \circ -))$ as maps $\mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_2)$. As $e_{02} \circ -: \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_2)$ is an isomorphism, we conclude that $e_{10} \circ (e_{01} \circ -): \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}_0)$ is homotopic to id as desired. $\square$

We are now ready to prove Proposition 6.17.

Proof of Proposition 6.17. We first check that the Fukaya category $\mathcal{W}$ has enough homotopy equivalences, see Definition 5.2. Let $\mathbb{L}_0 = (L_0 = (I, \gamma_0, P_0), \text{gr}_0, \mathcal{M}_0, \delta_0)$ be a Lagrangian brane and $\{\{\mathbb{L}_j^k\}_{k=0}^{l_j}\}_{j=1}^m$ be a set of transverse sets of Lagrangian branes. For each $j = 1, \dots, m$ let $\mathbb{L}_j$ be a composite Lagrangian brane of $\mathbb{L}_j^0, \dots, \mathbb{L}_j^{l_j}$ supported on the Lagrangian L_j .

By Lemma 6.18 it suffices to find a function $F \in C^\infty(I, \mathbb{R})$ such that $L(F, \text{id})$ is transverse to L_0 and each L_j . This condition defines an open dense subset in a neighbourhood of 0 in $C^\infty(I, \mathbb{R})$ hence the desired F exists.

Now let $\mathbb{L}_0 = (L_0 = (I, \gamma_0, P_0), \text{gr}_0, \mathcal{M}_0, \delta_0)$ be a Lagrangian brane and $L_1 = (I, \gamma_1, P_1)$ be a Lagrangian isotopic to L_0 . By Lemma 3.16 we can find a sequence of Lagrangians $L_0 = L'_0, L'_1, \dots, L'_n = L_1$ such that each L'_{i+1} is a transverse perturbation of L'_i up to a reparametrisation of the domain. It remains to prove that reparametrisation of the domain preserves homotopy equivalences.

Namely, let $\mathbb{L}_i = (L_i = (I_i, \gamma_i, P_i), \text{gr}_i, \mathcal{M}_i, \delta_i)$, $i = 0, 1$ be two transverse Lagrangian branes, $q \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$ and $p \in \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0)$ homotopy inverse morphisms, and $\varphi \in \text{Diff}(I_1)$ an orientation-preserving diffeomorphism. Then we get a Lagrangian pre-brane $\mathcal{L}'_1 = (I_1, \gamma_1 \circ \varphi, P_1 \circ \varphi, \varphi^* \text{gr}_1, \varphi^* \mathcal{M}_1)$ and the obstruction algebra does not change, $\mathcal{A}_{\mathcal{L}'_1} = \mathcal{A}_{\mathcal{L}_1}$, so we have a Lagrangian brane $\mathbb{L}'_1 = (\mathcal{L}'_1, \delta_1)$. For any Lagrangian brane $\mathbb{T}$ the set $\{\mathbb{T}, \mathbb{L}_1\}$ is transverse if and only if $\{\mathbb{T}, \mathbb{L}'_1\}$ is, and in this case $Z(\mathbb{T}, \mathbb{L}_1) = Z(\mathbb{T}, \mathbb{L}'_1)$ with the same modules $\mathcal{W}(z)$, so $\mathcal{W}(\mathbb{T}, \mathbb{L}_1) = \mathcal{W}(\mathbb{T}, \mathbb{L}'_1)$, and similarly $\mathcal{W}(\mathbb{L}_1, \mathbb{T}) = \mathcal{W}(\mathbb{L}'_1, \mathbb{T})$. Since reparametrising the domains of composite Lagrangian branes by φ does not change their obstruction algebras, these identifications commute with all μ . Therefore the morphisms $q' \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$ and $p' \in \mathcal{W}(\mathbb{L}'_1, \mathbb{L}_0)$ corresponding to q and p are homotopy inverse. $\square$

We prove that homotopy equivalences of local systems induce homotopy equivalences of Lagrangian branes.

Lemma 6.19. *Let $\mathbb{L}_0^i = (L_0, \text{gr}_0, \mathcal{M}_0^i, \delta_0^i)$ be two Lagrangian branes and $f: \mathcal{M}_0^0 \rightarrow \mathcal{M}_0^1$, $g: \mathcal{M}_0^1 \rightarrow \mathcal{M}_0^0$ closed degree zero morphisms of local systems compatible with brane structures. Suppose that there exist morphisms of local systems compatible with brane structures $H_i: \mathcal{M}_0^i \rightarrow \mathcal{M}_0^i$ such that $f \cdot g - \text{id} = dH_1$ and $g \cdot f - \text{id} = dH_0$. Then $\mathbb{L}_0^0$ and $\mathbb{L}_0^1$ are homotopy equivalent in $\mathcal{W}$.*

Proof. We have already proved Proposition 6.17, so $\mathcal{W}$ has enough homotopy equivalences and we can use the machinery of §5.1. By Corollary 6.16 there exist transverse perturbations $\mathbb{L}_1^1$ of $\mathbb{L}_0^1$, $\mathbb{L}_2^0$ of $\mathbb{L}_0^0$, and $\mathbb{L}_3^1$ of $\mathbb{L}_1^1$ such that $\{\mathbb{L}_0^0, \mathbb{L}_1^1, \mathbb{L}_2^0, \mathbb{L}_3^1\}$ is a transverse set and

$$g_{\mathbb{L}_1^1, \mathbb{L}_2^0} \circ f_{\mathbb{L}_0^0, \mathbb{L}_1^1} = (g \cdot f)_{\mathbb{L}_0^0, \mathbb{L}_2^0} \text{ and } f_{\mathbb{L}_2^0, \mathbb{L}_3^1} \circ g_{\mathbb{L}_1^1, \mathbb{L}_2^0} = (f \cdot g)_{\mathbb{L}_1^1, \mathbb{L}_3^1}.$$

By assumption and Lemma 6.13

$$(g \cdot f)_{\mathbb{L}_0^0, \mathbb{L}_2^0} - \text{id}_{\mathbb{L}_0^0, \mathbb{L}_2^0} \text{ and } (f \cdot g)_{\mathbb{L}_1^1, \mathbb{L}_3^1} - \text{id}_{\mathbb{L}_1^1, \mathbb{L}_3^1} \text{ are exact.}$$

By Lemma 5.5 composition with $(g \cdot f)_{\mathbb{L}_0^0, \mathbb{L}_2^0}$ on either side is homotopic to composition with $\text{id}_{\mathbb{L}_0^0, \mathbb{L}_2^0}$, and similarly for $(f \cdot g)_{\mathbb{L}_1^1, \mathbb{L}_3^1}$ and $\text{id}_{\mathbb{L}_1^1, \mathbb{L}_3^1}$. Hence by Lemmas 5.6 and 6.18

$$(g \cdot f)_{\mathbb{L}_0^0, \mathbb{L}_2^0} \text{ and } (f \cdot g)_{\mathbb{L}_1^1, \mathbb{L}_3^1} \text{ are homotopy equivalences.}$$

Finally, $f_{\mathbb{L}_0^0, \mathbb{L}_1^1}$ is a homotopy equivalence by Lemma 5.3, $\mathbb{L}_0^1$ is homotopy equivalent to $\mathbb{L}_1^1$ by Lemma 6.18, hence $\mathbb{L}_0^0$ and $\mathbb{L}_0^1$ are homotopy equivalent. $\square$

6.4. Strict units. Keep the notation $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ from before. Recall from Introduction that a collection of Lagrangian branes is called tame if the underlying Lagrangians form a transverse no more than countable set. In this subsection we prove the following.

Proposition 6.20. *There exists a saturated collection of systems of strict units $\mathcal{SU}$ (see Definition 5.13) on $\mathcal{W}$ such that for any tame collection $\mathbf{L}^0$ of Lagrangian branes there exists a system of strict units $\mathbf{L}^\bullet \in \mathcal{SU}$ on $\mathbf{L}^0$.*

Propositions 5.14 and 6.20 then give a strictly unital A_∞ -category $\mathcal{W}_{\mathcal{SU}} = \mathcal{W}_{\text{su}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$, well-defined up to homotopy equivalence.

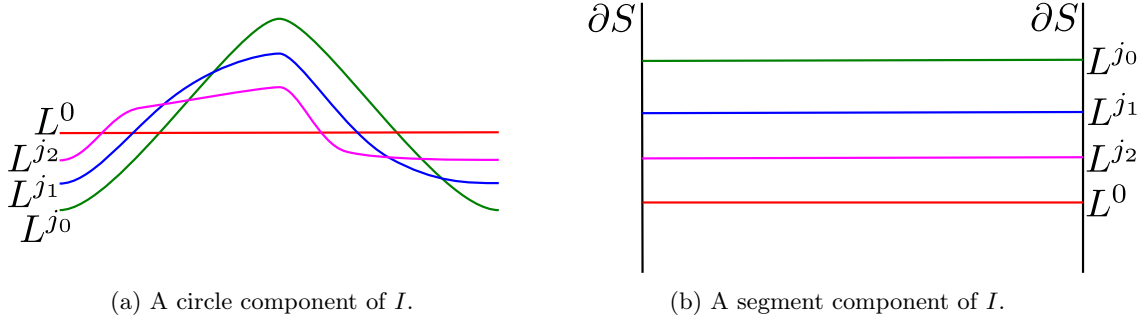
In §6.4.1 we describe how to get systems of strict units in $\mathcal{W}$. In §6.4.2 we describe $\mathcal{SU}$ and prove Proposition 6.20.

6.4.1. Systems of strict units. Let $L^0 = (I, \gamma^0, P^0)$ be a Lagrangian. Then there exists a transverse set $\{L^j\}_{j=0}^\infty$ such that

- (1) if $j_0 \neq j_1$, then L^{j_1} is a transverse perturbation of L^{j_0} ;
- (2) L^j tends to L^0 in $\text{Lag}_I(\mathcal{S})$ as $j \rightarrow \infty$;
- (3) for pairwise distinct $j_0, j_1, j_2 \geq 0$ there exists a composite Lagrangian $L = L_{h_0, h_1, h_2}(L^{j_0}, L^{j_1}, L^{j_2})$ that contains the transverse double $L_{h_0, h_1}(L^{j_0}, L^{j_1})$ of L^{j_0} nicely;
- (4) for any $j_0 < \dots < j_{l-2} < \min\{j_{l-1}, j_l\} < \max\{j_{l-1}, j_l\} < j_{l+1} < \dots < j_n$ there exists a composite Lagrangian $L = L_{h_0, \dots, h_n}(L^{j_0}, \dots, L^{j_n})$ that contains the transverse double $L_{h_{l-1}, h_l}(L^{j_{l-1}}, L^{j_l})$ nicely.

Such a transverse set can be obtained, for example, by perturbing each connected component C of I that is a circle as in Fig. 28(a) (the parts of L^j that are not depicted run parallel to L^0) in a small enough neighbourhood of a point $\gamma(t)$ for $t \in C$ and by perturbing each connected component C of I that is a segment as in Fig. 28(b) (the picture is of the pullback to a Weinstein neighbourhood of $\gamma|_C$).

Now let $\{L_k^0\}_{k \in K}$ be a transverse set of Lagrangians in $\mathcal{S}$ and $\{\mathbb{L}_i^0\}_{i \in I}$ a collection of Lagrangian branes in $(\mathcal{S}; \mathcal{C})$ such that each $\mathbb{L}_i^0$ is supported on some $L_{\kappa(i)}^0$. For each $k \in K$ pick a transverse set $\{L_k^j\}_{j=0}^\infty$ as above and for each $i \in I$ and $j \geq 0$ let $\mathbb{L}_i^j$ be the transverse perturbation of $\mathbb{L}_i^0$ supported on $L_{\kappa(i)}^j$. Note that $\mathbb{L}_i^{j_1}$ is a transverse perturbation of $\mathbb{L}_i^{j_0}$ for any $i \in I$ and $j_0 \neq j_1$ hence we have a homotopy equivalence $e_i^{j_0, j_1} = \text{id}_{\mathbb{L}_i^{j_0}, \mathbb{L}_i^{j_1}} \in \mathcal{W}(\mathbb{L}_i^{j_0}, \mathbb{L}_i^{j_1})$.

FIGURE 28. $L^0, L^{j_0}, L^{j_1}, L^{j_2}$ for $0 < j_0 < j_1 < j_2$

Lemma 6.21. $\mathbf{L}^\bullet = \left\{ \left(\left\{ \mathbb{L}_i^j \right\}_{j=0}^\infty, \left\{ e_i^{j_0, j_1} \right\}^{j_0 \neq j_1} \right) \right\}_{i \in I}$ is a system of strict units (see §5.3.1) on the collection $\mathbf{L}^0 = \{\mathbb{L}_i^0\}_{i \in I}$ of objects in $\mathcal{W}$.

Proof. We check properties (1)–(4) from §5.3.1. Property (1) follows from (3) above and Lemma 6.15.

To prove Property (2) fix $i_0, i_1 \in I$ and let $k_0 = \kappa(i_0)$ and $k_1 = \kappa(i_1)$. By Lemma 6.15 it suffices to show that there exists $N > 0$ such that for any $j_0, j_1, j'_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1 < j'_1$ there exists a composite Lagrangian $L = L_{h_0, h_1, h'_1}(L_{k_0}^{j_0}, L_{k_1}^{j_1}, L_{k_1}^{j'_1})$ that contains the transverse double $L_{h_1, h'_1}(L_{k_1}^{j_1}, L_{k_1}^{j'_1})$ nicely. If $k_0 = k_1$, then this follows from (3) above. If $k_0 \neq k_1$, then this is true since $L_{k_0}^0$ and $L_{k_1}^0$ are transverse, $L_{k_0}^j \xrightarrow{j \rightarrow \infty} L_{k_0}^0$, and $L_{k_1}^j \xrightarrow{j \rightarrow \infty} L_{k_1}^0$. Property (3) is proved analogously.

Property (4) follows from Lemma 6.14 and conditions (2) and (4) above. $\square$

6.4.2. Saturated collection of systems of strict units. Proposition 3.17 and the fact that there are countably many classes of immersed curves up to a reparametrisation of the domain and the action of $\text{Diff}(\mathcal{S})$ imply that the number of connected components of the space $\text{Lag}_I(\mathcal{S})$ of all Lagrangians with given domain I is countable. The number of different domains is also countable. Therefore, there exists a transverse set of Lagrangians $\{L_k^0\}_{k=0}^\infty$ in $\mathcal{S}$ such that for any Lagrangian L in $\mathcal{S}$ there exists a Lagrangian L_k^0 isotopic to L .

Let $\mathcal{SU}$ be the collection of all systems of strict units obtained as in §6.4.1 from a no more than countable transverse set of Lagrangians $\{L_k^0\}_{k=0}^\infty$ and some set of Lagrangian branes each of which is supported on one of L_k^0 .

Proof of Proposition 6.20. The extra condition on $\mathcal{SU}$ in Proposition 6.20 holds by construction, it remains to prove $\mathcal{SU}$ is saturated.

We prove $\mathcal{SU}$ satisfies (1) from Definition 5.13. Let $\mathbf{L}^\bullet \in \mathcal{SU}$ be a system of strict units on a collection $\mathbb{L}^0 = \{\mathbb{L}_i^0\}_{i \in I}$ of all Lagrangian brane structures on a transverse set of Lagrangians $\{L_k^0\}_{k=0}^\infty$ such that any Lagrangian in $\mathcal{S}$ is isotopic to one of L_k^0 . Then by construction and Proposition 6.17 for any Lagrangian brane $\mathbb{L}$ there exists a Lagrangian brane $\mathbb{L}_i^0, i \in I$ homotopy equivalent to $\mathbb{L}$.

We prove $\mathcal{SU}$ satisfies (2) from Definition 5.13. Recall the space $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ of Hamiltonian functions from §3.1.3 and note that the standard Frechét structure makes it into a Baire space. For $H \in C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ and a Lagrangian $L = (I, \gamma, P)$ define

$$H_*L = (I, \Phi_1^H \circ \gamma, P - g_H \circ \gamma) \text{ where } g_H = \int_0^1 (H_s + \lambda(X_s^H)) \circ \Phi_s^H ds$$

and Φ_1^H is the time-1 map of the Hamiltonian flow of H . Note that H_*L is a Lagrangian since $(\Phi_1^H)^*\lambda - \lambda = dg_H$.

Given any two transverse sets $\{L_k^i\}_{k=0}^\infty, i = 0, 1$ there exists a Hamiltonian $H \in C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ such that $\{H_*L_k^0\}_{k=0}^\infty \sqcup \{L_k^1\}_{k=0}^\infty$ is transverse. Indeed, for each $0 \leq k_0^0 < \dots < k_a^0$ and $0 \leq k_0^1 < \dots < k_b^1$

the condition that $\{H_*L_{k_0}^0, \dots, H_*L_{k_a}^0, L_{k_0}^1, \dots, L_{k_b}^1\}$ is transverse defines an open dense subset of $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$.

If now $\mathbf{L}_i^\bullet \in \mathcal{SU}$, $i = 0, 1$ are two systems of strict units supported on $\{L_k^i\}_{k=0}^\infty$, $i = 0, 1$, then by Lemmas 6.6 and 6.7 we have a strict autoequivalence $\mathcal{I}_H$ of $\mathcal{W}$ such that $\mathcal{I}_{H*}\mathbf{L}_0^\bullet \sqcup \mathbf{L}_1^\bullet \in \mathcal{SU}$. $\square$

6.5. Triangulated structure. Keep the notation $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{W} = \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ from before. In this subsection we prove the following statement.

Proposition 6.22. *The Fukaya category $\mathcal{W}$ is strongly triangulated (see §5.2.3).*

This is the last ingredient we need to prove Theorems 3 and 4 in §6.7.

We prove Proposition 6.22 by definition, i.e. we construct a strict totalisation functor $\mathcal{T}: \text{Tw } \mathcal{W} \rightarrow \mathcal{W}$ from the triangulated envelope defined in §5.2.1. In fact, we construct two of them, a geometric and an algebraic totalisation functors $\mathcal{T}_g$ and $\mathcal{T}_a$. $\mathcal{T}_g$ uses rotation of gr to define the shift and is defined for any $\mathcal{C}$, we construct it in §§6.5.1 and 6.5.2. To construct $\mathcal{T}_a$ we need $\mathcal{C}$ to have a shift. Then we can use the shift in $\mathcal{C}$ to define the shift in $\text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ and obtain the strict totalisation functor $\mathcal{T}_a: \text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C}) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$, we do this in §6.5.3. In §6.5.4 we prove that cones of degree zero closed morphisms in $\mathcal{C}$ give rise to cones over $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ in two equivalent ways, algebraic and geometric.

6.5.1. Shifts of Lagrangian pre-branes. Let $\mathcal{L} = (I, \gamma, P, \text{gr}, \mathcal{M})$ be a Lagrangian pre-brane in $(\mathcal{S}; \mathcal{C})$ and $\mathbf{m}: I \rightarrow \mathbb{Z}$ a locally constant function. We define the *geometric shift of $\mathcal{L}$ by $\mathbf{m}$* to be

$$\mathcal{L}[\mathbf{m}] = (I, \gamma, P, \text{gr}[\mathbf{m}], \mathcal{M})$$

where $\text{gr}[\mathbf{m}](t)$ is $\text{gr}(t)$ followed by a clockwise rotation of $\dot{\gamma}(t)$ by $\mathbf{m}(t)\pi$ for $t \in I$. We get the corresponding obstruction algebra $\mathcal{A}_{\mathcal{L}[\mathbf{m}]}$. Note that $|z|_{\mathcal{L}[\mathbf{m}]} = |z|_{\mathcal{L}} + \mathbf{m}(z_+) - \mathbf{m}(z_-)$ for any $z \in Z = Z_{\mathcal{L}}$. Unfortunately, the signs in $\mathcal{A}_{\mathcal{L}[\mathbf{m}]}$ are different from the signs appearing for twisted complexes in §5.2.1. We therefore define another algebra, the *Koszul shift $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ of $\mathcal{A}_{\mathcal{L}}$ by $\mathbf{m}$* . Namely,

$$\mathcal{A}_{\mathcal{L}}[\mathbf{m}] = \bigoplus_{z \in Z} \mathcal{A}_{\mathcal{L}}(z)[-|z|_{\mathcal{L}} + \mathbf{m}(z_-) - \mathbf{m}(z_+)]$$

as a graded module and $\mu_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]}$ is given by the Koszul sign rule as in (5.1), its components

$$\mu_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]}^{z^0; z^n, \dots, z^1}: \mathcal{A}_{\mathcal{L}}(z^n) \otimes \dots \otimes \mathcal{A}_{\mathcal{L}}(z^1) \rightarrow \mathcal{A}_{\mathcal{L}}(z^0)$$

are

$$\mu_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]}^{z^0; z^n, \dots, z^1}(\varphi^n, \dots, \varphi^1) = (-1)^{\sum_{1 \leq \alpha < \beta \leq n} (\mathbf{m}(z_-^\alpha) - \mathbf{m}(z_+^\alpha))(|\varphi^\beta| + |z^\beta|_{\mathcal{L}} - 1)} \mu_{\mathcal{A}_{\mathcal{L}}}^{z^0; z^n, \dots, z^1}(\varphi^n, \dots, \varphi^1).$$

Note that $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ is a structured module and the degree of $\mu_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]}^n$ is $2 - n$ since

$$\sum_{\alpha=1}^n (\mathbf{m}(z_-^\alpha) - \mathbf{m}(z_+^\alpha)) = \mathbf{m}(z_-^0) - \mathbf{m}(z_+^0)$$

whenever $\mu_{\mathcal{A}_{\mathcal{L}}}^{z^0; z^n, \dots, z^1} \neq 0$. Observe that the summands of $\mathcal{A}_{\mathcal{L}[\mathbf{m}]}$ and $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ corresponding to $z \in Z$ are the same graded module.

We now construct a naïve isomorphism $(\text{id}_Z, \mathcal{F}_{\mathcal{L}}): \mathcal{A}_{\mathcal{L}}[\mathbf{m}] \rightarrow \mathcal{A}_{\mathcal{L}[\mathbf{m}]}$, see §4.2.1 for the definition. For $z \in Z$ we set

$$(6.16) \quad \mathcal{F}_{\mathcal{L}}^z: \mathcal{A}_{\mathcal{L}}(z) \rightarrow \mathcal{A}_{\mathcal{L}[\mathbf{m}]}(z), \quad \varphi \mapsto (-1)^{(\mathbf{m}(z_-) - \mathbf{m}(z_+))|\varphi|} \nu(z)^{\mathbf{m}(z_+)} \varphi$$

where

$$\nu(z) = \begin{cases} (-1)^{|z|_{\mathcal{L}} - 1} & \text{if } \left(\frac{d\gamma}{dt} \Big|_{t=z_+}, \frac{d\gamma}{dt} \Big|_{t=z_-} \right) \text{ is a positive basis of } T_z S; \\ (-1)^{|z|_{\mathcal{L}}} & \text{otherwise} \end{cases}$$

if z is a double point and

$$\nu(z) = \begin{cases} (-1)^{|z|_{\mathcal{L}}} & \text{if } \frac{d\gamma}{dt} \Big|_{t=z_+} \text{ and } \frac{d\gamma}{dt} \Big|_{t=z_-} \text{ both point inside or outside } S; \\ (-1)^{|z|_{\mathcal{L}} - 1} & \text{otherwise} \end{cases}$$

if z is a Reeb chord.

Lemma 6.23. $(\text{id}_Z, \mathcal{F}_{\mathcal{L}}): \mathcal{A}_{\mathcal{L}}[\mathbf{m}] \rightarrow \mathcal{A}_{\mathcal{L}[\mathbf{m}]}$ defined above is a naïve isomorphism. In particular, $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ is a structured A_{∞} -algebra.

Proof. Since $|z|_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]} = |z|_{\mathcal{A}_{\mathcal{L}[\mathbf{m}]}} = |z|_{\mathcal{L}} + \mathbf{m}(z_+) - \mathbf{m}(z_-)$, and all $\mathcal{F}_{\mathcal{L}}^z$ are isomorphisms of graded modules, by Lemma 4.7 it remains to check that higher products are preserved.

Fix $z^0, \dots, z^n \in Z$ and $\varphi^i \in \mathcal{A}_{\mathcal{L}}(z^i)$, $i = 1, \dots, n$. We need to check

$$\mathcal{F}_{\mathcal{L}}^{z^0} \circ \mu_{\mathcal{A}_{\mathcal{L}}[\mathbf{m}]}^{z^0; z^n, \dots, z^1}(\varphi^n, \dots, \varphi^1) = \mu_{\mathcal{A}_{\mathcal{L}[\mathbf{m}]}}^{z^0; z^n, \dots, z^1}(\mathcal{F}_{\mathcal{L}}^{z^n}(\varphi^n), \dots, \mathcal{F}_{\mathcal{L}}^{z^1}(\varphi^1)).$$

We compute the sign by which both sides differ from $\mu_{\mathcal{A}_{\mathcal{L}}}^{z^0; z^n, \dots, z^1}(\varphi^n, \dots, \varphi^1)$. If $n = 1$ and $z^0 = z^1$, then from the definition in (4.5) we obtain that the sign for both sides is

$$(-1)^{(\mathbf{m}(z_-^0) - \mathbf{m}(z_+^0))(|\varphi^1| + 1)} \nu(z^0) \mathbf{m}(z_+^0).$$

To compute the sign in all other cases, since ν does not change when we replace a Reeb chord by a double point, by Lemma 3.18 and Proposition 4.8 we can assume that $\mathcal{L}$ is sufficiently self-wrapped. Then $z^0, \dots, z^n$ are all double points. Fix a polygon $\mathbf{u}$ contributing to $\mu_{\mathcal{A}_{\mathcal{L}}}^{z^0; z^n, \dots, z^1}$ by (4.6).

We can assume that all relevant components of I are oriented as sides of $\mathbf{u}$ since changing the orientation of one of the sides only changes the contribution of ν and of ε from Definition 4.4 to the signs, and they change by the same factor on both sides. A straightforward computation then shows that in this case the sign for both sides is

$$(-1)^{\mathbf{m}(z_-^0)|z^0|_{\mathcal{L}} + \sum_{0 \leq \alpha < \beta \leq n} (\mathbf{m}(z_-^\alpha) - \mathbf{m}(z_+^\alpha))(|\varphi^\beta| + |z^\beta|_{\mathcal{L}} - 1)}.$$

□

6.5.2. Geometric totalisation functor. We construct the strict *geometric totalisation functor* $\mathcal{T}_g: \text{Tw } \mathcal{W} \rightarrow \mathcal{W}$. Let $\mathbf{X} = (\bigoplus_{\alpha \in A} \mathbb{L}_{\alpha}[m_{\alpha}], \delta_{\mathbf{X}})$ be a twisted complex over $\mathcal{W}$ with $\mathbb{L}_{\alpha} = (\mathcal{L}_{\alpha} = (I_{\alpha}, \gamma_{\alpha}, P_{\alpha}, \text{gr}_{\alpha}, \mathcal{M}_{\alpha}), \delta_{\alpha})$. We define the Lagrangian brane $\mathcal{T}_g^0(\mathbf{X})$ as follows. First, identify A with $\{0, \dots, n\}$ in such a way that $\delta_{\mathbf{X}_{\alpha_0 \alpha_1}} \neq 0$ implies $\alpha_0 < \alpha_1$. Secondly, by Lemma 6.1 there exists a composite Lagrangian brane $(\mathcal{L}, \delta) = \mathbb{L}_{(0, h_1, \dots, h_n)}(\mathbb{L}_0, \dots, \mathbb{L}_n)$ that is adapted to $\delta_{\mathbf{X}}$. Then, by definition of a twisted complex, $\delta_{\mathbf{X}} + \delta$ is a bounding cochain in the Koszul shift $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ where

$$(6.17) \quad \mathbf{m}: \bigsqcup_{\alpha=0}^n I_{\alpha} \rightarrow \mathbb{Z}, \quad \mathbf{m}|_{I_{\alpha}} = m_{\alpha}.$$

We set $\mathcal{T}_g^0(\mathbf{X}) = (\mathcal{L}[\mathbf{m}], \mathcal{F}_{\mathcal{L}}(\delta_{\mathbf{X}} + \delta))$ where $\mathcal{F}_{\mathcal{L}}: \mathcal{A}_{\mathcal{L}}[\mathbf{m}] \rightarrow \mathcal{A}_{\mathcal{L}[\mathbf{m}]}$ is the naïve isomorphism from Lemma 6.23.

Now let $\mathbf{X}^i = (\bigoplus_{\alpha \in A^i} \mathbb{L}_{\alpha}^i[m_{\alpha}^i], \delta_{\mathbf{X}^i})$, $i = 0, 1$ be two twisted complexes over $\mathcal{W}$ with $\mathbb{L}_{\alpha}^i = (\mathcal{L}_{\alpha}^i = (I_{\alpha}^i, \gamma_{\alpha}^i, P_{\alpha}^i, \text{gr}_{\alpha}^i, \mathcal{M}_{\alpha}^i), \delta_{\alpha}^i)$. We define $\mathcal{T}_g^1: \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)$ where $\mathbb{L}^i = (\mathcal{L}^i = (I^i, \gamma^i, P^i, \text{gr}^i, \mathcal{M}^i), \delta^i) = \mathcal{T}_g^0(\mathbf{X}^i)$, $i = 0, 1$.

Unwrapping the definitions we see that

$$(6.18) \quad \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) = \bigoplus_{\substack{\alpha^0 \in A^0 \\ \alpha^1 \in A^1}} \bigoplus_{z \in Z(\mathbb{L}_{\alpha^0}^0, \mathbb{L}_{\alpha^1}^1)} \mathcal{W}(z)[-|z| + m_{\alpha^1}^1 - m_{\alpha^0}^0] = \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)$$

where

$$\mathcal{W}(z) = \mathcal{C}_{\gamma_{\alpha^1}^1(z_-)}(\mathcal{C}(z) \mathcal{M}_{\alpha^0}^0(z_+), \mathcal{M}_{\alpha^1}^1(z_-)).$$

Fix $z \in Z(\mathbb{L}_{\alpha^0}^0, \mathbb{L}_{\alpha^1}^1)$. By Lemma 6.1 we can find a composite Lagrangian brane $(\mathcal{L}, \delta) = \mathbb{L}_h(\mathbb{L}_{\alpha^0}^0, \mathbb{L}_{\alpha^1}^1)$ such that $A_h(z) > 0$. We set

$$\mathcal{T}_g^1|_{\mathcal{W}(z)} = \mathcal{F}_{\mathcal{L}}^z$$

where $\mathcal{F}_{\mathcal{L}}: \mathcal{A}_{\mathcal{L}}[\mathbf{m}] \rightarrow \mathcal{A}_{\mathcal{L}[\mathbf{m}]}$ is the naïve isomorphism from Lemma 6.23 for

$$\mathbf{m}: I_{\alpha^0}^0 \sqcup I_{\alpha^1}^1 \rightarrow \mathbb{Z}, \quad \mathbf{m}|_{I_{\alpha^i}^i} = m_{\alpha^i}^i.$$

The following lemma implies Proposition 6.22.

Lemma 6.24.

- (1) $\mathcal{T}_g^0(\mathbb{L}) = \mathbb{L}$ for any Lagrangian brane $\mathbb{L}$.
- (2) $\mathcal{T}_g^1$ is identity on $\mathrm{Tw} \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1) = \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)$ for each transverse pair $(\mathbb{L}^0, \mathbb{L}^1)$ in $\mathcal{W}$.
- (3) $\mathcal{T}_g^1: \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathcal{T}_g^0(\mathbf{X}^0), \mathcal{T}_g^0(\mathbf{X}^1))$ is an isomorphism of graded modules for each transverse pair $(\mathbf{X}^0, \mathbf{X}^1)$ in $\mathrm{Tw} \mathcal{W}$.
- (4) $\mathcal{T}_g: \mathrm{Tw} \mathcal{W} \rightarrow \mathcal{W}$ is a strict A_∞ -functor.

Proof. (1) follows from our choice of shift for $\mathbb{L}_0$ in the definition of $\mathcal{T}_g^0$. (2) holds since the sign in (6.16) is 1 when $\mathbf{m} = 0$. As $\mathcal{F}^z$ from Lemma 6.23 is an isomorphism, we obtain (3). It remains to prove (4).

Let $\{\mathbf{X}^i = (\bigoplus_{\alpha \in A^i} \mathbb{L}_\alpha^i[m_\alpha^i], \delta_{\mathbf{X}^i}) \mid i = 0, \dots, n\}$ be a transverse set of twisted complexes over $\mathcal{W}$ with $\mathbb{L}_\alpha^i = (\mathcal{L}_\alpha^i = (I_\alpha^i, \gamma_\alpha^i, P_\alpha^i, \mathrm{gr}_\alpha^i, \mathcal{M}_\alpha^i), \delta_\alpha^i)$. Then $\{\mathbb{L}^i = (\mathcal{L}^i, \delta^i) = \mathcal{T}_g^0 \mathbf{X}^i \mid i = 0, \dots, n\}$ is a transverse sequence since both transversality conditions are equivalent to

$$\{L_{\alpha^i}^i \mid i = 0, \dots, n; \alpha^i \in A^i\}$$

being a transverse set. Let $a^i \in \mathrm{Tw} \mathcal{W}(\mathbf{X}^{i-1}, \mathbf{X}^i)$, $i = 1, \dots, n$. We need to prove

$$(6.19) \quad \mathcal{T}_g^1 \mu_{\mathrm{Tw} \mathcal{W}}^n(a^n, \dots, a^1) = \mu_{\mathcal{W}}^n(\mathcal{T}_g^1 a^n, \dots, \mathcal{T}_g^1 a^1).$$

By Lemma 6.1 there exists a composite Lagrangian pre-brane $\mathcal{L} = \mathcal{L}_h(\mathcal{L}_{\alpha^i}^i)_{\alpha^i \in A^i}^{i=0, \dots, n}$ adapted to $a^n, \dots, a^1$ and all $\delta_{\mathbf{X}^i}$. As before,

$$\delta = \sum_{i=0}^n \left(\delta_{\mathbf{X}^i} + \sum_{\alpha \in A^i} \delta_\alpha^i \right)$$

is a bounding cochain in $\mathcal{A}_{\mathcal{L}}[\mathbf{m}]$ where

$$\mathbf{m}: \bigsqcup_{i=0}^n \bigsqcup_{\alpha^i \in A^i} I_{\alpha^i}^i \rightarrow \mathbb{Z}, \quad \mathbf{m}|_{I_{\alpha^i}^i} = m_{\alpha^i}^i.$$

By Lemma 6.5 the left-hand side of (6.19) is equal to $\mathcal{F}_{\mathcal{L}} \circ \mu_{\delta}^n(a^n, \dots, a^1)$, while the right-hand side is equal to $\mu_{\mathcal{F}_{\mathcal{L}}(\delta)}^n(\mathcal{F}_{\mathcal{L}}(a^n), \dots, \mathcal{F}_{\mathcal{L}}(a^1))$. Since $\mathcal{F}_{\mathcal{L}}$ is a naïve morphism, these are equal. $\square$

6.5.3. Algebraic totalisation functor. Tw defines a strict 2-functor $\mathrm{dgc}at_{\mathbf{k}} \rightarrow \mathrm{dgc}at_{\mathbf{k}}$. Let $\mathrm{Tw} \mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \mathrm{dgc}at_{\mathbf{k}}$ be the weighted system of dg-categories given by $\mathrm{Tw} \circ \mathcal{C}$. Tw has a subfunctor Σ , called *the shift enlargement*, consisting of twisted complexes of the form $(X[m], 0)$ for $m \in \mathbb{Z}$.

Let $\mathcal{L} = (I, \gamma, P, \mathrm{gr}, \mathcal{M})$ be a Lagrangian pre-brane in $(\mathcal{S}; \Sigma \mathcal{C})$ and $\mathbf{m}: I \rightarrow \mathbb{Z}$ a locally constant function. We define the *algebraic shift of $\mathcal{L}$ by $\mathbf{m}$* to be

$$\mathcal{L}\{\mathbf{m}\} = (I, \gamma, P, \mathrm{gr}, \mathcal{M}[\mathbf{m}])$$

where $\mathcal{M}[\mathbf{m}](t) = \mathcal{M}(t)[\mathbf{m}(t)]$ for $t \in I$ and $\mathcal{M}[\mathbf{m}](a) = \mathcal{M}(a)$ for any path $a: [0, 1] \rightarrow I$ (this makes sense since $\mathbf{m}(a(0)) = \mathbf{m}(a(1))$ as $\mathbf{m}$ is locally constant). A straightforward computation shows that the algebraic shift is the same as the Koszul shift on the obstruction algebra, $\mathcal{A}_{\mathcal{L}\{\mathbf{m}\}} = \mathcal{A}_{\mathcal{L}}[\mathbf{m}]$.

We construct the *algebraic strict totalisation functor* $\mathcal{T}_a: \mathrm{Tw} \mathcal{W} \rightarrow \mathcal{W}$ where $\mathcal{W} = \mathcal{W}_{\mathrm{pre}}^{\mathrm{geom}}(\mathcal{S}; \Sigma \mathcal{C})$. The construction is similar to the construction in §6.5.2 but simpler since the Koszul and algebraic shifts coincide. Let $\mathbf{X} = (\bigoplus_{\alpha \in A} \mathbb{L}_\alpha[m_\alpha], \delta_{\mathbf{X}})$ be a twisted complex over $\mathcal{W}$ with $\mathbb{L}_\alpha = (\mathcal{L}_\alpha = (I_\alpha, \gamma_\alpha, P_\alpha, \mathrm{gr}_\alpha, \mathcal{M}_\alpha), \delta_\alpha)$. We define the Lagrangian brane $\mathcal{T}_a^0(\mathbf{X})$ as follows. Identify A with $\{0, \dots, n\}$ in such a way that $\delta_{\mathbf{X}_{\alpha_0 \alpha_1}} \neq 0$ implies $\alpha_0 < \alpha_1$ and take a composite Lagrangian brane $(\mathcal{L}, \delta) = \mathbb{L}_{(0, h_1, \dots, h_n)}(\mathbb{L}_0, \dots, \mathbb{L}_n)$ that is adapted to $\delta_{\mathbf{X}}$. Then, by definition of a twisted complex, $\delta_{\mathbf{X}} + \delta$ is a bounding cochain in the Koszul shift $\mathcal{A}_{\mathcal{L}}[\mathbf{m}] = \mathcal{A}_{\mathcal{L}\{\mathbf{m}\}}$ where $\mathbf{m}$ is as in (6.17). We set $\mathcal{T}_a^0(\mathbf{X}) = (\mathcal{L}\{\mathbf{m}\}, \delta_{\mathbf{X}} + \delta)$.

Now let $\mathbf{X}^i = (\bigoplus_{\alpha \in A^i} \mathbb{L}_\alpha^i[m_\alpha^i], \delta_{\mathbf{X}^i})$, $i = 0, 1$ be two twisted complexes over $\mathcal{W}$ with $\mathbb{L}_\alpha^i = (\mathcal{L}_\alpha^i = (I_\alpha^i, \gamma_\alpha^i, P_\alpha^i, \mathrm{gr}_\alpha^i, \mathcal{M}_\alpha^i), \delta_\alpha^i)$. We define $\mathcal{T}_a^1: \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)$ where

$\mathbb{L}^i = (\mathcal{L}^i = (I^i, \gamma^i, P^i, \text{gr}^i, \mathcal{M}^i), \delta^i) = \mathcal{T}_a^0(\mathbf{X}^i)$, $i = 0, 1$. The formula (6.18) still holds and we set $\mathcal{T}_a^1|_{\mathcal{W}(z)} = \text{id}$. Arguing analogously to Lemma 6.24 we obtain the following.

Lemma 6.25.

- (1) $\mathcal{T}_a^0(\mathbb{L}) = \mathbb{L}$ for any Lagrangian brane $\mathbb{L}$.
- (2) $\mathcal{T}_a^1$ is identity on $\text{Tw } \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1) = \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)$ for each transverse pair $(\mathbb{L}^0, \mathbb{L}^1)$ in $\mathcal{W}$.
- (3) $\mathcal{T}_a^1: \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathcal{T}_a^0(\mathbf{X}^0), \mathcal{T}_a^0(\mathbf{X}^1))$ is an isomorphism of graded modules for each transverse pair $(\mathbf{X}^0, \mathbf{X}^1)$ in $\text{Tw } \mathcal{W}$.
- (4) $\mathcal{T}_a: \text{Tw } \mathcal{W} \rightarrow \mathcal{W}$ is a strict A_∞ -functor.

We therefore have two strict totalisation functors $\mathcal{T}_g, \mathcal{T}_a: \text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Sigma \mathcal{C}) \rightarrow \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \Sigma \mathcal{C})$ and Lemma 5.10 applies. In particular, if $\mathbb{L}$ is a Lagrangian brane and $m \in \mathbb{Z}$, then the geometric shift $\mathcal{T}_g^0(\mathbb{L}[m])$ is homotopy equivalent to the algebraic shift $\mathcal{T}_a^0(\mathbb{L}[m])$.

6.5.4. From algebraic to geometric cones. Let $\mathbb{L}_i = (L, \mathcal{M}_i, \delta_i)$, $i = 0, 1$ be two Lagrangian branes in $(\mathcal{S}; \mathcal{C})$ supported on the same graded Lagrangian $L = (I, \gamma, P, \text{gr})$ and $f: \mathcal{M}_0 \rightarrow \mathcal{M}_1$ a morphism of local systems compatible with brane structures, closed of degree zero.

To f we associate an *algebraic cone* $\mathbb{C}_a(f)$ in $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \text{Tw } \mathcal{C})$ and a *geometric cone* $\mathbf{C}_g(f)$ in $\text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ and prove that they are homotopy equivalent in $\text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \text{Tw } \mathcal{C})$.

The algebraic cone is the Lagrangian brane $\mathbb{C}_a(f) = (L, \text{Cone}(f), \delta_{\mathbb{C}_a(f)})$ in $(\mathcal{S}; \text{Tw } \mathcal{C})$ where

$$\text{Cone}(f)(t) = (\mathcal{M}_0(t)[1] \oplus \mathcal{M}_1(t), f(t)[-1]) \in \text{Tw } \mathcal{C}_{\gamma(t)} \text{ for any } t \in I$$

with the transition maps $\mathcal{M}_0(a) \oplus \mathcal{M}_1(a)$ for any path $a: [0, 1] \rightarrow I$, and $\delta_{\mathbb{C}_a(f)} = \delta_0 \oplus \delta_1$. The compatibility of f with the brane structure implies that $\delta_{\mathbb{C}_a(f)}$ is indeed a bounding cochain.

To construct the geometric cone we choose a transverse perturbation $\mathbb{L}_1^1$ of $\mathbb{L}_1$ and set $\mathbf{C}_g(f) = (\mathbb{L}_0[1] \oplus \mathbb{L}_1^1, f_{\mathbb{L}_0, \mathbb{L}_1^1}[-1])$. Lemma 6.13 implies that $f_{\mathbb{L}_0, \mathbb{L}_1^1} \in \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})(\mathbb{L}_0, \mathbb{L}_1^1)$ is a closed morphism of degree 0 and hence $\mathbf{C}_g(f)$ is indeed a twisted complex over $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$.

Lemma 6.26. *Let $\mathbb{L}_0$, $\mathbb{L}_1$, and f be as above. Then the algebraic cone $\mathbb{C}_a(f)$ is homotopy equivalent to the geometric cone $\mathbf{C}_g(f)$ in $\mathcal{W} = \text{Tw } \mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \text{Tw } \mathcal{C})$.*

Proof. Let $\left\{ \left(\left\{ \mathbb{L}_i^j \right\}_{j=0}^\infty, \left\{ e_i^{j_0, j_1} \right\}^{j_0 \neq j_1} \right) \right\}_{i=0,1,a}$ be a system of strict units as in Lemma 6.21 with $\mathbb{L}_0^0 = \mathbb{L}_0$, $\mathbb{L}_0^1 = \mathbb{L}_1$, $\mathbb{L}_a^0 = \mathbb{C}_a(f)$, and $\mathbb{L}_a^1$ as in the construction of $\mathbf{C}_g(f)$. Let $\iota^0: \mathcal{M}_0 \rightarrow \text{Cone}(f)$ and $\iota^1: \mathcal{M}_1 \rightarrow \text{Cone}(f)$ be the inclusions, and $p^0: \text{Cone}(f) \rightarrow \mathcal{M}_0$ and $p^1: \text{Cone}(f) \rightarrow \mathcal{M}_1$ the projections. These are morphisms of local systems compatible with bounding cochains. ι^1 and p^1 have degree 0, $|\iota^0| = -1$, and $|p^0| = 1$. ι^1 and p^0 are closed, $d\iota^0 = \iota^1 \cdot f$, and $dp^1 = f \cdot p^0$.

Consider the following chain of morphisms in $\mathcal{W}$

$$\left(\begin{array}{c} \mathbb{L}_0^0[1] \oplus \mathbb{L}_1^1 \\ f_{\mathbb{L}_0, \mathbb{L}_1^1}[-1] \end{array} \right) \xrightarrow{\left(\begin{array}{cc} \iota_{\mathbb{L}_0, \mathbb{L}_a^2}^0[-1] & \iota_{\mathbb{L}_1, \mathbb{L}_a^2}^1 \end{array} \right)} \mathbb{L}_a^2 \xrightarrow{\left(\begin{array}{c} p_{\mathbb{L}_a^2, \mathbb{L}_0^3}^0[1] \\ p_{\mathbb{L}_a^2, \mathbb{L}_1^4}^1 \end{array} \right)} \left(\begin{array}{c} \mathbb{L}_0^3[1] \oplus \mathbb{L}_1^4 \\ f_{\mathbb{L}_0^3, \mathbb{L}_1^4}[-1] \end{array} \right) \xrightarrow{\left(\begin{array}{cc} \iota_{\mathbb{L}_0^3, \mathbb{L}_a^5}^0[-1] & \iota_{\mathbb{L}_1^4, \mathbb{L}_a^5}^1 \end{array} \right)} \mathbb{L}_a^5.$$

By Lemmas 6.13 and 6.15 and Corollary 6.16 these three morphisms are closed of degree 0. By Lemmas 6.14 and 6.15 the composition of the first two morphisms is

$$\left(\begin{array}{cc} \text{id}_{\mathbb{L}_0^0, \mathbb{L}_0^3} & 0 \\ 0 & \text{id}_{\mathbb{L}_1^1, \mathbb{L}_1^4} \end{array} \right)$$

and of the second two morphisms is $\text{id}_{\mathbb{L}_a^2, \mathbb{L}_a^5}$. The former is a homotopy equivalence by Lemmas 5.11 and 6.18 applied to $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \text{Tw } \mathcal{C})$, which is strongly triangulated with enough homotopy equivalences by Propositions 6.17 and 6.22, and the latter is a homotopy equivalence by Lemma 6.18. As $\mathcal{W}$ has enough homotopy equivalences by Proposition 5.8, by Lemma 5.3 these three morphisms are homotopy equivalences. In particular, the first morphism is a homotopy equivalence from $\mathbf{C}_g(f)$ to a Lagrangian brane homotopy equivalent to $\mathbb{C}_a(f)$ hence $\mathbf{C}_g(f)$ and $\mathbb{C}_a(f)$ are homotopy equivalent. $\square$

6.6. Embedding of topological Fukaya category. Fix a weighted Liouville surface $(\mathcal{S} = (S, Q, \eta, \lambda); \mathcal{C}: \mathcal{P}_2^+(\mathcal{S}) \rightarrow \text{dgc} \text{cat}_k)$.

In §6.6.1 we prove that the A_∞ -category $\mathcal{V}_\Gamma$ of an arc brane system defined in §1.2 is strictly equivalent to the A_∞ -category $\mathcal{W}_{\Gamma^\bullet}$ of the corresponding tame collection of Lagrangian branes Γ^0 . In §6.6.2 we prove that disc sequences induce morphisms of twisted complexes in $\mathcal{V}_\Gamma$. In §6.6.3 we prove that the subcategories generated by full arc brane systems coincide, see Lemma 6.29.

6.6.1. Arc brane systems and tame collections of Lagrangian branes. Note that an arc brane (γ, gr, M) from §1.2.1 gives a Lagrangian brane $([0, 1], \gamma, P, \text{gr}, \mathcal{M}, 0)$ from Definition 4.11 via choosing arbitrary $P: [0, 1] \rightarrow \mathbb{R}$ satisfying $\gamma^* \lambda + dP = 0$ and turning M into a local system on $[0, 1]$ by transition functors of $\gamma^* \mathcal{C}$.

Let $\Gamma = \{\mathbb{L}_i\}_{i \in I}$ be an arc brane system in $(\mathcal{S}; \mathcal{C})$. Then the corresponding collection of Lagrangian branes $\Gamma^0 = \{\mathbb{L}_i^0\}_{i \in I}$ is tame and hence there exists a system of strict units

$$\mathbf{L}^\bullet = \left\{ \left(\left\{ \mathbb{L}_i^j \right\}_{j=0}^\infty, \left\{ e_i^{j_0, j_1} \right\}^{j_0 \neq j_1} \right) \right\}_{i \in I} \in \mathcal{SU}.$$

We construct a strict equivalence $\mathcal{F}$ from $\mathcal{V}_\Gamma$ (which we have not yet proved to satisfy A_∞ -relations) into the strictly unital A_∞ -category $\mathcal{W}_{\mathbf{L}^\bullet}$. On objects it is $\mathcal{F}^0(\mathbb{L}_i) = \mathbb{L}_i^0$. To define $\mathcal{F}$ on morphisms note that $\mathbb{L}_i^j$ is given by Fig. 28(b) and for $i_0, i_1 \in I$ and $j \gg 0$ the underlying curves of $\mathbb{L}_{i_0}^0$ and $\mathbb{L}_{i_1}^j$ do not intersect and a Reeb chord y from $\mathbb{L}_{i_0}^0$ to $\mathbb{L}_{i_1}^j$ gives a boundary path $\tilde{y}$ from $\mathbb{L}_{i_0}$ to $\mathbb{L}_{i_1}$ of the same degree, and the correspondence is a bijection after we have identified the two constant boundary paths. Moreover, we have an isomorphism of dg-modules

$$\mathcal{F}^y: \mathcal{V}_\Gamma(\tilde{y}) = \mathcal{C}_{\tilde{y}(1)}(\mathcal{C}(\tilde{y})(M_{i_0}^{y+}, M_{i_1}^{y-})) \xrightarrow{\mathcal{C}(s_y)} \mathcal{C}_{y(1)}(\mathcal{C}(y)(M_{i_0}^{y+}, \mathcal{C}(s_y)(M_{i_1}^{y-}))) = \mathcal{W}(y)$$

where s_y is the short segment of ∂S from $\tilde{y}(1)$ to $y(1)$, so that $\mathcal{C}(y) = \mathcal{C}(s_y) \circ \mathcal{C}(\tilde{y})$. The maps above assemble into an isomorphism of graded modules

$$\mathcal{F}^1: \mathcal{V}_\Gamma(\mathbb{L}_{i_0}, \mathbb{L}_{i_1}) \rightarrow \mathcal{W}(\mathbb{L}_{i_0}^0, \mathbb{L}_{i_1}^j) = \mathcal{W}_{\mathbf{L}^\bullet}(\mathbb{L}_{i_0}^0, \mathbb{L}_{i_1}^0)$$

Lemma 6.27. $\mathcal{F}$ defined above is a strict full equivalence $\mathcal{V}_\Gamma \rightarrow \mathcal{W}_{\mathbf{L}^\bullet}$. In particular, $\mathcal{V}_\Gamma$ is a strictly unital A_∞ -category.

Proof. We need to check that for $i_0, \dots, i_n \in I$ and $0 = j_0 \ll j_1 \ll \dots \ll j_n$

$$\mathcal{F}^{y^0} \circ \mu_{\mathcal{V}_\Gamma}^{\tilde{y}^0: \tilde{y}^n, \dots, \tilde{y}^1}(\varphi^n, \dots, \varphi^1) = \mu_{\mathcal{W}}^{y^0: y^n, \dots, y^1}(\mathcal{F}^{y^n}(\varphi^n), \dots, \mathcal{F}^{y^1}(\varphi^1))$$

where $\varphi^\alpha \in \mathcal{V}_\Gamma(\tilde{y}^\alpha) \subset \mathcal{V}_\Gamma(\mathbb{L}_{i_{\alpha-1}}, \mathbb{L}_{i_\alpha})[[y^i]]$ and $\mathcal{F}^{y^\alpha}(\varphi^\alpha) \in \mathcal{W}(y^\alpha) \subset \mathcal{W}(\mathbb{L}_{i_{\alpha-1}}^{j_{\alpha-1}}, \mathbb{L}_{i_\alpha}^{j_\alpha})[[y^i]]$ for $\alpha = 1, \dots, n$ and $\mathcal{F}^{y^0}: \mathcal{V}_\Gamma(\tilde{y}^0) \subset \mathcal{V}_\Gamma(\mathbb{L}_{i_0}, \mathbb{L}_{i_n})[[y^0]] \rightarrow \mathcal{W}(y^0) \subset \mathcal{W}(\mathbb{L}_{i_0}^{j_0}, \mathbb{L}_{i_n}^{j_n})[[y^0]]$. This is indeed the case as (1.4) corresponds to (4.5), (1.5) corresponds to (4.6) for a degenerate polygon, while (1.6) and (1.7) are particular cases of (4.6) for a non-degenerate polygon. $\square$

6.6.2. From disc sequence to homotopy equivalence. Let Γ be an arc brane system, $n \geq 2$, $\mathbb{L}_i = (\gamma_i, \text{gr}_i, M_i)$, $i = 1, \dots, n$ arc branes from Γ , $(y^n, \dots, y^1)$ a disc sequence bounding a region cut out by Γ_Γ with boundary path y^i from γ_{i-1} to γ_i (the index is mod n), and $\varphi^i \in \mathcal{V}_\Gamma(y^i)$, $i = 1, \dots, n$ closed morphisms. Suppose $\sum_{i=1}^n (|\varphi^i| + |y^i|) = n-2$ and set $\psi^i = \mu^{c^i: y^{i+n}, \dots, y^{i+1}}(\varphi^{i+n}, \dots, \varphi^{i+1}) \in \mathcal{V}_\Gamma(\mathbb{L}_i, \mathbb{L}_i)$ where c^i is the constant boundary path at γ_i . A straightforward computation similar to [25, p. 275] gives the following.

Lemma 6.28. In the setup above,

$$\mathbf{L} = \left(\bigoplus_{i=1}^{n-1} \mathbb{L}_i \left[\sum_{1 \leq j \leq i} (|\varphi^j| + |y^j|) - i + 1 \right], \sum_{2 \leq i < n} \varphi^i [|\varphi^i| + |y^i| - 1] \right)$$

is a twisted complex. The morphisms

$$\varphi^1 = \varphi^1 [|\varphi^1| + |y^1|]: \mathbb{L}_n \rightarrow \mathbf{L} \text{ and } \varphi^n = \varphi^n [|\varphi^n| + |y^n|]: \mathbf{L} \rightarrow \mathbb{L}_n$$

are closed of degree zero. Moreover,

$$\mu^2(\varphi^1, \varphi^n) = (-1)^\dagger \text{diag}(\psi^1, \dots, \psi^{n-1}) \text{ and } \mu^2(\varphi^n, \varphi^1) = (-1)^\dagger \psi^n$$

where $\dagger = |\varphi^1| + |y^1| - 1 + \#\{i \in \{1, \dots, n\} : |\varphi^i| + |y^i| \text{ is even}\}/2$.

Note that Lemma 6.28 implies that $\mathbf{L}$ is homotopy equivalent to $\mathbb{L}_n$ if all $H^*\varphi^i$ are invertible and the weight of the disc bounded by $(y^n, \dots, y^1)$ is an invertible morphism of dg-functors.

6.6.3. Image of the topological Fukaya category. Let $\mathbf{L}^\bullet \in \mathcal{SU}$ be a system of strict units on a full tame collection of Lagrangian branes $\mathbf{L}^0$ and Γ a full arc brane system such that the corresponding tame collection of Lagrangian branes Γ^0 is contained in $\mathbf{L}^0$. Consider the strict full embedding $\mathcal{V}_\Gamma \hookrightarrow \mathcal{W}_{\mathbf{L}^\bullet}$ given in §6.6.1. Denote by $\mathcal{W}_\Gamma$ the smallest full A_∞ -subcategory of $\mathcal{W}_{\mathbf{L}^\bullet}$ containing the image of $\mathcal{V}_\Gamma$ and closed under taking shifts, cones, and homotopy equivalent objects.

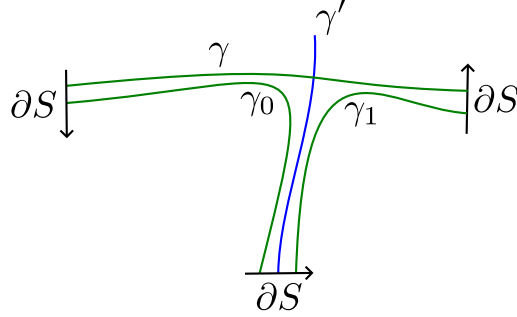


FIGURE 29. Replacing γ with γ_0 and γ_1 .

Lemma 6.29. *Let $\mathbb{L}$ be a Lagrangian brane corresponding to an arc brane. Then $\mathbb{L}$ is homotopy equivalent in $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$ to an object of $\mathcal{W}_\Gamma$.*

Proof. By Propositions 5.14 and 6.20 we can assume that $\mathcal{W}_{\mathbf{L}^\bullet}$ contains all Lagrangian branes supported on any no more than countable transverse set of Lagrangians. Replacing $\mathbb{L}$ with an isotopic brane we can obtain a brane $\mathbb{L}'$ which is transverse to all branes from Γ . By Proposition 6.17 $\mathbb{L}'$ is homotopy equivalent to $\mathbb{L}$ in $\mathcal{W}_{\text{pre}}^{\text{geom}}(\mathcal{S}; \mathcal{C})$. So we need to prove $\mathbb{L}' \in \mathbf{L}^0$ is an object of $\mathcal{W}_\Gamma$.

As $\mathcal{W}_{\mathbf{L}^\bullet}$ is triangulated by Lemma 5.15 and Proposition 6.22, any twisted complex on objects of $\mathcal{W}_\Gamma$ is homotopy equivalent to an object of $\mathcal{W}_\Gamma$. By Lemmas 6.19, 6.25, and 6.26 any Lagrangian brane on an arc from Γ lies in $\mathcal{W}_\Gamma$, so we can assume that Γ contains all arc branes supported on an arc from Γ . We can also assume that Γ is supported on a finite full arc system Γ since any full arc system has a finite full arc subsystem.

We prove that $\mathbb{L}'$ lies in $\mathcal{W}_\Gamma$ by induction on the number of intersection points between its underlying curve γ' and arcs from Γ . If $\mathbb{L}'$ does not intersect any of the arcs in Γ , then we are done by Lemma 6.28. If $t \in [0, 1]$ is the smallest with the property that $\gamma'(t)$ lies on some $\gamma \in \Gamma$, then consider the arcs γ_0 and γ_1 that run close to $\text{Im } \gamma \cup \text{Im } \gamma'$ as in Fig. 29. Again by Lemma 6.28 all Lagrangian branes supported on γ_0 and γ_1 lie in $\mathcal{W}_\Gamma$ so we can replace Γ with another full arc system $(\Gamma \setminus \{\gamma\}) \cup \{\gamma_0, \gamma_1\}$ and reduce the number of intersections of γ' with arcs from Γ by one. $\square$

6.7. Proofs of the main theorems. We collect the results of this section to prove Theorems 1 and 3 to 5.

Proof of Theorem 3. (1) follows from Propositions 6.4, 6.17, and 6.22. (2) follows from Proposition 6.17. (3) follows from Lemma 6.7 and Corollary 6.8. $\square$

Proof of Theorem 4. (1) follows from Proposition 4.8, definition of $\mathcal{W}_{\mathbf{L}^\bullet}$ in §5.3.2 and the construction of $\mathcal{SU}$ since L_i^j tends to L_i^0 as $j \rightarrow \infty$. The homotopy equivalence part follows from Proposition 5.12.

(2) is immediate from construction and (3) is part of the definition of a saturated collection of systems of strict units. (4) follows formally, see §5.3.4. (5) follows from Lemma 5.15. $\square$

Proof of Theorem 5. (1) is Lemma 6.27. (2) follows from Lemma 6.29 since given any two full arc brane systems Γ_0, Γ_1 satisfying the assumption we have $\Gamma_{1-i}^0 \subset \text{Ob } \mathcal{W}_{\Gamma_i}$ for $i = 0, 1$. $\square$

Proof of Theorem 1. (1) follows from Lemma 6.27. (2) follows from Theorem 5 (2) since $\mathcal{W}_{\mathbf{L}^\bullet}$ is triangulated for a full tame collection $\mathbf{L}^0$ by Theorem 4 (5) and for any two arc brane systems Γ_0, Γ_1 we can embed $\mathcal{V}_{\Gamma_0}$ and $\mathcal{V}_{\Gamma_1}$ in the same $\mathcal{W}_{\mathbf{L}^\bullet}$ for a full tame collection $\mathbf{L}^0$ by Theorem 4 (3). $\square$

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