

Relative Wrapped Fukaya Categories of Surfaces

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I, Daniil Mamaev, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

Given a compact oriented surface S with stops on the boundary and finitely many interior points weighted by elements of a commutative ring R , we construct the relative wrapped Fukaya category $\mathcal{W}$. The construction uses a compatible Liouville form on S , though $\mathcal{W}$ is independent of this choice. The Fukaya category $\mathcal{W}$ is an R -linear triangulated unital A_∞ -pre-category. All objects of $\mathcal{W}$ are Lagrangian branes: exact properly immersed curves equipped with a Legendrian lift to the contactisation, grading data, a local system of R -modules, and a bounding cochain solving the Maurer–Cartan equation in an obstruction algebra. This obstruction algebra generalises the Chekanov–Eliashberg dg-algebra of a Legendrian link by incorporating wrapping. We also construct from $\mathcal{W}$ a strictly unital A_∞ -category.

Our second main result shows that Legendrian isotopy induces a homotopy equivalence in the Fukaya category between the corresponding Lagrangian branes. This generalises the invariance of the quasi-isomorphism class of a Lagrangian brane under Hamiltonian isotopy and the invariance of the stable class of the Chekanov–Eliashberg dg-algebra under Legendrian isotopy.

Impact statement

I expect the results of this thesis to have an impact in various areas of algebra and geometry. The thesis uses techniques from symplectic topology, contact geometry, and homotopical algebra. Fukaya categories have been used to tackle questions on the topology of symplectic manifolds, homological mirror symmetry, and representation theory. The relative wrapped Fukaya categories of surfaces introduced in this thesis are expected to be useful in all of the above. For example, they should make the topology of Lagrangians in exact symplectic manifolds admitting an algebraic torus fibration more accessible and give new examples of homological mirror symmetry in families.

As an indication of interest in my work, I have been invited to give several talks on the subject in the UK and internationally.

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Introduction

Outline

Let S be an oriented surface, R a unital commutative ring, $\Lambda \subset \partial S$ (*the stop*) and $D \subset \text{Int } S$ (*the divisor*) finite sets, $w: D \rightarrow R$ a *weight function*, and η (*the grading*) a line field on S . In this thesis we define and study a Fukaya A_∞ -category associated to the above data. The objects arise from immersed curves in $S \setminus (D \cup \Lambda)$. The morphism space between two objects is generated by intersection points between the underlying curves and paths in $\partial S \setminus \Lambda$ between the ends of the curves. The A_∞ -operations are given by counts of immersed discs bounded by the curves, a disc u is counted with weight $\prod_{q \in D} w(q)^{\deg(u,q)}$.

To make the counts finite we choose a Liouville form λ (i.e. a 1-form such that $d\lambda$ is a volume form inducing the orientation of S) satisfying certain conditions along ∂S , Λ , and D , and only consider exact curves γ (i.e. such that $\gamma^*\lambda$ is exact). A choice of λ can be made if each connected component of S has non-empty boundary. We call the data $(\mathcal{S} = (S; \Lambda, D; \lambda; \eta), w)$ as above a *weighted surface* and the resulting category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ the *relative wrapped Fukaya category* of $(\mathcal{S}, w)$. The choice of λ is required for the construction, but $\mathcal{W}$ is independent of it.

Let $\gamma: (I, \partial I) \rightarrow (S \setminus D, \partial S \setminus \Lambda)$ be a compact properly immersed exact curve in general position. To get an object of $\mathcal{W}$ from γ we first choose a Legendrian lift of γ to the contactisation $S \times \mathbb{R}$, a local system of R -modules on the domain I , and a *grading*, which relates the tangent lines of γ to the line field η . We call γ with the above data a *Lagrangian pre-brane* $\mathcal{L}$. Then we define a curved A_∞ -algebra $\mathcal{A}_\mathcal{L}$, the *obstruction algebra* of $\mathcal{L}$. The elements of $\mathcal{A}_\mathcal{L}$ are given by double points of γ and Reeb chords of positive action between the ends of γ (each such point or chord gives an R -module of maps between respective fibres of the local systems), and the A_∞ -structure comes from counts (weighted by w) of immersed polygons with boundary on $\gamma \cup \partial S$. Finally, an object of $\mathcal{W}$, called a *Lagrangian brane*, is a pair $(\mathcal{L}, \delta)$ where $\mathcal{L}$ is a Lagrangian pre-brane and $\delta \in \mathcal{A}_\mathcal{L}$ is a bounding cochain (i.e. a solution to the Maurer–Cartan equation $\mu^0(1_R) + \mu^1(\delta) + \mu^2(\delta, \delta) + \dots = 0$).

The morphism space $\mathcal{W}((\mathcal{L}_0, \delta_0), (\mathcal{L}_1, \delta_1))$ in the Fukaya category $\mathcal{W}$ is only defined when $\mathcal{L}_0$ and $\mathcal{L}_1$ are transverse in a suitable sense. In this case it is given by the intersection points between $\mathcal{L}_0$ and $\mathcal{L}_1$ and the Reeb chords from

an end of $\mathcal{L}_0$ to an end of $\mathcal{L}_1$. The A_∞ -structure is given by counts of immersed polygons with boundary on Lagrangian branes; these counts are now weighted not only by the weight function w but also by the bounding cochains.

Our first result is that the thus defined relative wrapped Fukaya category enjoys the desired properties.

THEOREM 1. *Let $(\mathcal{S}, w)$ be a weighted surface. Then*

- (1) *The relative wrapped Fukaya category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ is a triangulated $\mathbb{Z}$ -graded R -linear A_∞ -pre-category with enough homotopy equivalences.*
- (2) *$\mathcal{W}$ does not depend on the choice of a compatible Liouville form λ .*
- (3) *There exists a strictly unital A_∞ -category $\tilde{\mathcal{W}}$ with $\text{Ob } \tilde{\mathcal{W}} = \text{Ob } \mathcal{W}$, well-defined up to homotopy equivalence of A_∞ -categories, such that two Lagrangian branes are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\tilde{\mathcal{W}}$.*

The difference between an A_∞ -pre-category and an A_∞ -category is that in the former the A_∞ -operations $\tilde{\mu}^n, n \geq 1$ are defined only for some $(n + 1)$ -tuples of objects, in our case these are the tuples of transverse Lagrangian branes. A homotopy equivalence in an A_∞ -pre-category is a closed morphism of degree zero invertible up to homotopy. Two objects are homotopy equivalent if they can be connected by a chain of homotopy equivalences, this is stronger than being quasi-isomorphic. Having enough homotopy equivalences is an appropriate notion of unitality for A_∞ -pre-categories over general commutative rings. By triangulated we mean that there is a strict totalisation functor from the category of twisted complexes over $\mathcal{W}$ to $\mathcal{W}$.

To our knowledge, this is the first definition of a relative Fukaya category over a non-complete ring. Moreover, the A_∞ -pre-category $\mathcal{W}$ and the A_∞ -category $\tilde{\mathcal{W}}$ from Theorem 1 are constructed directly from geometric data, without the use of algebraic gadgets such as A_∞ -quotients, (co)limits, or triangulated envelopes.

The second result of the thesis is that Legendrian isotopic Lagrangian pre-branes give homotopy equivalent objects in the Fukaya category. We say that two Lagrangian pre-branes with the same domain and local system of R -modules are *Legendrian isotopic* if their Legendrian lifts are isotopic in the class of embedded Legendrians in the contactisation such that the actions of Reeb chords between the ends of their projections to S do not vanish.

THEOREM 2. *Let $(\mathcal{L}_0, \delta_0)$ be a Lagrangian brane in $\mathcal{S}$ and $\mathcal{L}_1$ a Lagrangian pre-brane that is Legendrian isotopic to $\mathcal{L}_0$. Then there exists a homotopy equivalence of obstruction algebras $\mathcal{F}_1: \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ such that $(\mathcal{L}_0, \delta_0)$ and $(\mathcal{L}_1, (\mathcal{F}_1)_*(\delta_0))$ are homotopy equivalent objects in $\mathcal{W}(\mathcal{S})$.*

That the obstruction algebras of Legendrian isotopic Lagrangian pre-branes are homotopy equivalent is not very surprising, it is a variation on the invariance of the ‘stable type’ of the Chekanov–Eliashberg dg-algebra of a Legendrian knot, see Chekanov [15]. That certain isotopies of immersed Lagrangians more general than global Hamiltonian isotopies give equivalent objects in the Fukaya category was conjectured by Akaho–Joyce [5, §13]. This has been established for different classes of isotopies of *geometrically unobstructed* immersed Lagrangians by e.g. Abouzaid [1] and Biran–Cornea [8]. The significant difference of Theorem 2 to the previous results is that the objects can be supported on geometrically obstructed curves, with the bounding cochain resolving this obstruction.

Background

We give a brief and incomplete history of Fukaya categories to place our results in context. Fukaya categories were introduced by Fukaya [22] building on the works of Donaldson, Floer, Gromov, Hofer, Salamon, Ono, and others. Since then many flavours of Fukaya categories have been developed, but the basic idea remained the same. A Fukaya category is an A_∞ -category $\mathcal{F}$ associated to a symplectic manifold M . The objects are Lagrangian branes in M , i.e. Lagrangian submanifolds in M with additional data. The morphism space $\mathcal{F}(L_0, L_1)$ comes from $L_0 \cap L_1$ when L_0 and L_1 are transverse. The A_∞ -operations $\mu^n: \mathcal{F}(L_{n-1}, L_n) \otimes \dots \otimes \mathcal{F}(L_0, L_1) \rightarrow \mathcal{F}(L_0, L_n)$ are given by counts of pseudo-holomorphic polygons with sides on $L_0, \dots, L_n$. These operations satisfy certain higher associativity relations, called the A_∞ -relations. These are quadratic relations, the first states that μ^1 is a differential on $\mathcal{F}(L_0, L_1)$ (its cohomology gives the Lagrangian Floer cohomology $\mathrm{HF}(L_0, L_1)$), the second that μ^2 satisfies the Leibniz rule with respect to μ^1 , the third that μ^2 is associative up to a homotopy given by μ^3 , and so on.

There are many technical issues with setting up the counts of pseudo-holomorphic polygons above for general symplectic manifolds M , see Fukaya–Oh–Ohta–Ono [23]. In this thesis we do not discuss these since we instead count smooth immersed discs for $L_0, \dots, L_n$ in general position, which is equivalent to counting pseudo-holomorphic discs when M is 2-dimensional, see Seidel [46, §13]. The issue of what to do when the Lagrangians are not in general position, however, remains. There are two ways to deal with this. One is to hard-code into the definition Hamiltonian perturbations making any tuple of Lagrangians transverse, see e.g. [46, §9]. Another is to only define the A_∞ -operations for Lagrangians in general position, as proposed by Kontsevich–Soibelman [31,

§4]. We choose the latter approach to avoid making choices. Theorem 1 (3) shows that the former approach can be recovered from ours.

Since their inception, Fukaya categories have been tied to homological mirror symmetry (HMS), which links the symplectic geometry of a space X^A to the algebraic geometry of a mirror X^B via an equivalence of triangulated categories. The subject began with Kontsevich’s conjecture [30]: for any Calabi–Yau manifold X^A there exists a complex algebraic manifold X^B of the same dimension such that the a Fukaya category of X^A is equivalent to the bounded derived category of coherent sheaves on X^B . Thirty years later, the conjecture remains open in general, but HMS has grown into a broad field extending beyond the Calabi–Yau setting and yielding both heuristic and rigorous insights into the geometry of mirror pairs.

Relative Fukaya categories were introduced by Seidel [45] to relate a Fukaya category of a closed symplectic manifold to a Fukaya category of an open submanifold in order to establish HMS, e.g. for the quartic surface [47]. A similar idea of weighting disc counts according to how many times a disc passes through a point goes back to the work of Ozsváth–Szabó on Heegaard Floer homology [40]. Relative Fukaya categories are usually defined in a formal neighbourhood of a point, i.e. R is a complete local ring, see e.g. Perutz–Sheridan [42]. In our setting it is possible to define a relative category for any commutative ring R .

Partially wrapped Fukaya categories for symmetric products of surfaces were introduced by Auroux [6] to give a symplectic interpretation for Bordered Heegaard Floer homology. Fully wrapped Fukaya categories were defined by Abouzaid–Seidel [4] (the difference between the partially and fully wrapped categories is the absence of stops in the latter). HMS applications followed, see e.g. Abouzaid–Auroux–Efimov–Katzarkov–Orlov [3], Bocklandt [9], Chan–Pomerleano–Ueda [13], Lekili–Polishchuk [34, 36]. Then Sylvan [48] and Ganatra–Pardon–Shende [24, 25] defined wrapped Fukaya categories in full generality.

Another view on Fukaya categories is that they refine invariants of Lagrangians in symplectic manifolds. The study of these invariants predates HMS, it started with the fundamental observation by Floer that Lagrangian Floer (co)homology is invariant with respect to Hamiltonian diffeomorphism applied to *one* of the arguments. That is, if L_0 and L_1 are Lagrangian branes in the symplectic manifold M and φ is a Hamiltonian diffeomorphism of M , then $\mathrm{HF}(\varphi_*L_0, L_1) \cong \mathrm{HF}(L_0, L_1)$. This observation can then be promoted to the following: if L_0 and L_1 are Hamiltonian isotopic, then they are quasi-isomorphic in the Fukaya category. Theorem 2 is a generalisation of this,

with weaker premise (Legendrian instead of Hamiltonian isotopy) and slightly stronger conclusion (homotopy equivalence of corresponding objects in the Fukaya category instead of quasi-isomorphism).

In the case of surfaces a combinatorial description of the wrapped Fukaya category is given by Haiden–Katzarkov–Kontsevich [26]. This description leads to an equivalence of wrapped Fukaya categories of surfaces and derived categories of graded gentle algebras, which is further explored by Lekili–Polishchuk [35], Oppen–Plamondon–Schroll [39], and others. A more geometric way to define wrapped Fukaya categories of surfaces over a field is sketched by Haiden [27].

Relative wrapped Fukaya categories of surfaces were introduced by Lekili–Perutz [32] and then developed by Lekili–Polishchuk [33] to prove HMS between the ‘ n -Tate curve’ and the n -punctured torus over $\mathbb{Z}[[t_1, \dots, t_n]]$. Bocklandt–van de Kreeke [10, 11] gave an algebraic definition of relative wrapped Fukaya category linear over $\mathbb{C}[[q_1, \dots, q_n]]$ where $q_1, \dots, q_n$ are punctures. Evans–Lekili [21] gave a combinatorial definition of the relative wrapped Fukaya category over an arbitrary commutative ring, but were only able to prove that the resulting category does not depend on choices after completion.

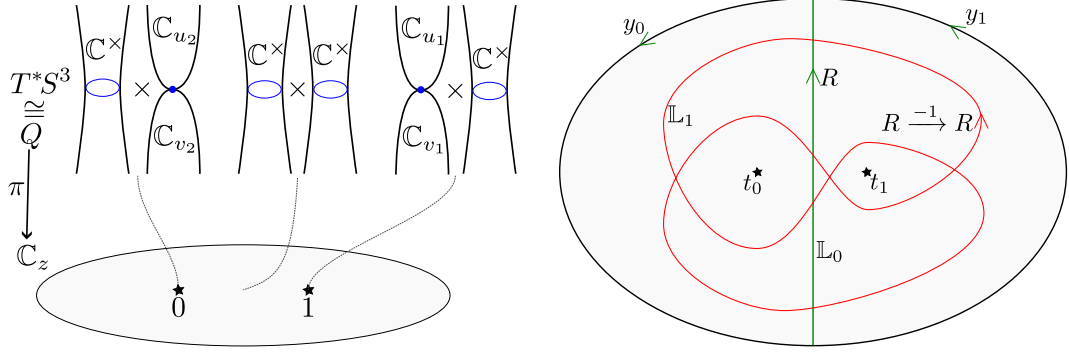
The differences of our construction with those already present in the literature are the following.

- The relative wrapped Fukaya category is defined over an arbitrary commutative ring, not necessarily complete.
- All the objects of the category are given by Lagrangian branes and the resulting category is triangulated without the need to take triangulated envelope.
- The morphisms are given by intersection points and Reeb chords, without the need of taking (homotopy) colimits.
- The A_∞ -operations are defined by counts of immersed polygons, without the need of passing to A_∞ -quotients.

Motivation

We give two examples of how relative wrapped Fukaya categories of surfaces appear in homological mirror symmetry based on the proposals by Lekili–Segal [37] and Evans–Lekili [21].

Algebraic torus fibrations. Lekili–Segal [37] conjectured that the relative wrapped Fukaya category of the base of an algebraic torus fibration is equivalent (possibly after completion) to the wrapped Fukaya category of the total space of the fibration. We illustrate this conjecture with an example.



(a) Algebraic torus fibration on T^*S^3 . Fibres over 0, 1, and a third point are in black. The intersection of $\mu^{-1}(0)$ with the fibres is in blue.

(b) Disc with a two-point divisor. $\mathbb{L}_0$ is the green curve with trivial local system with fibre R . $\mathbb{L}_1$ is the red curve with local system with fibre R and monodromy -1 . Bounding cochains for $\mathbb{L}_0$ and $\mathbb{L}_1$ are zero.

FIGURE 1. Algebraic torus fibration and weighted surface structure on its base.

The cotangent bundle of S^3 is exact symplectomorphic to a non-singular quadric in $\mathbb{C}^4$:

$$\begin{array}{ccc}
 T^*S^3 = \{(q, p) \in \mathbb{C}^2 \times \mathbb{C}^2 : |q| = 1, \operatorname{Re}(q_1 \bar{p}_1 + q_2 \bar{p}_2) = 0\} & & (q, p) \\
 \downarrow \cong & & \downarrow \\
 & & \sqrt{2} \left(q \cosh \frac{|p|}{2} + i \frac{p}{|p|} \sinh \frac{|p|}{2}, \bar{q} \cosh \frac{|p|}{2} + i \frac{\bar{p}}{|\bar{p}|} \sinh \frac{|p|}{2} \right) \\
 Q & \xlongequal{\quad} & \{(u, v) \in \mathbb{C}^2 \times \mathbb{C}^2 : u_1 v_1 + u_2 v_2 = 1\}.
 \end{array}$$

The projection

$$\pi: Q \rightarrow \mathbb{C}_z, \quad (u_1, v_1, u_2, v_2) \mapsto u_2 v_2$$

is an algebraic torus fibration, i.e. we have a fibre-preserving action

$$(\mathbb{C}^\times)^2 \curvearrowright Q, \quad (\xi_1, \xi_2) \cdot (u_1, v_1, u_2, v_2) = (\xi_1 u_1, \xi_1^{-1} v_1, \xi_2 u_2, \xi_2^{-1} v_2)$$

that is simply transitive on all but finitely many fibres. The fibres of this projection are

$$Q_t \cong \begin{cases} \mathbb{C}^\times \times \{(u_2, v_2) \in \mathbb{C}^2 : u_2 v_2 = 0\} & \text{if } t = 0; \\ \{(u_1, v_1) \in \mathbb{C}^2 : u_1 v_1 = 0\} \times \mathbb{C}^\times & \text{if } t = 1; \\ \mathbb{C}^\times \times \mathbb{C}^\times & \text{otherwise.} \end{cases}$$

Restricting the action to the real torus $\mathbb{T}^2 \subset (\mathbb{C}^\times)^2$ we get a Hamiltonian action with moment map

$$\mu: Q \rightarrow \mathbb{R}^2, \quad \mu(u_1, v_1, u_2, v_2) = (|u_1|^2 - |v_1|^2, |u_2|^2 - |v_2|^2).$$

See Fig. 1(a) for a sketch of $\pi: Q \rightarrow \mathbb{C}_z$ and the zero locus of μ .

Note that the action of $\mathbb{T}^2$ on $\mu^{-1}(0)$ is not free over $z \in \{0, 1\}$, so $\mathbb{C}_z$ is not quite the symplectic reduction of Q . Also observe that different factors degenerate when we approach $z = 0$ and $z = 1$. To capture this we consider the weighted surface $(\mathcal{S}, w)$ where S is a disc in $\mathbb{C}_z$ containing 0 and 1, $\Lambda = \emptyset$, $D = \{0, 1\}$, $R = k[t_0, t_1]$, η is the vertical line field, and $w: D \rightarrow R$ is given by $w(0) = t_0$ and $w(1) = t_1$, see Fig. 1(b).

The curves $\mathbb{L}_0$ and $\mathbb{L}_1$ in Fig. 1(b) are obtained from a cotangent fibre and a Hamiltonian perturbation of the zero section in T^*S^3 by the standard symplectic reduction of Lagrangian submanifolds. Namely, the cotangent fibre over $q = (1/2, 1/2, 1/2, 1/2)$ intersects $\mu^{-1}(0)$ transversely and the intersection projects to the line $\{\operatorname{Re} z = 1/2\}$ in $\mathbb{C}_z$. The zero section in T^*S^3 is not transverse to $\mu^{-1}(0)$ (it is equal to $\mu^{-1}(0) \cap \pi^{-1}([0, 1])$), but a graph of e.g. $d(q_1 + q_2 + q_3)$ intersects $\mu^{-1}(0)$ transversely and projects to the red curve in Fig. 1(b).

In Example 1.30 we compute the endomorphism algebra of the Lagrangian brane $\mathbb{L}_0$ in the relative wrapped Fukaya category $\tilde{\mathcal{W}}_w(\mathcal{S})$ and show that it is quasi-isomorphic to the endomorphism algebra of the cotangent fibre in the wrapped Fukaya category $\mathcal{W}(T^*S^3)$. Under this equivalence the zero section in T^*S^3 corresponds to $\mathbb{L}_1$ from Fig. 1(b).

Relative curves. Evans–Lekili [21] proved that the derived category of coherent sheaves on a crepant resolution of the singularity $uv = t_0 \dots t_n$ is quasi-equivalent to a subcategory of the relative wrapped Fukaya category $\mathcal{W}_w(\mathcal{S})$ where S is an annulus, $\Lambda = \emptyset$, $D = \{q_0, \dots, q_n\}$, the line field is radial, $R = k[t_0, \dots, t_n]$ and $w(q_i) = t_i$. It is expected that the above is in fact a quasi-equivalence with $\mathcal{W}_w(\mathcal{S})$, we will address the generation in relative wrapped Fukaya categories in another paper. The geometric heuristic for the above result is as follows.

Let R be a finite type algebra over an algebraically closed field k and $X^B \xrightarrow{\pi} \operatorname{Spec} R$ a relative curve, i.e. a flat morphism such that each geometric fibre is an algebraic curve. Then π^* gives the dg-category of perfect complexes $\operatorname{Perf} X^B$ an R -linear structure (the reason to consider Perf instead of $D^b \operatorname{Coh}$ is that the former behaves better in families) and we have a functor $\operatorname{Perf} X^B \rightarrow \operatorname{Perf} X_{\mathfrak{m}}^B$ for each geometric point $\mathfrak{m}$ of $\operatorname{Spec} R$ (i.e. a maximal ideal of R). On the other hand, if $(\mathcal{S}, w: D \rightarrow R)$ is a weighted surface, then the relative wrapped Fukaya category $\mathcal{W}_w(\mathcal{S})$ is linear over R and for each maximal ideal $\mathfrak{m}$ we get a functor $\mathcal{W}_w(\mathcal{S}) \rightarrow \mathcal{W}_{w_{\mathfrak{m}}}(\mathcal{S})$ where $w_{\mathfrak{m}} = (R \rightarrow R/\mathfrak{m}) \circ w$. Since the mirrors to some algebraic curves over a field are given by wrapped Fukaya categories of surfaces, one can hope that mirrors to some relative curves would be given

by relative wrapped Fukaya categories of surfaces. We illustrate this with a simple example.

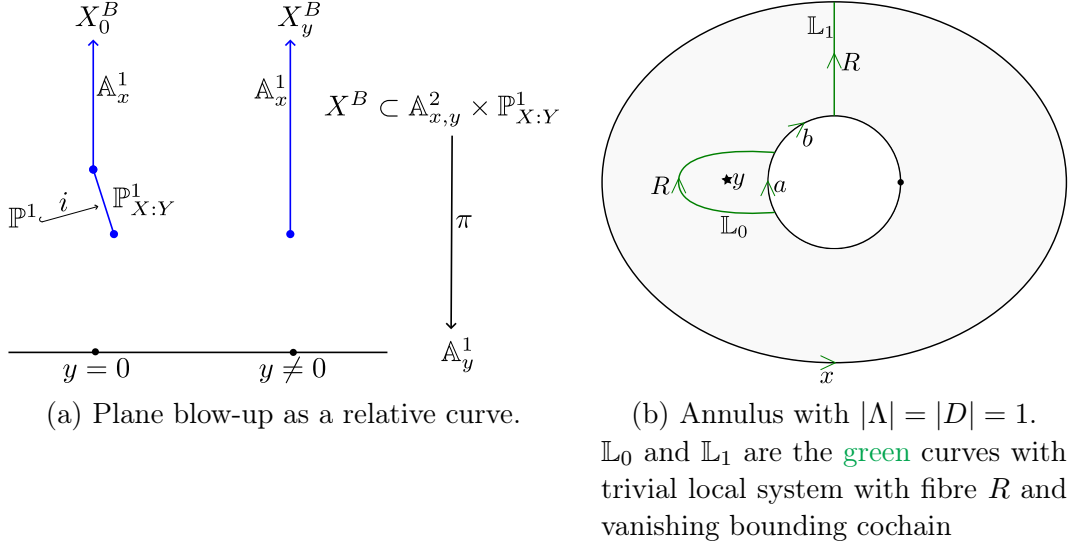


FIGURE 2. Plane blow-up and its mirror.

Let k be an algebraically closed field and $X^B = \{(x, y, X : Y) \in \mathbb{A}^2 \times \mathbb{P}^1 : xY = Xy\}$ the blow-up of the affine plane at the origin. The projection $\pi : X^B \rightarrow \mathbb{A}^1_y$ to the second coordinate is a relative curve. For a closed point y of $\mathbb{A}^1$ the fibre over y is

$$X_y^B \cong \begin{cases} \mathbb{A}^1_x & \text{if } y \neq 0; \\ \{(x, X : Y) \in \mathbb{A}^1 \times \mathbb{P}^1 : xY = 0\} & \text{if } y = 0. \end{cases}$$

See Fig. 2(a) for a sketch of $\pi : X^B \rightarrow \mathbb{A}^1_y$.

The mirrors to the fibres are well-known and are given by the usual surfaces with stops:

$$X_y^A = \begin{cases} (S = [0, 1] \times \mathbb{R}/\mathbb{Z}, \Lambda = \{(0, 0)\}; \eta = T[0, 1]) & \text{if } y \neq 0; \\ (S = [0, 1] \times \mathbb{R}/\mathbb{Z} \setminus \{q\}, \Lambda = \{(0, 0)\}; \eta = T[0, 1]) & \text{if } y = 0. \end{cases}$$

That is, we have $\text{Perf } X_y^B \cong \mathcal{W}(X_y^A)$. We assemble a weighted surface $(\mathcal{S}, w)$ from it by setting $S = [0, 1] \times \mathbb{R}/\mathbb{Z}$, $\Lambda = \{(0, 0)\}$, $D = \{q\}$, $\eta = T[0, 1]$, $R = k[y]$, and $w(q) = y$, as in Fig. 2(b).

In Example 1.31 we will show that $D^b \text{Coh } X^B = \text{Perf } X^B$ is quasi-isomorphic to a subcategory of $\tilde{\mathcal{W}}_w(\mathcal{S})$ with $i_* \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{X^B}$ corresponding to $\mathbb{L}_0$ and $\mathbb{L}_1$ from Fig. 2(b) respectively.

Structure of the thesis

In Chapter 1 we give all definitions needed to construct the relative wrapped Fukaya category of a weighted surface and to state Theorems 1 and 2 precisely.

The exposition there is self-contained and includes examples to facilitate understanding. All proofs are postponed to later chapters.

In Chapter 2 we study Liouville structures on a given surface, Lagrangian structures on a given curve, and their deformation. We describe the moduli space of graded Liouville marked structures on a given surface in Proposition 2.1 and prove that diffeomorphisms of marked surfaces deform to symplectomorphisms in Proposition 2.3. We characterise Lagrangian structures on a given curve in terms of linear inequalities on actions and areas in Lemmas 2.9 and 2.11. In Lemmas 2.12 and 2.13 we characterise Lagrangian and Legendrian isotopies respectively. In Proposition 2.20 we explain how to replace all Reeb chords by double points by a Legendrian isotopy.

In Chapter 3 we study A_∞ -pre-categories with enough homotopy equivalences. After establishing basic facts about homotopy equivalences in Lemmas 3.1 to 3.4, we explain how to get a strictly unital A_∞ -category from a certain A_∞ -pre-category in Lemma 3.6 and give a sufficient condition for this construction to be canonical in Lemma 3.8. After that we study triangulated A_∞ -pre-categories.

In Chapter 4 we prove Theorem 1. We prove that the obstruction algebra of a Lagrangian brane is a curved A_∞ -algebra in Proposition 4.2. In Proposition 4.4 we prove that the relative wrapped Fukaya category $\mathcal{W}$ of a weighted surface is an A_∞ -pre-category and that it does not depend on the choice of a compatible Liouville form. In Propositions 4.11 and 4.19 we prove that $\mathcal{W}$ has enough homotopy equivalences and is triangulated. In Proposition 4.21 we prove Theorem 1 (3).

In Chapter 5 we prove Theorem 2. We recall the changes in the spaces of polygons bounded by a Lagrangian under Legendrian Reidemeister moves in Lemmas 2.19 and 5.1 to 5.3. We prove that a homotopy equivalence between an A_∞ -algebra and its stabilisation can be chosen to respect a given bounding cochain in Lemma 5.4. In Proposition 5.5 we prove that obstruction algebras of Legendrian isotopic Lagrangian pre-branes are homotopy equivalent and that this homotopy equivalence can be chosen to respect a given bounding cochain. Proposition 5.10 finishes the proof of Theorem 2.

Conventions

We adopt the following conventions throughout.

For geometric considerations we work in the smooth category unless otherwise stated. Manifold means (smooth) manifold with (possibly empty) boundary. A smooth map on a subset A of a manifold is the restriction of a smooth map on a neighbourhood of A .

Given a smooth map $\varphi: M \rightarrow N$ the pullback on n -forms is $\varphi^*: \Omega^n(N) \rightarrow \Omega^n(M)$, for any other fibre bundle E on N the pullback on sections is $\varphi^*: C^\infty(N, E) \rightarrow C^\infty(M, \varphi^*E)$. The pullback of a along an embedding is denoted by $a|_M$.

A component of a topological space T is its path-connected component, the set of all components is denoted by $\pi_0 T$.

For algebraic considerations we work over a unital commutative ring R . A linear map is a morphism of (left) R -modules. The tensor product over R is denoted by $\otimes$.

Compositions of morphisms and concatenations of paths are written from right to left, i.e. the composition of $A \xrightarrow{x} B \xrightarrow{y} C$ is the morphism $A \xrightarrow{y \circ x} C$.

CHAPTER 1

From surface to category

1.1. Surfaces, curves, and polygons

In this section we set up the geometric underpinning for relative wrapped Fukaya categories of surfaces.

Let S be a compact 2-dimensional manifold, $\Lambda \subset \partial S$ a finite set, called the *stop*, and $D \subset \text{Int } S = S \setminus \partial S$ a finite set, called the *divisor*. We call such data $(S; \Lambda, D)$ a *marked surface*.

In §1.1.1 we choose a Liouville form on a marked surface satisfying certain conditions along the boundary, the stop, and the divisor; this choice is contractible. In §1.1.2 we define Lagrangians in a Liouville marked surface, which are certain immersed curves with additional data, these will give rise to objects in the Fukaya category. In §1.1.3 we describe the self-intersection information of a Lagrangian, consisting of its double points and Reeb chords between its ends, these will give morphisms in the Fukaya category. In §1.1.4 we describe the polygons bounded by a Lagrangian, their counts will give rise to A_∞ -operations. In §1.1.5 we add the grading information to surfaces and curves to make the Fukaya category $\mathbb{Z}$ -graded.

1.1.1. Surface. We describe a Liouville structure on S compatible with the marked structure (Λ, D) .

Recall that a 1-form λ on a $2n$ -dimensional manifold M is called a *Liouville form* if $(d\lambda)^{\wedge n}$ is a volume (i.e. nowhere vanishing) form on M . The *Liouville vector field* $Z = Z_\lambda$ is defined by $d\lambda(Z, -) = \lambda$, and its flow is called the *Liouville flow*. Note that if M admits a Liouville form, then it is orientable (by the volume form) and M cannot be closed (by the Stokes theorem).

Let λ be a Liouville form on the surface S . We impose the following conditions on λ along ∂S , Λ , and D .

The first set of conditions ensures that the Liouville flow Φ_Z^t is defined for all $t \leq 0$ and that $D \cup \Lambda$ does not move under the flow.

- (1) λ (equivalently, Z) vanishes on D .
- (2) $\Lambda = \{p \in \partial S : \lambda(p) = 0\}$.
- (3) Z points outwards along $\partial S \setminus \Lambda$.

Note that there can be zeroes of λ in $\text{Int } S$ not lying in D , but all zeroes of λ on ∂S lie in Λ . Observe that condition (3) is equivalent to $\lambda|_{\partial S \setminus \Lambda}$ being a volume

form that induces the orientation of $\partial S \setminus \Lambda$ agreeing with the orientation of ∂S as the boundary of S .

We impose one more condition to ensure that introducing λ does not affect the mapping class group. On one hand, the natural notion of automorphism of the marked surface $(S; \Lambda, D)$ is an orientation-preserving diffeomorphism φ of S such that $\varphi(\Lambda) = \Lambda$ and $\varphi(D) = D$. Let $\text{Diff}(S; \Lambda, D)$ be the group of such maps φ . On the other hand, an automorphism of the exact symplectic manifold $(S; \lambda)$ is a diffeomorphism φ of S such that $\varphi^*\lambda - \lambda$ is exact. Let $\text{Symp}_{\text{ex}}(S; \lambda)$ be the group of such maps φ . It is therefore natural to define

$$(1.1) \quad \text{Symp}_{\text{ex}}(S; \Lambda, D; \lambda) = \text{Diff}(S; \Lambda, D) \cap \text{Symp}_{\text{ex}}(S; \lambda).$$

We would like the mapping class group $\pi_0 \text{Diff}(S; \Lambda, D) = \text{MCG}(S; \Lambda, D)$ to be equal to the mapping class group $\pi_0 \text{Symp}_{\text{ex}}(S; \Lambda, D; \lambda) = \text{MCG}(S; \Lambda, D; \lambda)$, but there is an obstruction to that. Namely, if C_0 and C_1 are two components of ∂S with $|C_0 \cap \Lambda| = |C_1 \cap \Lambda|$, then there exists $\varphi \in \text{Diff}(S; \Lambda, D)$ mapping C_0 to C_1 , but if $\varphi \in \text{Symp}_{\text{ex}}(S; \lambda)$, then $\int_{C_0} \lambda = \int_{C_1} \lambda$. We require this obstruction to vanish.

(4) $\int_C \lambda = 1$ for each connected component C of ∂S .

This is enough for $\text{Symp}_{\text{ex}}(S; \Lambda, D; \lambda)$ to be a strong deformation retract of $\text{Diff}(S; \Lambda, D)$, see Proposition 2.3.

We summarise the discussion above into the following definition.

Definition 1.1. A *Liouville structure* on the marked surface $(S; \Lambda, D)$ is a Liouville form λ on S satisfying conditions (1)–(4) above. $\mathcal{S} = (S; \Lambda, D; \lambda)$ is then called a *Liouville marked surface*.

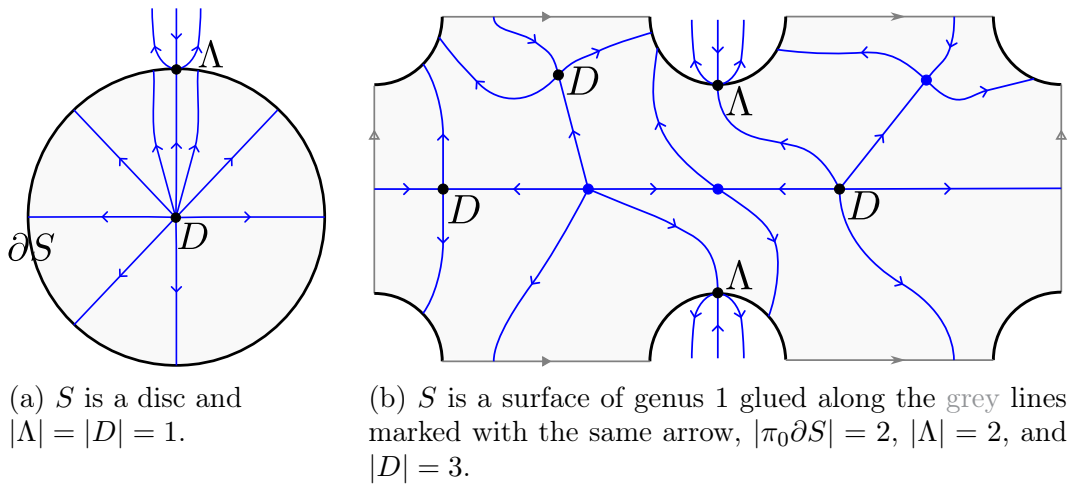


FIGURE 3. Examples of Liouville marked surfaces.

The dots are the zeroes and the blue lines are the integral lines of the Liouville vector field. The zeroes that belong to Λ or D are marked by the corresponding letter.

In Fig. 3 the zeroes and integral lines of the Liouville vector field are sketched for two Liouville marked surfaces.

If S is orientable and $\partial S \neq \emptyset$ (which is necessary for the existence of a Liouville form on S), then the marked surface $(S; \Lambda, D)$ admits a Liouville structure. Moreover, the space of Liouville structures on $(S; \Lambda, D)$ such that $d\lambda$ induces a given orientation of S is convex in $\Omega^1(S)$, see Proposition 2.1.

Liouville structures are closely connected to contact structures. Recall that a 1-form α on a $(2n+1)$ -dimensional manifold N is called a *contact form* if $\alpha \wedge (d\alpha)^n$ is a volume form on N . The pair (N, α) is then called a *(co-oriented) contact manifold*. The *Reeb vector field* R is defined by $d\alpha(R, -) = 0$ and $\alpha(R) = 1$, its integral curves are called *Reeb trajectories*.

An n -dimensional submanifold of a $(2n+1)$ -dimensional contact manifold (N, α) is called a *Legendrian* if it is everywhere tangent to the *contact distribution* $\ker \alpha$. A Reeb trajectory connecting two points of a Legendrian is called a *Reeb chord*. The *length* of a Reeb chord $y: [0, l] \rightarrow N$ is $l = \int_0^l y^* \alpha$. Unless stated otherwise, we only consider non-constant Reeb chords.

Given a Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda)$ we obtain two contact manifolds from it. The first one is $(\partial S \setminus \Lambda, \lambda|_{\partial S \setminus \Lambda})$, which is a disjoint union of open intervals and circles. The second is the contact 3-manifold

$$(1.2) \quad (S^+, \lambda^+) = (S \setminus (\Lambda \cup D) \times \mathbb{R}_z, \lambda + dz),$$

called the *contactisation* of $\mathcal{S}$.

1.1.2. Curves. Fix a Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda)$. We describe the curves in S that will give rise to objects in the Fukaya category.

Let I be a 1-dimensional compact manifold, i.e. a finite disjoint union of circles and closed segments. Consider an immersion $\gamma: I \rightarrow S$. The first set of conditions we impose is that γ is a generic properly immersed curve in $S \setminus (\Lambda \cup D)$:

- (1) $\gamma(I) \cap (\Lambda \cup D) = \emptyset$.
- (2) $\gamma^{-1}(\partial S) = \partial I$ and γ is transverse to ∂S ;
- (3) $\gamma|_{\partial I}$ is an embedding into ∂S ;
- (4) all singularities of γ are transverse double points.

The curve γ is called an *exact immersion* if there exists a function $P: I \rightarrow \mathbb{R}$ such that

$$(5) \quad \gamma^* \lambda + dP = 0.$$

Note that this implies that the *Legendrian lift* $\gamma^+ = (\gamma, P): I \rightarrow S^+$ is indeed Legendrian in the contactisation (S^+, λ^+) , i.e. γ^+ is tangent to the contact distribution $\ker \lambda^+$.

We impose further general position requirements, now on P :

- (6) for any double point of γ the values of P at its two preimages are different;
- (7) for any Reeb chord $y: [0, l] \rightarrow \partial S \setminus \Lambda$ connecting $\gamma(t_0) = y(0)$ and $\gamma(t_1) = y(l)$, the value $P(t_1)$ is different from $P(t_0) - l$.

Note that (6) ensures the Legendrian lift γ^+ is embedded and that (7) is in a sense a variant of (6): for $t_0 \in \partial I$ if we attach the maximally extended Reeb trajectory $y: [0, l_0) \rightarrow \partial S \setminus \Lambda$ with $y(0) = t_0$ to γ and extend the potential to satisfy (5), then the potential at a point $l \in [0, l_0)$ will be equal to $P(t_0) - \int_0^l y^* \lambda = P(t_0) - l$.

Combining all conditions together we obtain the following definition.

Definition 1.2. A (exact immersed) *Lagrangian* L in the Liouville marked surface $\mathcal{S}$ is the data of

- a 1-dimensional compact manifold I , called the *domain*;
- an immersion $\gamma: I \rightarrow S$, called the *underlying curve*;
- a function $P: I \rightarrow \mathbb{R}$, called the *potential*

satisfying conditions (1)–(7) above.

Note that the potential is part of the data. For a generic exact immersion $\gamma: I \rightarrow S$ the choice of potential P is parametrised by the complement of a hyperplane arrangement in $\mathbb{R}^{\pi_0 I}$, see §4.2.1. Choices of the potential from different connected components of this complement can give non-equivalent obstruction algebras and hence different objects in the Fukaya category.

1.1.3. Double points and Reeb chords. Let $L = (I, \gamma, P)$ be a Lagrangian in a Liouville marked surface $\mathcal{S}$. We describe the self-intersection information of L . It consists of double points of γ and certain Reeb chords connecting the ends of L . They will give rise to the obstruction algebra and to morphism spaces in the Fukaya category.

Let $X = X_L$ be the set of double points of γ . Fix $x \in X$ and let

$$(1.3) \quad x_+, x_- \in I \text{ be such that } \gamma(x_+) = x = \gamma(x_-) \text{ and } P(x_+) > P(x_-).$$

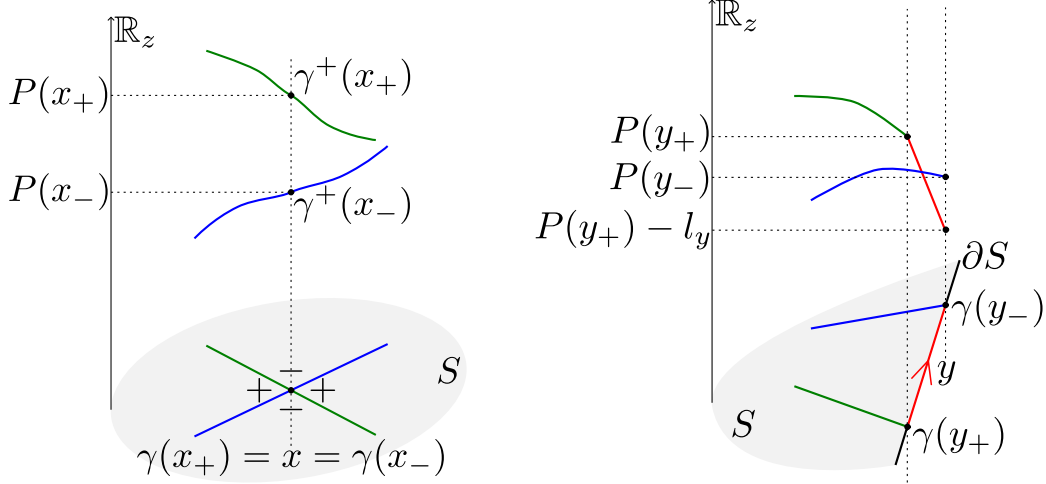
This makes sense by condition (6). The *action* of x is

$$(1.4) \quad A(x) = P(x_+) - P(x_-) > 0.$$

Note that double points of L are in bijection with Reeb chords of its Legendrian lift L^+ and that the action of a double point is equal to the length of the corresponding Reeb chord.

The image of a small neighbourhood of x_+ (resp. x_-) under γ is called the *overstrand* (resp. *understrand*) at x . To arrange the counts of discs later it is convenient to encode the over- and understrand information at x by *Reeb signs*. These are the assignments of $+$ or $-$ to each of the four quadrants cut

out by the tangent lines $\dot{\gamma}(x_+)$ and $\dot{\gamma}(x_-)$ from $T_x S$. Namely, we assign $+$ (resp. $-$) to the two quadrants that lie to the left of the curve that first follows the understrand (resp. overstrand) and then jumps to the overstrand (resp. understrand) at x , see Fig. 4(a). The quadrants with Reeb sign $+$ (resp. $-$) are called *positive* (resp. *negative*).



(a) Double point x of L with Reeb signs indicated.

(b) Reeb chord y between two ends of L and two Legendrian lifts of y .

FIGURE 4. Lifts of a double point and a Reeb chord.

Lagrangian $L = (I, \gamma, P)$ in S and its Legendrian lift $\gamma^+ = (\gamma, P)$ in the contactisation S^+ around a double point and a Reeb chord.

Note that $\gamma|_{\partial I}$ gives a Legendrian submanifold in the contact manifold $(\partial S \setminus \Lambda, \lambda|_{\partial S \setminus \Lambda})$. Let $Y = Y_L$ be the set of Reeb chords $y: [0, l_y] \rightarrow \partial S \setminus \Lambda$ that connect two ends of L . Consider a Reeb chord $y \in Y$. Let

$$(1.5) \quad y_+, y_- \in \partial I \text{ be such that } \gamma(y_+) = y(0) \text{ and } \gamma(y_-) = y(l_y).$$

The *action* of y is

$$(1.6) \quad A(y) = P(y_+) - l_y - P(y_-).$$

Note that $A(y)$ is always non-zero by condition (7).

The definitions of action for double points and Reeb chords are compatible in the following sense. Given a Reeb chord $y \in Y$, consider the Reeb trajectory $\tilde{y}: [0, l) \rightarrow \partial S \setminus \Lambda$ that maximally extends y . Extend $\gamma: I \rightarrow S$ to a piece-wise smooth curve $\tilde{\gamma} = \gamma \cup \tilde{y}: I \cup_{x_+=0} [0, l) \rightarrow S$ and $P: I \rightarrow \mathbb{R}$ to a continuous function $\tilde{P}: I \cup_{x_+=0} [0, l) \rightarrow \mathbb{R}$ satisfying (5) everywhere except $x_+ = 0$. Then $\tilde{\gamma}(y_-) = \tilde{\gamma}(l_y)$ is a self-intersection point of $\tilde{\gamma}$ and $A(y) = \tilde{P}(l_y) - \tilde{P}(y_-)$, see Fig. 4(b).

Let $y^i: [0, l_i] \rightarrow \partial S \setminus \Lambda$, $i = 0, 1$ be two Reeb chords connecting the ends of L such that $y^0(l_i) = y^1(0)$. Their *concatenation* is the Reeb chord

$$(1.7) \quad y^1 y^0: [0, l_0 + l_1] \rightarrow \partial S \setminus \Lambda, \quad s \mapsto \begin{cases} y^0(s), & \text{if } s \in [0, l_0]; \\ y^1(s - l_0), & \text{if } s \in [l_0, l_0 + l_1]. \end{cases}$$

The action is additive with respect to concatenation.

The set X is finite by compactness and so is the set $Y^{>0}$ of Reeb chords of positive action, see Lemma 2.10. We denote $Z = Z_L = X \cup Y^{>0}$.

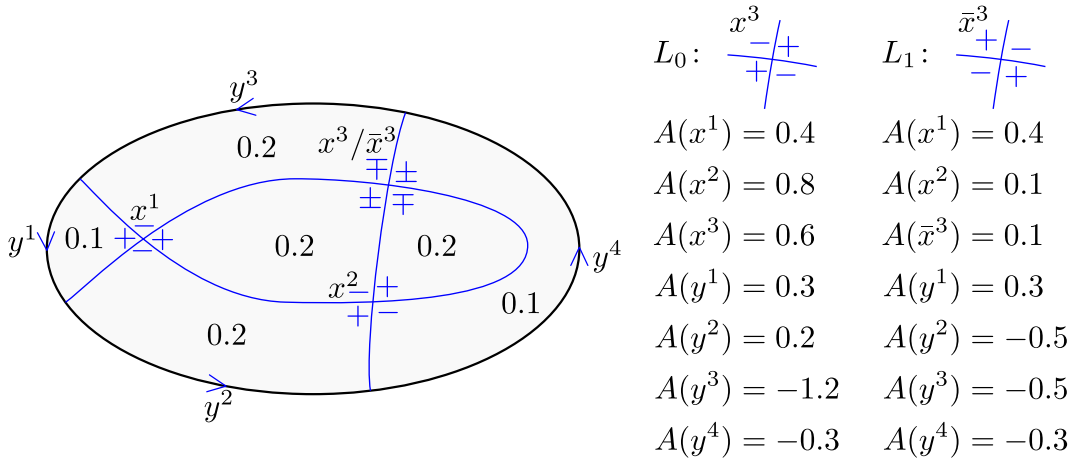


FIGURE 5. Two Lagrangians with the same underlying curve.

The blue curve is the underlying curve of both L_0 and L_1 , the blue signs at each double point are Reeb signs, the numbers inside the regions cut out from the disc by the curve are their areas.

Example 1.3. In Fig. 5 two Lagrangians L_0 and L_1 in a disc with some Liouville form λ are depicted. The domain of both Lagrangians is the disjoint union of two segments and their underlying curves are the same. L_1 is obtained from L_0 by changing the potential on one of the segments by a constant. We have $Z_{L_0} = \{x^1, x^2, x^3, y^1, y^2, y^2 y^1, y^4 y^2 y^1\}$ and $Z_{L_1} = \{x^1, x^2, \bar{x}^3, y^1\}$.

This example is continued in Example 1.6.

The Liouville form in Fig. 5 is only represented by areas (i.e. integrals of $d\lambda$) and actions. In fact, this information is enough, an immersed curve satisfying conditions (1)–(4) from §1.1.2 can be smoothly isotoped (in the class of such curves) to be the underlying curve of a Lagrangian with prescribed Z and Reeb signs at each double point if and only if a certain system of linear equations and inequalities involving the actions of double points and Reeb chords and areas of the regions cut out from the surface by the curve has a solution, see Lemmas 2.9 and 2.11.

1.1.4. Polygons. Fix a Lagrangian $L = (I, \gamma, P)$ in a graded marked surface $\mathcal{S} = (S; \Lambda, D; \lambda)$. We describe immersed polygons bounded by L . Their counts will give A_∞ -operations.

Definition 1.4. A *polygon bounded by $L = (I, \gamma, P)$* is a tuple (Π, u, v) such that

- Π is the compact region in $\mathbb{R}^2$ bounded by a piecewise-smooth Jordan curve;
- $u: (\Pi, \partial\Pi) \rightarrow (S, \text{Im } \gamma \cup (\partial S \setminus \Lambda))$ is an immersion;
- v is a vertex (i.e. non-smooth point) of Π .

Two such polygons (Π_i, u_i, v_i) , $i = 0, 1$ are *equivalent* if there exists a diffeomorphism $\phi: \Pi_0 \rightarrow \Pi_1$ such that $u_1 \circ \phi = u_0$ and $\phi(v_0) = \phi(v_1)$.

Let X be the set of self-intersection points of L and Y the set of Reeb chords between the ends of L , as in §1.1.3. Fix a polygon $\mathbf{u} = (\Pi, u, v^0)$ bounded by L . Note that the set of vertices V of Π is mapped by u to $X \cup \gamma(\partial I)$. Let $W(\mathbf{u}) = \{w^0, \dots, w^n\}$ be the set of connected components of $V \cup u^{-1}(\partial S)$, numbered in the counter-clockwise order on $\partial\Pi$, with $v^0 \in w^0$. For each $w \in W(\mathbf{u})$ either $u(w)$ is a double point of L or $u(w)$ is a Reeb chord between the ends of L , i.e. $u(w) \in X \cup Y$.

Let v be a vertex of Π . We say that v is *convex* (resp. *concave*) if the angle of the tangent cone of Π at v is in $(0, \pi)$ (resp. in $(\pi, 2\pi)$). Note that concave vertices of Π are necessarily mapped to X by u . Call the number of concave vertices of Π the *virtual dimension of $\mathbf{u}$* and denote it by $\text{vdim } \mathbf{u}$. To define the A_∞ -operations we will only need polygons of virtual dimension zero. To prove the A_∞ -relations we will also need polygons of virtual dimension one.

Fix $w \in W(\mathbf{u})$ such that $u(w)$ is a double point of L (i.e. $u(w) \in X$). Then the image under u of a small neighbourhood of w in Π covers one quadrant of $u(w)$ if w is convex and three quadrants of $u(w)$ if w is concave. If the majority of these quadrants is positive (negative), then we say that $\mathbf{u}$ has a *positive (negative) corner* at w . We obtain the partition

$$(1.8) \quad \{w \in W(\mathbf{u}) : u(w) \in X\} = W^+(\mathbf{u}) \sqcup W^-(\mathbf{u})$$

into positive and negative corners. Our definition of positive and negative corners follows the convention in Legendrian contact homology, the convention in Lagrangian Floer theory is, unfortunately, the opposite.

The Stokes theorem gives

$$(1.9) \quad \sum_{w \in W^+(\mathbf{u})} A(u(w)) - \sum_{\substack{w \in W(\mathbf{u}) : \\ u(w) \in Y}} A(u(w)) - \sum_{w \in W^-(\mathbf{u})} A(u(w)) = \int_{\Pi} u^* d\lambda > 0.$$

This suggests that the Reeb chords on the boundary of an immersed polygon can be thought of as negative corners. In fact, certain Reeb chords not on

the boundary of an immersed polygon can be thought of as its positive corners. In the definition below we make this idea precise by grouping immersed polygons bounded by L based on where their negative and positive ‘corners’ are.

Definition 1.5. Fix $n \geq 0$, and $z^0, z^1, \dots, z^n \in Z = X \cup Y^{>0}$. We define the set $\Delta(L; z^0; z^n, \dots, z^1)$ of polygons bounded by L with a ‘positive corner’ at z^0 and ‘negative corners’ at $z^1, \dots, z^n$.

- (1) If z^0 is a double point of L (i.e. $z^0 \in X$), let $\Delta(L; z^0; z^n, \dots, z^1)$ be the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^n\}$, $W^+(\mathbf{u}) = w^0$, and $u(w^i) = z^i$ for $i = 0, \dots, n$.
- (2) If z^0 is a Reeb chord between the ends of L (i.e. $z^0 \in Y$), let $\Delta(L; z^0; z^n, \dots, z^1)$ be the union of the following:
 - (a) if $z^1 \in Y^{>0}$ and there exists (necessarily unique) $y \in Y$ such that the concatenation yz^0 is equal to z^1 , then the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^{n-1}\}$, $W^+(\mathbf{u}) = \emptyset$, $u(w^0) = y$, and $u(w^i) = z^{i+1}$ for $i = 1, \dots, n-1$;
 - (b) if $z^n \in Y^{>0}$ and there exists (necessarily unique) $y \in Y$ such that the concatenation z^0y is equal to z^n , then the set of equivalence classes of polygons $\mathbf{u} = (\Pi, u, v^0)$ bounded by L such that $W(\mathbf{u}) = \{w^0, w^1, \dots, w^{n-1}\}$, $W^+(\mathbf{u}) = \emptyset$, $u(w^i) = z^{i+1}$ for $i = 0, \dots, n-2$, and $u(w^{n-1}) = y$;
 - (c) if $n = 2$, $z^1, z^2 \in Y^{>0}$, and $z^0 = z^2z^1$, then a singleton $\{*\}$. Set $\text{vdim}(*) = 0$.

For $k \geq 0$ denote by $\Delta^k(L; z^0; z^n, \dots, z^1)$ the subset of $\Delta(L; z^0; z^n, \dots, z^1)$ consisting of the elements of virtual dimension k .

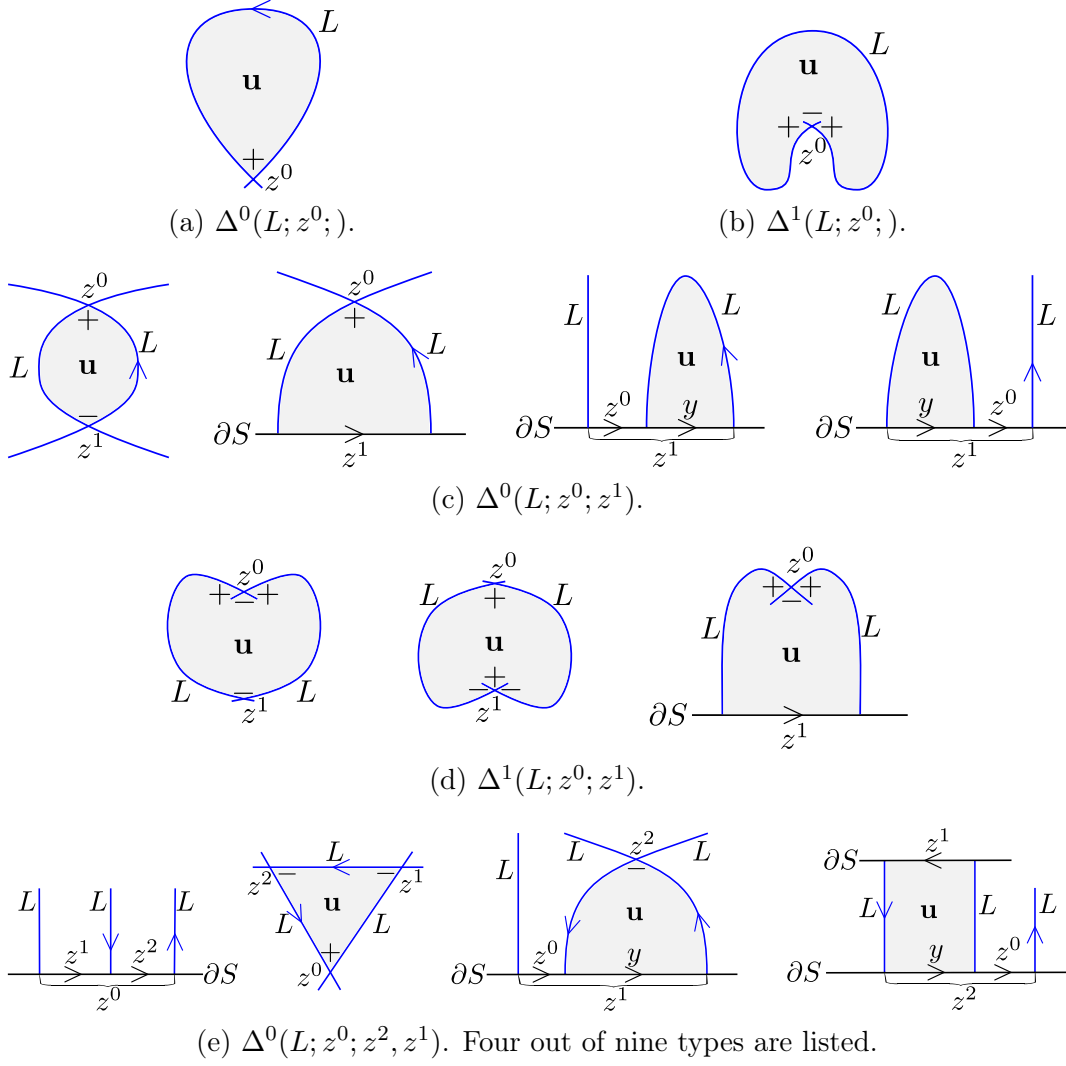
See Fig. 6 for examples of polygons from Definition 1.5. The pictures of polygons bounded by L that we draw are of the pull-back of a neighbourhood of the polygon $\mathbf{u}$ and Reeb chords involved from S to $\mathbb{R}^2$ via an extension of u .

Additivity of action with respect to concatenation of Reeb chords and (1.9) imply that

$$(1.10) \quad 0 < \sum_{i=1}^n A(z^i) \leq A(z^0) \text{ whenever } \Delta(L; z^0; z^n, \dots, z^1) \neq \emptyset$$

and hence that

$$\text{the set } \bigcup_{\substack{n \geq 0, \\ z^0, z^1, \dots, z^n \in Z}} \Delta(L; z^0; z^n, \dots, z^1) \text{ is finite,}$$

FIGURE 6. Examples of polygons in $\Delta(L; z^0; z^n, \dots, z^1)$.

see Lemma 2.16 for details. If we were to consider the moduli spaces of holomorphic discs, then the finite set $\Delta^k(L; z^0; z^n, \dots, z^1)$ would match the set of connected components of the corresponding moduli space.

For a polygon $\mathbf{u} = (\Pi, u, v^0) \in \Delta(L; z^0; z^n, \dots, z^1)$ the vertex v^0 is uniquely determined by the rest of the data, so we suppress it from the notation and write $\mathbf{u} = (\Pi, u)$.

Example 1.6. We continue Example 1.3. For the Lagrangian L_0 in Fig. 5 only the following $\Delta^0(L_0; \dots)$ are non-empty:

$$\begin{aligned} &\Delta^0(L_0; x^1;), \Delta^0(L_0; x^2; x^3), \\ &\Delta^0(L_0; x^1; y^1), \Delta^0(L_0; x^2; y^2 y^1), \Delta^0(L_0; x^3; y^2), \Delta^0(L_0; x^3; y^4 y^2 y^1), \\ &\Delta^0(L_0; y^2 y^1; y^4 y^2 y^1), \Delta^0(L_0; y^2 y^1; y^2, y^1), \Delta^0(L_0; x^2; y^2, x^1). \end{aligned}$$

For the Lagrangian L_1 in Fig. 5 only the following $\Delta^0(L_1; \dots)$ are non-empty:

$$\Delta^0(L_1; x^1;), \Delta^0(L_1; x^1; y^1), \Delta^0(L_1; x^1; \bar{x}^3, x^2).$$

These non-empty sets are all singletons.

This example is continued in Example 1.8.

1.1.5. Grading. We describe the last piece of geometric data, which is needed to make obstruction algebras and Fukaya categories $\mathbb{Z}$ -graded.

Definition 1.7. A *grading* on a Liouville marked surface $(S; \Lambda, D; \lambda)$ is a *line field* η on S , i.e. an element of $C^\infty(S, \mathbb{P}TS)$.

A *grading* on a Lagrangian $L = (I, \gamma, P)$ in a graded Liouville marked surface is a homotopy class of paths from $\gamma^*\eta$ to the tangent line $\dot{\gamma}$, i.e. an element $\text{gr} \in \Pi_1 C^\infty(I, \gamma^* \mathbb{P}TS)(\gamma^*\eta, \dot{\gamma})$.

Let $L = (I, \gamma, P; \text{gr})$ be a graded Lagrangian in a graded Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda; \eta)$. We define the degree of double points of L and Reeb chords between the ends of L .

The *degree* of a double point $x \in X_L$ is an integer $|x|$ determined as follows. First, obtain an element in $\pi_1(\mathbb{P}T_x \mathcal{S}, \eta(x))$ as a concatenation of $\text{gr}(x_-)$ (which rotates $\eta(x)$ to $\dot{\gamma}(x_-)$), followed by the smallest counter-clockwise rotation from $\dot{\gamma}(x_-)$ to $\dot{\gamma}(x_+)$, and then followed by the inverse of $\text{gr}(x_+)$. Now identify $\pi_1(\mathbb{P}T_x \mathcal{S}, \eta(x))$ with $\mathbb{Z}$ by sending the counter-clockwise rotation by π to 1 to obtain the degree of x .

The *degree* of a Reeb chord $y \in Y_L$ is an integer $|y|$ determined as follows. First, obtain a path from $\eta(\gamma(y_-))$ to $\dot{y}(l_y)$ in $\mathbb{P}(T_{\gamma(y_-)} S)$ as a concatenation of $\text{gr}(y_-)$ and the smallest counter-clockwise rotation from $\dot{\gamma}(y_-)$ to $\dot{y}(l_y)$. Secondly, transport it along $[0, l_y]$ to a path from $\eta(\gamma(y_+))$ to $\dot{y}(0)$ in $\mathbb{P}(T_{\gamma(y_+)} S)$. Thirdly, compose it with the smallest clockwise rotation from $\dot{y}(0)$ to $\dot{\gamma}(y_+)$ and then with the inverse of $\text{gr}(y_+)$ to obtain an element in $\pi_1(\mathbb{P}T_{\gamma(y_+)} S, \eta(\gamma(y_+)))$. Finally, identify $\pi_1(\mathbb{P}T_{\gamma(y_+)} S, \eta(\gamma(y_+)))$ with $\mathbb{Z}$ by sending the counter-clockwise rotation by π to 1 to obtain the degree of y . Note that the degree is additive with respect to concatenation of Reeb chords.

The definitions of degree for self-intersection points of L and Reeb chords between the ends of L are compatible in the following sense. Allow for a moment $\gamma|_{\partial I}$ not to be an embedding and let $z_+, z_- \in \partial I$ be such that $\gamma(z_+) = \gamma(z_-) = z \in \partial S$ and $P(z_+) > P(z_-)$. The two definitions of degree agree if and only if the composition of the smallest counter-clockwise rotation from $\dot{\gamma}_{l_0}(z_-^0)$ to $T_z \partial S$ with the smallest clockwise rotation from $T_z \partial S$ to $\dot{\gamma}_{l_0}(z_+^0)$ is homotopic to the smallest counter-clockwise rotation from $\dot{\gamma}_{l_0}(z_-^0)$ to $\dot{\gamma}_{l_0}(z_+^0)$. This is equivalent to the unique quadrant of z contained in S being negative.

A simple computation shows

$$(1.11) \quad |z^0| = 2 - n + k + |z^1| + \dots + |z^n| \text{ whenever } \Delta^k(L; z^0; z^n, \dots, z^1) \neq \emptyset,$$

see Lemma 2.16 for details.

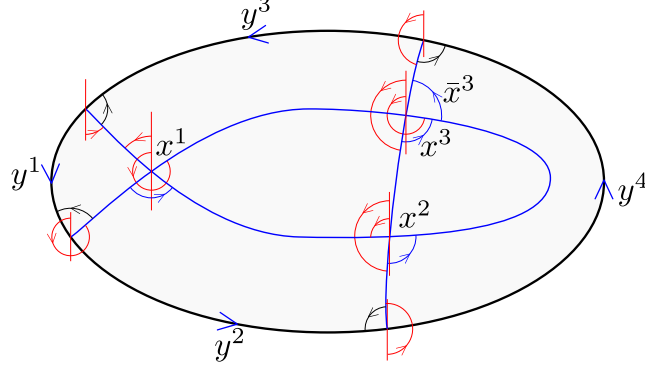


FIGURE 7. Graded Lagrangian in a graded surfaces.

The vertical **red** lines are the line field η , the **red** arcs are the paths from η to the tangent line, the **blue** arcs are the smallest counter-clockwise rotations of the understrand to the overstrand, the black arcs are the smallest counter-clockwise rotations from the curve to the boundary.

Example 1.8. We continue Example 1.6.

We fix a line field on the disc to be the vertical line at every point and pick the same grading for L_0 and L_1 , as indicated by the red arcs in Fig. 7. The only difference between L_0 and L_1 that affects the degree is that the understrand and the overstrand for x^3 and $\bar{x}^3$ are swapped, the corresponding smallest counter-clockwise rotations are marked with x^3 for L_0 and $\bar{x}^3$ for L_1 .

The degrees of the elements of Z_{L_0} are

$$|x^1| = 2, \quad |x^2| = 1, \quad |x^3| = 0, \quad |y^1| = 1, \quad |y^2| = -1, \quad |y^2 y^1| = 0, \quad |y^4 y^2 y^1| = -1.$$

The degrees of the elements of Z_{L_1} are

$$|x^1| = 2, \quad |x^2| = 1, \quad |\bar{x}^3| = 1, \quad |y^1| = 1.$$

This example is continued in Example 1.11.

1.2. Lagrangian pre-branes and branes

We add algebraic data to the geometric setup described in §1.1.

Definition 1.9. A *weighted surface* is a graded Liouville marked surface $(S; \Lambda, D; \lambda; \eta)$ (see Definitions 1.1 and 1.7) together with a *weight function* $w: D \rightarrow R$ where R is a unital commutative ring.

Throughout this section we work in a fixed weighted surface $(\mathcal{S} = (S; \lambda, \Lambda, D; \eta), w: D \rightarrow R)$.

In §1.2.1 we add local systems of R -modules to Lagrangians to obtain Lagrangian pre-branes. In §1.2.2 we recall the notion of curved A_∞ -algebras. In §1.2.3 we associate to a Lagrangian pre-brane a curved A_∞ -algebra, called its obstruction algebra. In §1.2.4 we recall the notion of bounding cochains in curved A_∞ -algebras and define a Lagrangian brane as a Lagrangian pre-brane with a bounding cochain in its obstruction algebra. Lagrangian branes will be the objects of the Fukaya category.

1.2.1. Lagrangian pre-branes. Recall that a *local system of R -modules* on a topological space I is a functor from the fundamental groupoid $\Pi_1(I)$ to the category of R -modules. The *fibre* E_t of a local system E at $t \in I = \text{Ob } \Pi_1(I)$ is the R -module $E(t)$. The *transition map* along a path $a: [0, 1] \rightarrow I$ is the isomorphism $E([a]): E_{a(0)} \rightarrow E_{a(1)}$ where $[a]$ is the homotopy-relative-endpoints class of a .

Definition 1.10. A *Lagrangian pre-brane* in the weighted surface $(\mathcal{S}, w)$ is the data of

- a graded Lagrangian L in $\mathcal{S}$ (see Definitions 1.2 and 1.7);
- a local system E of R -modules on the domain of L .

Note that our Lagrangian pre-branes are neither oriented nor Pin. The reason for the former is that there is no natural orientation if the line field η is not induced by a nowhere vanishing vector field on S . The reason for the latter is that the signs coming from the non-triviality of the Pin structure are incorporated into the transition maps of the local system.

Fix a Lagrangian pre-brane $\mathcal{L} = (L = (I, \gamma, P, \text{gr}), E)$ in $(\mathcal{S}, w)$. The *R -orientation line* of a component C of I is the free R -module of rank 1

$$(1.12) \quad |C|_R = H_1(C, \partial C; R).$$

For $t \in I$ denote by C_t the component of I with $t \in C_t$. Note that a non-zero vector in $T_{\gamma(t)}S$ tangent to γ gives an orientation of C_t , i.e. a generator of $H_1(C_t, \partial C_t; \mathbb{Z})$, and that in turn gives a generator of $|C_t|_R$ via the canonical isomorphism $H_1(C_t, \partial C_t; R) \cong H_1(C_t, \partial C_t; \mathbb{Z}) \otimes_{\mathbb{Z}} R$. As there is no preferred orientation on I , the orientation lines are a natural and convenient way to keep track of the signs in A_∞ -operations.

Let $\mathbf{u} = (\Pi, u)$ be a polygon bounded by L . As u preserves orientation, for any $q \in \text{Int } S \setminus u(\partial \Pi)$ the degree $\deg(u, q)$ of u at q is equal to $|u^{-1}(q)|$ hence non-negative. Therefore, we can define the *weight* of $\mathbf{u}$,

$$(1.13) \quad w(\mathbf{u}) = \prod_{q \in D} w(q)^{\deg(u, q)} \in R.$$

If $\mathbf{u}$ is the unique element of $\Delta(L; z^0, z^2, z^1)$ from the case (c) of Definition 1.5 (i.e. z^0, z^1, z^2 are Reeb chords and $z^0 = z^2 z^1$), then we set $w(\mathbf{u}) = 1$.

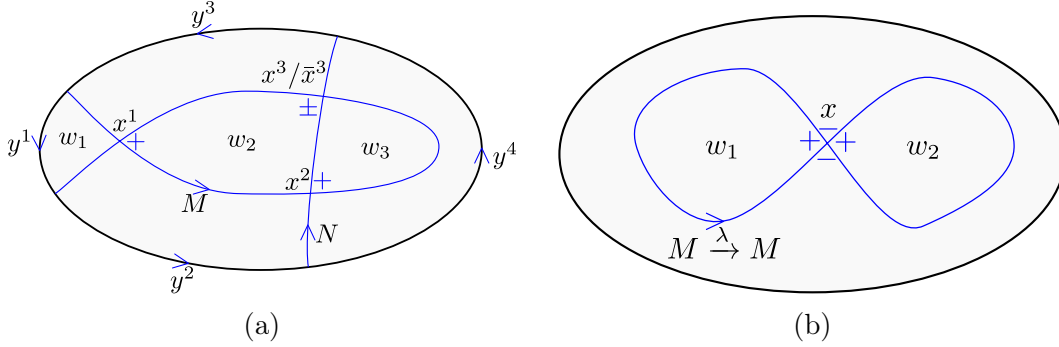


FIGURE 8. Lagrangian pre-branes in weighted surfaces.
The blue arrows on blue curves show the chosen orientations.

Example 1.11. We continue Example 1.8.

Suppose that $|D| = 3$ and the points of D are weighted by $w_1, w_2, w_3 \in R$ as in Fig. 8(a). Since the domain of L_0 (or L_1) is given by disjoint union of two segments, any local system on it can be trivialised, so we can assume that it is given by R -modules M and N as indicated. We obtain the Lagrangian pre-branes $\mathcal{L}_0$ and $\mathcal{L}_1$. To do the computations it is also convenient to choose an orientation of the domain of the Lagrangian, this gives an isomorphism of R -modules $R \rightarrow |C|_R$ for each component C of the domain. The orientations we choose are indicated in the picture.

This example is continued in Example 1.17.

Example 1.12. Let S be a disk, $|D| = 2$, $\Lambda = \emptyset$, η the vertical line field and $w: D \rightarrow R$ be given by $w_1, w_2 \in R$ as in Fig. 8(b). Consider the Lagrangian L depicted by the blue curve, its Legendrian lift is a Legendrian unknot (the Reeb signs at the unique double point are necessarily as depicted since the other option contradicts the action inequality (1.10)). We have $Z_L = \{x\}$. This Lagrangian can be graded and $|x| = 2$ for any grading. The domain of L is a circle hence any local system on it can be trivialised on the complement of one point t_0 to be equal to an R -module M and when traversing t_0 in the direction of the orientation the transition map is given by an automorphism λ of M . We always put the point t_0 where the arrow indicating orientation is. We denote by $\mathcal{L}_\lambda$ the Lagrangian pre-brane supported on L with the transition map λ .

This example is continued in Example 1.18

1.2.2. Curved A_∞ -algebras. Our primary reference for curved A_∞ -algebras is Fukaya–Oh–Ohta–Ono [23, §§3.2, 3.6, 4.2, 4.3]. The difference with [23] is that there the curved A_∞ -algebras are filtered and supported on a

free graded module over the Novikov ring, while our curved A_∞ -algebras are structured and supported on an arbitrary graded module over an arbitrary commutative ring R .

A ($\mathbb{Z}$ -graded) curved A_∞ -algebra (over R) is a $\mathbb{Z}$ -graded R -module

$$\mathcal{A} = \bigoplus_{d \in \mathbb{Z}} \mathcal{A}^d$$

together with a collection of linear maps $\mu^n: \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ of degree $2 - n$ for each $n \geq 0$ such that the A_n -relation

$$(1.14) \quad \sum_{n \geq j \geq i \geq 0} (-1)^\dagger \mu^{n-j+1+i} (a^n, \dots, a^{j+1}, \mu^{j-i}(a^j, \dots, a^{i+1}), a^i, \dots, a^1) = 0$$

where $\dagger = |a^i| + \dots + |a^1| - i$ holds for any $n \geq 0$ and homogeneous elements $a^1, \dots, a^n \in \mathcal{A}$.

$\mu^0()$ in (1.14) is interpreted as $\mu^0(1_R) \in \mathcal{A}^2$. The A_0 -relation (1.14) reads $\mu^1(\mu^0()) = 0$ and the non-vanishing of μ^0 is the obstruction to μ^1 being a differential as $\mu^1(\mu^1(a)) = (-1)^{|a|} \mu^2(\mu^0(), a) - \mu^2(a, \mu^0())$ by the A_1 -relation.

An A_∞ -algebra is a curved A_∞ -algebra with $\mu^0 = 0$. We sometimes use the term *uncurved A_∞ -algebra* for emphasis. In an uncurved A_∞ -algebra the A_0 -relation is vacuous and the first three non-trivial relations are

$$(A_1) \quad \mu^1(\mu^1(a_1)) = 0;$$

$$(A_2) \quad \mu^1(\mu^2(a_2, a_1)) + \mu^2(a_2, \mu^1(a_1)) + (-1)^{|a_1|-1} \mu^2(\mu^1(a_2), a_1) = 0;$$

$$(A_3) \quad \begin{aligned} & \mu^2(a_3, \mu^2(a_2, a_1)) + (-1)^{|a_1|-1} \mu^2(\mu^2(a_3, a_2), a_1) + \mu^1(\mu^3(a_3, a_2, a_1)) \\ & + \mu^3(a_3, a_2, \mu^1(a_1)) + (-1)^{|a_1|-1} \mu^3(a_3, \mu^1(a_2), a_1) + \\ & (-1)^{|a_1|+|a_2|-2} \mu^3(\mu^1(a_3), a_2, a_1) = 0. \end{aligned}$$

These relations imply that $a \cdot b = (-1)^{|b|} \mu^2(a, b)$ satisfies the graded Leibniz rule with respect to the differential $da = (-1)^{|a|} \mu^1(a)$ and descends to an associative product on the cohomology of $(\mathcal{A}, d)$.

In curved A_∞ -algebras without any additional structure many natural notions (such as bounding cochains, which we define in §1.2.4) do not make sense since they involve infinite sums. The additional structure we use is the following.

Definition 1.13. A *structured module* is a graded R -module $\mathcal{A}$ together with the data of

- a finite set Z and an R -module $\mathcal{A}(z)$ for each $z \in Z$;
- a *degree* function $|\cdot|: Z \rightarrow \mathbb{Z}$;

- a decomposition $\mathcal{A} = \bigoplus_{z \in Z} \mathcal{A}(z)[-|z|]$.

The data of a structured module $\mathcal{A}$ and a curved A_∞ -algebra $(\mathcal{A}, \mu)$ is called a *structured A_∞ -algebra* if there exists an *action* function $A: Z \rightarrow \mathbb{R}_{>0}$ such that if the entry

$$\mu^{z^0; z^n, \dots, z^1}: \mathcal{A}(z^n) \otimes \dots \otimes \mathcal{A}(z^1) \rightarrow \mathcal{A}(z^0)$$

of μ^n is non-zero, then $A(z^0) \geq A(z^n) + \dots + A(z^1)$ and the inequality is strict except possibly for $n = 2$.

Note that the action function is not part of the data in Definition 1.13, we explain the reason for that in §1.5.4. Observe that in a structured A_∞ -algebra $\mu^n = 0$ for $n \gg 0$ and moreover any nested expression involving μ with sufficiently many occurrences of μ or sufficiently many inputs vanishes.

Recall that a curved A_∞ -algebra $(\mathcal{A}, \mu)$ is called *nilpotent* if it possesses a filtration $\mathcal{A} = F^1 \mathcal{A} \supset F^2 \mathcal{A} \supset \dots \supset F^N \mathcal{A} = 0$ such that

$$\mu^n(F^{i_1} \mathcal{A} \otimes \dots \otimes F^{i_n} \mathcal{A}) \subset F^{i_1 + \dots + i_n} \mathcal{A}.$$

The following examples shows that a nilpotent A_∞ -algebra supported on a structured module is not necessarily structured.

Example 1.14. Let $R = \mathbb{Z}/4\mathbb{Z}$, $Z = \{a, b\}$, $\mathcal{A}(a) = \mathcal{A}(b) = R$, $|b| = |a| + 1 = 0$ and the only non-zero entries of μ be

$$\mu^{b;a} = 2 \text{ and } \mu^{a;b,b,b} = 2.$$

The A_∞ -relations are satisfied since they are quadratic and $2^2 = 0$. $\mathcal{A}$ is nilpotent as witnessed by the filtration

$$\mathcal{A}(a)[1] \oplus \mathcal{A}(b) \supset 2\mathcal{A}(a)[1] \supset 2\mathcal{A}(a)[1] \supset 0.$$

However, $\mathcal{A}$ is not structured since we would have $A(b) > A(a) > 3A(b) > 0$ if it was.

1.2.3. Obstruction algebra. Let $\mathcal{L} = (L = (I, \gamma, P, \text{gr}), E)$ be a Lagrangian pre-brane in the weighted surface $(\mathcal{S}, w)$. We define the *obstruction algebra* $\mathcal{A}_{\mathcal{L}}$, which is a $\mathbb{Z}$ -graded structured A_∞ -algebra over R .

Let Z be the union of the set X of double points of L and the set $Y^{>0}$ of Reeb chords of positive action between the ends of L , as in §1.1.3. Recall that Z is a finite set. We defined the action function $A: Z \rightarrow \mathbb{R}_{>0}$ in §1.1.3 and the degree function $|\cdot|: Z \rightarrow \mathbb{Z}$ in §1.1.5. As structured R -module we define

$$(1.15) \quad \mathcal{A}_{\mathcal{L}} = \bigoplus_{z \in Z} \mathcal{A}_{\mathcal{L}}(z)[-|z|]$$

where

$$(1.16) \quad \mathcal{A}_{\mathcal{L}}(z) = \text{Hom}_R(E_{z_+}, E_{z_-}) \otimes (|C_{z_-}|_R)^{\otimes |z|}.$$

The structure maps $\mu^n: \mathcal{A}_{\mathcal{L}}^{\otimes n} \rightarrow \mathcal{A}_{\mathcal{L}}[2-n]$ are defined entry-wise. For $n \geq 0$ and $z^0, \dots, z^n \in Z_L$ the entry

$$(1.17) \quad \mu^{z^0; z^n, \dots, z^1}: \mathcal{A}_{\mathcal{L}}(z^n) \otimes \dots \otimes \mathcal{A}_{\mathcal{L}}(z^1) \rightarrow \mathcal{A}_{\mathcal{L}}(z^0)$$

is given by

$$\sum_{\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)} \mu^{\mathbf{u}}$$

where for $\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)$ the map $\mu^{\mathbf{u}}$ is defined as explained below.

Note that $\Delta^0(L; z^0; z^n, \dots, z^1)$ is non-empty only if $z_+^0 \in C_0$, $z_-^0 \in C_n$ and $z_+^i \in C_{i-1}$, $z_-^i \in C_i$ for $i = 1, \dots, n$ for some components $C_0, \dots, C_n$ of I . Since $|C_i|_R$ is a free R -module of rank 1 for any $i = 1, \dots, n$, the map $\mu^{\mathbf{u}}$ is determined by the values of

$$\mu^{\mathbf{u}}(\varphi^n \otimes r_n^{\otimes |z^n|} \otimes \dots \otimes \varphi^1 \otimes r_1^{\otimes |z^1|})$$

for all choices of $\varphi^i \in \text{Hom}_R(E_{z_+^i}, E_{z_-^i})$, $i = 1, \dots, n$ and a fixed choice of orientation classes $r_i \in |C_i|_R$ for $i = 1, \dots, n$. We describe such a choice that extends the one from Seidel [46, Figure 13.1].

Construction 1.15.

- Suppose $\mathbf{u}$ is the unique element of $\Delta(L; z^0; z^2, z^1)$ from the case (c) of Definition 1.5, i.e. $z^0, z^1, z^2 \in Y^{>0}$ and $z^0 = z^2 z^1$. Let $r_0 = r_2 \in |C_2|_R$ be the orientation induced by a tangent vector to $\gamma(C_2)$ at $\gamma(z_-^0) \in \partial S$ pointing inside S . Let $r_1 \in |C_1|_R$ be the orientation induced by a tangent vector to $\gamma(C_1)$ at $\gamma(z_-^1) \in \partial S$ pointing outside S .
- Suppose $\mathbf{u} = (\Pi, u)$ is a polygon bounded by L . For $i = 1, \dots, n$ (resp. $i = 0$) let

$$r_i \in |C_i|_R \quad (\text{resp. } r_0 \in |C_n|_R)$$

be the orientation of C_i (resp. C_n) induced by the orientation of $\partial\Pi$ if there is a vertex of $\partial\Pi$ corresponding to z_-^i (resp. z_-^0), or if this is not the case, i.e. we need to determine the orientation of C_n in the case (b) of Definition 1.5, then the orientation of C_n induced by a tangent vector to $\gamma(C_n)$ at $\gamma(z_-^i) \in \partial S$ pointing inside S .

See the blue arrows in Fig. 6 for examples of orientations defined above.

For $\varphi^i \in \text{Hom}_R(E_{z_+^i}, E_{z_-^i})$, $i = 1, \dots, n$ set

$$(1.18) \quad \mu^{\mathbf{u}}(\varphi^n \otimes r_n^{\otimes |z^n|} \otimes \dots \otimes \varphi^1 \otimes r_1^{\otimes |z^1|}) = w(\mathbf{u}) \cdot \varphi^0 \otimes r_0^{\otimes |z^0|}$$

where $\varphi^0 \in \text{Hom}_R(E_{z_+^0}, E_{z_-^0})$ is the composition

$$(1.19) \quad E_{z_+^0} \rightarrow E_{z_+^1} \xrightarrow{\varphi^1} E_{z_-^1} \rightarrow \dots \rightarrow E_{z_+^n} \xrightarrow{\varphi^n} E_{z_-^n} \rightarrow E_{z_-^0}$$

such that the unnamed maps are the transition maps of E along the (pullback to I of the) boundary of $\mathbf{u}$ (some of these maps are the identity for the case (2) of Definition 1.5).

We prove that $\mathcal{A}_{\mathcal{L}}$ is indeed a structured A_∞ -algebra in Proposition 4.2.

Remark 1.16. Applying the component-wise R -linear dual to the bar construction on the obstruction algebra $(\mathcal{A}_{\mathcal{L}}, \mu)$ we get a differential graded algebra structure on

$$B\mathcal{A}_{\mathcal{L}}^\vee = \bigoplus_{d=0}^{\infty} (\mathcal{A}_{\mathcal{L}}^\vee[-1])^{\otimes d}.$$

More concretely, the algebra structure comes from the tensor product and the differential is given by

$$\partial(az^0) = \sum_{\mathbf{u} \in \Delta^0(L; z^0, z^n, \dots, z^1)} \pm(\mu^{\mathbf{u}})^\vee(a) \cdot z^1 \dots z^n \text{ for } z^0, \dots, z^n \in Z_L.$$

The A_∞ -relations translate to the graded Leibniz rule and hence the formula above determines a differential on $B\mathcal{A}_{\mathcal{L}}^\vee$.

What we get is a version of the Chekanov–Eliashberg algebra. Namely, if S is a disc, $R = \mathbb{F}_2$, E is the trivial local system $\mathbb{F}_2$, then

- when $D = \Lambda = \emptyset$ and the domain of L is closed, we obtain the DGA of a Legendrian link;
- when $D = \emptyset$, $|\Lambda| = 2$, the domain of L is a segment, and the two ends of L lie in different components of $\partial S \setminus \Lambda$, we obtain the DGA of a long Legendrian knot

both defined by Chekanov [15]. More generally, modulo sign conventions, when $D = \Lambda = \emptyset$, R is any commutative ring, and the domain of L is closed, we obtain the DGA defined by Ekholm–Etnyre–Sullivan [18, 19].

Example 1.17. We continue Example 1.11. Denote the R -modules of linear maps by $G_B^A = \text{Hom}_R(A, B)$ for $A, B \in \{M, N\}$.

We compute the obstruction algebra $\mathcal{A}_{\mathcal{L}_0}$. Using the description of Z_{L_0} from Example 1.3 and trivialisations $R \rightarrow |C|_R$ for components of the domain of L_0 from Example 1.11 we identify

$$\mathcal{A}_{\mathcal{L}_0} = G_M^M x^1 \oplus G_N^M x^2 \oplus G_N^M x^3 \oplus G_M^M y^1 \oplus G_N^M y^2 \oplus G_N^M y^2 y^1 \oplus G_N^M y^4 y^2 y^1$$

where the elements of Z_{L_0} denote formal variables of degrees given in Example 1.8. The sets of polygons from Example 1.6 give

$$\begin{aligned}\mu^0() &= w_2 w_3 1_M \cdot x^1, \quad \mu^1(ax^3) = -w_3 a \cdot x^2, \quad \mu^1(ay^1) = -w_1 a \cdot x^1, \\ \mu^1(ay^2) &= w_2 a \cdot x^3, \quad \mu^1(ay^2 y^1) = w_1 a \cdot x^2, \quad \mu^1(ay^4 y^2 y^1) = -w_1 a \cdot x^3 - w_3 a \cdot y^2 y^1, \\ \mu^2(by^2, ay^1) &= b \circ a \cdot y^2 y^1, \quad \mu^2(by^2, ax^1) = b \circ a \cdot x^2.\end{aligned}$$

We compute the obstruction algebra $\mathcal{A}_{\mathcal{L}_1}$. As a graded R -module we have

$$\mathcal{A}_{\mathcal{L}_0} = G_M^M x^1 \oplus G_N^M x^2 \oplus G_M^N \bar{x}^3 \oplus G_M^M y^1.$$

The sets of polygons from Example 1.6 give

$$\mu^0() = w_2 w_3 1_M \cdot x^1, \quad \mu^1(ay^1) = -w_1 a \cdot x^1, \quad \mu^2(b\bar{x}^3, ax^2) = w_2(b \circ a) \cdot x^1.$$

This example is continued in Example 1.22

Example 1.18. We continue Example 1.12. Denote $G = \text{End}_R(M)$.

We compute the obstruction algebra $\mathcal{A}_{\mathcal{L}_\lambda}$. As a graded R -module we have $\mathcal{A}_{\mathcal{L}_\lambda} = Gx$. The two teardrops in Fig. 8(b) give the only non-trivial A_∞ -operation

$$\mu^0() = (w_1 \lambda + w_2 1_M) \cdot x.$$

This example is continued in Example 1.23.

1.2.4. Lagrangian branes. Before defining Lagrangian branes we recall what bounding cochains in curved A_∞ -algebras are.

Let $\mathcal{A}$ be a structured A_∞ -algebra. Given any elements $\delta_0, \dots, \delta_n \in \mathcal{A}^1$ we can define a map $\mu_{\delta_n, \dots, \delta_0}^n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ by

$$(1.20) \quad \mu_{\delta_n, \dots, \delta_0}^n(a^n, \dots, a^1) = \sum_{k_0, \dots, k_n \geq 0} \mu^{n+k_n+\dots+k_0}(\underbrace{\delta_n, \dots, \delta_n}_{k_n}, a^n, \dots, a^2, \underbrace{\delta_1, \dots, \delta_1}_{k_1}, a^1, \underbrace{\delta_0, \dots, \delta_0}_{k_0}).$$

The sum in (1.20) is finite since $\mathcal{A}$ is structured. These maps satisfy a version of the A_∞ -relation:

$$\sum_{n \geq j \geq i \geq 0} (-1)^\dagger \mu_{\delta_n, \dots, \delta_j, \delta_i, \dots, \delta_0}^{n-j+1+i} \left(a^n, \dots, a^{j+1}, \mu_{\delta_j, \dots, \delta_i}^{j-i}(a^j, \dots, a^{i+1}), a^i, \dots, a^1 \right) = 0.$$

Specialising to $\delta_0, \dots, \delta_n = \delta$ we get a nilpotent A_∞ -algebra structure

$$(1.21) \quad \mu_\delta = \{\mu_{\delta, \dots, \delta}^n\}_{n=0}^\infty \text{ on } \mathcal{A}.$$

A *bounding cochain* in $\mathcal{A}$ is an element $\delta \in \mathcal{A}^1$ such that $(\mathcal{A}, \mu_\delta)$ is uncurved, i.e. such that the *Maurer–Cartan equation*

$$(1.22) \quad \mu_\delta^0() = \sum_{n=0}^\infty \mu^n(\underbrace{\delta, \dots, \delta}_n) = 0$$

holds. Note that $(\mathcal{A}, \mu_\delta)$ is structured, with the same Z , action, and degree.

Definition 1.19. A *Lagrangian brane* in the weighted surface $(\mathcal{S}, w)$ is the data of

- a Lagrangian pre-brane $\mathcal{L}$ in $(\mathcal{S}, w)$;
- a bounding cochain δ in the obstruction algebra $\mathcal{A}_\mathcal{L}$.

Lagrangian branes will be the objects of the Fukaya category. For a Lagrangian brane $\mathbb{L} = (\mathcal{L}, \delta)$ denote by $(\mathcal{A}_\mathbb{L}, \mu_\mathbb{L})$ the uncurved A_∞ -algebra $(\mathcal{A}_\mathcal{L}, \mu_\delta)$.

Remark 1.20. Classically (see Akaho–Joyce [5]), given a Lagrangian pre-brane $\mathcal{L}$, one looks for bounding cochains in a much larger algebra, the curved A_∞ -endomorphism algebra of $\mathcal{L}$, which we will not define in our setting, but only explain what the underlying R -module is. It is the direct sum of the following:

- (1) for each circle component C of the domain I the cohomology space $H^*(C; E|_C)$;
- (2) for each self-intersection point x of $\mathcal{L}$ the spaces $\text{Hom}(E_{x_+}, E_{x_-}) \otimes (|C_{x_-}|_R)^{\otimes |x|} [-|x|]$ and $\text{Hom}(E_{x_-}, E_{x_+}) \otimes (|C_{x_+}|_R)^{\otimes (1-|x|)} [|x| - 1]$;
- (3) for each Reeb chord y between the ends of $\mathcal{L}$ (including the constant Reeb chords) the space $\text{Hom}(E_{y_+}, E_{y_-}) \otimes (|C_{y_-}|_R)^{\otimes |y|} [-|y|]$.

It is clear that the obstruction algebra is a submodule of the above, and it can be shown that it is a curved A_∞ -subalgebra, see Haiden [27, Proposition 3.1] for the proof in a related setting. The benefit of working with the obstruction algebra instead of the curved endomorphism algebra is not only that the former is smaller, but also that one does not need to perturb the Lagrangian to define the A_∞ -operations.

Remark 1.21. Recall the relation between obstruction algebra and Chekanov–Eliashberg algebra from Remark 1.16. Assume for concreteness that $R = \mathbb{F}_2$ and E is the trivial local system $\mathbb{F}_2$ and consider a Lagrangian pre-brane $\mathcal{L}$ with closed domain. Then $\mathcal{A}_\mathcal{L}^1$ can be identified with the degree 1 maps from $\mathbb{F}_2$ to $\mathcal{A}_\mathcal{L}$ and taking the dual of such map gives a degree 0 map from $\mathcal{BA}_\mathcal{L}^\vee$ to $\mathbb{F}_2$. We therefore get a map

$$\mathcal{A}_\mathcal{L}^1 \rightarrow \text{Hom}_{\mathbb{F}_2}(\mathcal{BA}_\mathcal{L}^\vee, \mathbb{F}_2), \quad \delta \mapsto \epsilon_\delta$$

that establishes a bijection between bounding cochains in $\mathcal{A}_\mathcal{L}$ and augmentations of $\mathcal{BA}_\mathcal{L}^\vee$, i.e. maps of dg-algebras $(\mathcal{BA}_\mathcal{L}^\vee, \partial) \rightarrow (\mathbb{F}_2, 0)$.

The linearised contact cohomology $LCH_{\epsilon_\delta}^*(\mathcal{L})$ is then identified with the cohomology of $(\mathcal{A}_\mathcal{L}, \mu_\delta)$, see Etnyre–Ng [20, §4] for details.

Example 1.22. We continue Example 1.17.

A bounding cochain in $\mathcal{A}_{\mathcal{L}_0}$ is given by $\delta = a \cdot x^2 + b \cdot y^1$ for $a \in G_N^M$ and $b \in G_M^M$ satisfying $w_2 w_3 1_M - w_1 b = 0$.

A bounding cochain in $\mathcal{A}_{\mathcal{L}_1}$ is given by $\delta = a \cdot x^2 + b \cdot \bar{x}^3 + c \cdot y^1$ for $a \in G_N^M$, $b \in G_M^N$, and $c \in G_M^M$ satisfying $w_2 w_3 1_M + w_2(b \circ a) - w_1 c = 0$.

For instance, if $R = \mathbb{C}[t]$, $w_1 = t^3$, $w_2 = t$, $w_3 = t$, and $M = R$, then $\mathcal{L}_0$ does not support a Lagrangian brane, while $(\mathcal{L}_1, x^2 - t\bar{x}^3)$ and $(\mathcal{L}_1, (1+t)x^2 - t\bar{x}^3 - y^1)$ are Lagrangian branes.

Example 1.23. We continue Example 1.18.

Since $\mathcal{A}_{\mathcal{L}_\lambda}^1 = 0$, the only possible bounding cochain in $\mathcal{A}_{\mathcal{L}_\lambda}$ is $\delta = 0$. It is indeed a bounding cochain if and only if $w_1 \lambda + w_2 1_M = 0$.

For instance, if $R = \mathbb{C}[t]$, $w_1 = w_2 = t$, $M = (\mathbb{C}[t]/(t^2))^{\oplus 2}$, and $\lambda = \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}$, then $(\mathcal{L}_\lambda, 0)$ is a Lagrangian brane.

1.3. Relative wrapped Fukaya category of a weighted surface

In this section we define the relative wrapped Fukaya category $\mathcal{W}_w(\mathcal{S})$ of a weighted surface $(\mathcal{S} = (S; \lambda, \Lambda, D; \eta), w: D \rightarrow R)$ as a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category.

In §1.3.1 we discuss transverse Lagrangians, i.e. the tuples of Lagrangians for which that A_∞ -operations are defined. In §1.3.2 we recall the definition of A_∞ -pre-categories. In §1.3.3 we define the relative wrapped Fukaya category.

1.3.1. Transverse Lagrangians. Let $L_i = (I_i, \gamma_i, P_i)$, $i = 0, \dots, n$ be Lagrangians in $\mathcal{S}$ such that

$$\gamma = \bigsqcup_{i=0}^n \gamma_i: I = \bigsqcup_{i=0}^n I_i \rightarrow S$$

has at worst transverse double points and $\gamma|_{\partial I}$ is an embedding. In this case we say that $L_0, \dots, L_n$ are *transverse*. Note that γ is an immersion satisfying conditions (1)–(4) from §1.1.2.

Given a vector $h = (h_0, \dots, h_n) \in \mathbb{R}^{n+1}$ define

$$P_h: I \rightarrow \mathbb{R}, P_h(t) = P_i(t) + h_i \text{ if } t \in I_i.$$

The data $L_h = L_h(L_0, \dots, L_n) = (I, \gamma, P_h)$ satisfies all conditions on being a Lagrangian in Definition 1.2 except that P_h can fail to satisfy conditions (6) and (7). Call the vector $h \in \mathbb{R}^{n+1}$ *admissible* for $L_0, \dots, L_n$ if L_h is a Lagrangian and let $\mathcal{H}(L_0, \dots, L_n)$ be the set of admissible shift vectors. For $h \in \mathcal{H}(L_0, \dots, L_n)$ we call the Lagrangian L_h the *composite Lagrangian* of $L_0, \dots, L_n$ with *shift vector* h .

For $i, j \in \{0, \dots, n\}, i \neq j$ let $Z(L_i, L_j)$ be the union of the set $X(L_i, L_j)$ of intersection points of γ_i and γ_j and the set $Y(L_i, L_j)$ of Reeb chords from an end of γ_i to an end of γ_j . Note that $Z(L_i, L_j)$ is infinite if L_i and L_j both have ends on a component C of ∂S with $C \cap \Lambda = \emptyset$. The set $Z(L_i, L_j)$ will give rise to the morphism space in the Fukaya category between Lagrangian branes supported on L_i and L_j .

For $z \in X(L_i, L_j)$ let $z_+ \in I_i$ and $z_- \in I_j$ be such that $\gamma_i(z_+) = z = \gamma_j(z_-)$, and for $z \in Y(L_i, L_j)$ let $z_+ \in I_i$ and $z_- \in I_j$ be such that z goes from $\gamma_i(z_+)$ to $\gamma_j(z_-)$.

Given a (not necessarily admissible) shift vector h and $z \in Z(L_i, L_j)$ for $i \neq j$, we define the action

$$(1.23) \quad A_h(z) = \begin{cases} P_h(z_+) - P_h(z_-), & \text{if } z \text{ is an intersection point;} \\ P_h(z_+) - \int_z \lambda - P_h(z_-), & \text{if } z \text{ is a Reeb chord.} \end{cases}$$

Note that h being admissible is equivalent to the non-vanishing of A_h for all z , but, unlike the action in (1.4), $A_h(z)$ can be negative for intersection points. If h is an admissible vector, then z can also be viewed as an element of Z_{L_h} for the composite brane L_h and hence we get a definition of $z_{\pm}$ as in §1.1.3 and we can pick the admissible vector h so that the two definition of $z_{\pm}$ agree for any $z \in Z(L_i, L_j), i < j$. However, if $Z(L_i, L_j)$ is infinite for some $i < j$, then it is not possible to pick an admissible vector h so that $A_h(z) = A_L(z) > 0$ for any $z \in Z(L_i, L_j)$, though this can be achieved for a finite subset of $Z(L_i, L_j)$, see Lemma 4.5 for the precise statement.

If we are given a grading gr_i for each L_i , then we get a grading gr on γ and hence can define the degree $|z|$ as in §1.1.5, but with the new definition of $z_{\pm}$.

Let $\mathcal{L}_i = (I_i, \gamma_i, P_i, \text{gr}_i, E_i), i = 0, \dots, n$ be Lagrangian pre-branes such that their underlying Lagrangians are transverse and $h \in \mathbb{R}^{n+1}$ an admissible shift vector, i.e. $L = L_h(L_0, \dots, L_n)$ is a Lagrangian. Then $\mathcal{L} = (L, \sqcup E_i) = \mathcal{L}_h(\mathcal{L}_0, \dots, \mathcal{L}_n)$ is a Lagrangian pre-brane, so the obstruction algebra $(\mathcal{A}_{\mathcal{L}}, \mu_{\mathcal{L}})$ is defined. For any $i = 0, \dots, n$ and $z \in Z_{L_i}$ we have $\mathcal{A}_{\mathcal{L}}(z) = \mathcal{A}_{\mathcal{L}_i}(z)$, and if $z^0, \dots, z^k \in Z_L$ are such that $z^j \in Z_{L_{i_j}}$ for each $j = 0, \dots, k$, then

$$\mu_{\mathcal{L}}^{z^0, z^k, \dots, z^1} = \begin{cases} \mu_i^{z^0, z^k, \dots, z^1}, & \text{if } i_0 = \dots = i_k = i; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\bigoplus_{i=0}^n (\mathcal{A}_{\mathcal{L}_i}, \mu_i)$ is an A_{∞} -subalgebra of $(\mathcal{A}_{\mathcal{L}}, \mu_{\mathcal{L}})$.

If $\delta_i \in \mathcal{A}_{\mathcal{L}_i}, i = 0, \dots, n$ are bounding cochains, i.e. $\mathbb{L}_i = (\mathcal{L}_i, \delta_i)$ are Lagrangian branes, then we obtain a bounding cochain $\delta = \delta_1 + \dots + \delta_n \in \mathcal{A}_{\mathcal{L}}$. Indeed, δ lies in the A_{∞} -subalgebra $\bigoplus_{i=0}^n (\mathcal{A}_{\mathcal{L}_i}, \mu_i)$ and hence $\mu_{\delta}^0() = \mu_{\delta_1}^0() +$

$\dots + \mu_{\delta_n}^0() = 0$. We call the resulting Lagrangian brane $\mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n) = (\mathcal{L}, \delta)$ the *composite* Lagrangian brane of $\mathbb{L}_0, \dots, \mathbb{L}_n$ with the shift vector h . The A_∞ -operations in the Fukaya category will be computed in A_∞ -algebras $(\mathcal{A}_{\mathbb{L}}, \mu_{\mathbb{L}})$ for composite Lagrangian branes $\mathbb{L}$.

1.3.2. A_∞ -pre-categories. We recall the definition of A_∞ -pre-category by Kontsevich–Soibelman [31, §4.3]. Let R be a unital commutative ring. A ($\mathbb{Z}$ -graded R -linear) A_∞ -pre-category $\mathcal{W}$ is the following data.

- A class of objects $\text{Ob } \mathcal{W}$.
- For each $n \geq 1$ a subclass $\text{Ob}_{tr}^n \mathcal{W}$ of $(\text{Ob } \mathcal{W})^n$, called the class of *transverse sequences*, such that $\text{Ob}_{tr}^1 \mathcal{W} = \text{Ob } \mathcal{A}$ and subsequences of transverse sequences are transverse.
- For each pair $(X_0, X_1) \in \text{Ob}_{tr}^2 \mathcal{W}$ a $\mathbb{Z}$ -graded R -module of morphisms $\mathcal{W}(X_0, X_1)$.
- For each $n \geq 1$ and a transverse sequence $(X_0, \dots, X_n)$ an R -linear map

$$\tilde{\mu}^n: \mathcal{W}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{W}(X_0, X_1) \rightarrow \mathcal{W}(X_0, X_n)$$

of degree $2 - n$ satisfying the A_∞ -relation (1.14) for all chains of morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$.

Note that, unlike A_∞ -algebras, all A_∞ -pre-categories are uncurved, i.e. $\tilde{\mu}^0 = 0$.

An A_∞ -category is an A_∞ -pre-category such that any finite sequence of objects is transverse. Given a transverse sequence $\mathbf{X} = (X_0, \dots, X_n)$ in an A_∞ -pre-category $\mathcal{W}$, we define the *directed* A_∞ -category $\mathcal{W}|_{\mathbf{X}}$ by setting $\text{Ob } \mathcal{W}|_{\mathbf{X}} = \{X_0, \dots, X_n\}$,

$$\mathcal{W}|_{\mathbf{X}}(X_i, X_j) = \begin{cases} \mathcal{W}(X_i, X_j), & \text{if } i < j; \\ 0, & \text{otherwise.} \end{cases}$$

and the A_∞ -operations given by that in $\mathcal{W}$.

Let $\mathcal{W}$ be an A_∞ -pre-category, I a set and $\mathbf{X}_I = \{X_i\}_{i \in I}$ a collection of objects in $\mathcal{W}$. We say that $\mathbf{X}_I$ is a *transverse set* (this is a slight abuse of language since objects of $\mathcal{W}$ can be repeated in $\mathbf{X}_I$) if for any finite sequence of pairwise distinct elements $i_0, \dots, i_n \in I$ the sequence $(X_{i_0}, \dots, X_{i_n})$ is transverse. Unless stated otherwise, our standing assumption is that transverse sequences in A_∞ -pre-categories *form* transverse sets, that is, any permutation of a transverse sequence is a transverse sequence.

1.3.3. Fukaya category. Let $(\mathcal{S}, w)$ be a weighted surface (see Definition 1.9). We define the *relative wrapped Fukaya category* $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$, which is a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category.

Objects. An object of $\mathcal{W}$ is a Lagrangian brane $\mathbb{L} = (I, \gamma, P, \text{gr}, E, \delta)$ in $(\mathcal{S}, \mathbf{w})$.

Transverse sequences. A sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n) \in (\text{Ob } \mathcal{W})^{n+1}$ is transverse if the underlying Lagrangians $L_0, \dots, L_n$ are transverse (see §1.3.1).

Morphisms. For a transverse pair of objects $(\mathbb{L}_0, \mathbb{L}_1)$ let $Z = Z(\mathbb{L}_0, \mathbb{L}_1)$ be the set of intersection points between γ_0 and γ_1 and Reeb chords from an end of γ_0 to an end of γ_1 , see §1.3.1. The morphism space is the graded R -module

$$(1.24) \quad \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1) = \bigoplus_{z \in Z} \mathcal{W}(z)[-|z|]$$

where

$$(1.25) \quad \mathcal{W}(z) = \text{Hom}_R((E_0)_{z_+}, (E_1)_{z_-}) \otimes (|C_{z_-}|_R)^{\otimes |z|}$$

A_∞ -operations. For $n \geq 1$, a transverse sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n)$, and intersection points or Reeb chords

$$z^0 \in Z(\mathbb{L}_0, \mathbb{L}_n), \quad z^i \in Z(\mathbb{L}_{i-1}, \mathbb{L}_i), \quad i = 1, \dots, n$$

the entry

$$(1.26) \quad \tilde{\mu}^{z^0; z^n, \dots, z^1} : \mathcal{W}(z^n) \otimes \dots \otimes \mathcal{W}(z^1) \rightarrow \mathcal{W}(z^0)$$

of the structure map $\tilde{\mu}^n$ is defined as follows. Pick an admissible shift vector $h \in \mathcal{H}(L_0, \dots, L_n)$ such that $A_h(z^0), \dots, A_h(z^n) > 0$ and define

$$\tilde{\mu}^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1}$$

where $\mathbb{L} = \mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n)$ is the composite Lagrangian brane of $\mathbb{L}_0, \dots, \mathbb{L}_n$ with the shift vector h .

See Proposition 4.4 for the proof that the choices required to define the structure maps can be made and do not affect the result and that we get a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category where transverse sequences form transverse sets.

Remark 1.24. One could allow η to be a line field only on $S \setminus D$, R to be a graded ring, and $\mathbf{w}: D \rightarrow R$ to be such that $\mathbf{w}(q)$ is homogeneous of degree $2 - \text{ind}_q \eta$ for any $q \in D$ where $\text{ind}_q \eta$ is the index of η around q ($\text{ind}_q \eta = 2$ if and only if η extends to q). All the constructions can then be performed in this graded setting and one would obtain a Fukaya category linear over the graded ring R .

The interesting case for homological mirror symmetry is when R is non-negatively graded and is finitely generated by R^1 over Noetherian R^0 . Serre's theorem on Proj then realises $\text{Coh Proj } R$ as the quotient of the category of finitely generated modules over R by torsion modules, and so the quotient

of the Fukaya category by the corresponding torsion objects becomes linear over $\text{Coh Proj } R$. The details on these relative wrapped Fukaya categories over projective varieties will appear elsewhere.

Remark 1.25. A related approach to defining Fukaya category of surfaces is outlined by Haiden [27]. The setup there is different in that R is a field, $D = \emptyset$, and the Lagrangians are only allowed to end in a fixed finite set of points on ∂S , so Reeb chords on the boundary do not enter the construction. The resulting category is called an infinitesimal Fukaya category of a surface.

The idea of using a version of Chekanov–Eliashberg algebra as an obstruction algebra also appears in Chantraine–Dimitroglou–Rizell–Ghiggini–Golovko [14].

Remark 1.26. If S is a disc, $D = \Lambda = \emptyset$, R is a field, and L is a graded Lagrangian whose domain is closed, then the positive augmentation A_∞ -category $\text{Aug}_+(L)$ of Ng–Rutherford–Shende–Sivek–Zaslow [38] is defined. Modulo sign conventions, the A_∞ -operation $\tilde{\mu}^n$ on a transverse sequence of Lagrangian branes $\mathbb{L}_0, \dots, \mathbb{L}_n$ such that each $\mathbb{L}_i$ is supported on a perturbation of L can be computed in the augmentation category $\text{Aug}_+(L)$.

Example 1.27. We do a variation on Example 1.22 that illustrates the computation of a morphism space in the Fukaya category.

We can view the curves in Fig. 8(a) as the underlying curves $\gamma_M: [0, 1] \rightarrow S$ and $\gamma_N: [0, 1] \rightarrow S$ of two Lagrangians L_M and L_N respectively. Consider Lagrangian branes $\mathbb{L}_M = (L_M, M, by^1)$ where $b \in G_M^M$ satisfies $w_2 w_3 1_M = w_1 b$ and $\mathbb{L}_N = (L_N, N, 0)$. Set $G = G_N^M = \text{Hom}_R(M, N)$. The morphism space from $\mathbb{L}_M$ to $\mathbb{L}_N$ in the Fukaya category is given by

$$\begin{aligned} \mathcal{W}(\mathbb{L}_M, \mathbb{L}_N) &= Gx^2 \oplus Gx^3 \oplus \\ &\bigoplus_{n \geq 0} (Gy^2 z_1^n \oplus Gy^2 y^1 z_0^n \oplus Gy^4 y^2 z_1^n \oplus Gy^4 y^2 y^1 z_0^n) \end{aligned}$$

where $z_0 = y^3 y^4 y^2 y^1$ (resp. $z_1 = y^1 y^3 y^4 y^2$) is the Reeb chord from $\gamma_M(0)$ (resp. $\gamma_M(1)$) to itself going around ∂S once. The degrees determined by Fig. 7 are

$$\begin{aligned} |x^2| &= 1, \quad |x^3| = 0, \quad |y^2 z_1^n| = -1 - 2n, \\ |y^2 y^1 z_0^n| &= -2n, \quad |y^4 y^2 z_1^n| = -2 - 2n, \quad |y^4 y^2 y^1 z_0^n| = -1 - 2n. \end{aligned}$$

To compute the A_∞ -operation $\tilde{\mu}^1: \mathcal{W}(\mathbb{L}_M, \mathbb{L}_N) \rightarrow \mathcal{W}(\mathbb{L}_M, \mathbb{L}_N)$ on Gy for a Reeb chord y from an end of L_M to an end of L_N we shift the Lagrangian L_N down (i.e. subtract $C \gg 0$ from the potential P_N) so that the action of y

becomes positive. This yields

$$\begin{aligned}
\tilde{\mu}^1(ax^2) &= 0, \quad \tilde{\mu}^1(ax^3) = -w_3a \cdot x^2, \\
\tilde{\mu}^1(ay^2) &= w_2a \cdot x^3 + ab \cdot y^2y^1 \text{ and } \tilde{\mu}^1(a \cdot y^2z_1^n) = ab \cdot y^2y^1z_0^n \text{ for } n \geq 1, \\
\tilde{\mu}^1(a \cdot y^2y^1) &= w_1a \cdot x^2 \text{ and } \tilde{\mu}^1(a \cdot y^2y^1z_0^n) = 0 \text{ for } n \geq 1, \\
\tilde{\mu}^1(a \cdot y^4y^2z_1^n) &= -w_3a \cdot y^2z_1^n + ab \cdot y^4y^2y^1z_0^n \text{ for } n \geq 0, \\
\mu^1(a \cdot y^4y^2y^1) &= -w_1a \cdot x^3 - w_3a \cdot y^2y^1 \text{ and} \\
\tilde{\mu}^1(a \cdot y^4y^2y^1z_0^n) &= -w_3a \cdot y^2y^1z_0^n \text{ for } n \geq 1.
\end{aligned}$$

1.4. Setup for Theorem 1

In this section we first give the necessary definitions for Theorem 1 in §§1.4.1 to 1.4.3, and then sketch the proof of Theorem 1 in §1.4.4. In §1.4.5 we do the computations for the two examples from the introduction.

1.4.1. Homotopy equivalences in A_∞ -pre-category. Let $\mathcal{W}$ be an A_∞ -pre-category, see §1.3.2. The structure maps $\tilde{\mu}^1$ make morphism spaces into cochain complexes by the A_1 -relation. The A_2, A_3 -relations imply that $a \circ b = (-1)^{|b|}\tilde{\mu}^2(a, b)$ satisfies the graded Leibniz rule with respect to the differential $da = (-1)^{|a|}\tilde{\mu}^1(a)$ and descends to an associative (partially defined) composition on cohomology. We will use the associated homological algebra language (closed, cohomologous, etc.) for morphisms in $\mathcal{W}$. Denote the degree k component of $\mathcal{W}(X_0, X_1)$ by $\mathcal{W}^k(X_0, X_1)$.

Recall our standing assumption that transverse sequences in $\mathcal{W}$ form transverse sets. Recall from [31, Definition 9] that a closed morphism $q^1 \in \mathcal{W}^0(X_0, X_1)$ is called a *quasi-isomorphism* if for any transverse set $\{T, X_0, X_1\}$ the maps

$$(\mathcal{W}(T, X_0), d) \xrightarrow{q^1 \circ -} (\mathcal{W}(T, X_1), d) \text{ and } (\mathcal{W}(X_1, T), d) \xrightarrow{- \circ q^1} (\mathcal{W}(X_0, T), d)$$

are quasi-isomorphisms. Since we are working with arbitrary modules over an arbitrary commutative ring, quasi-isomorphisms do not necessarily have quasi-inverses. We therefore introduce a more convenient notion of a homotopy equivalence in A_∞ -pre-category.

Definition 1.28. Let $\{X_0, X_1\}$ be a transverse set in $\mathcal{W}$ and $q^1 \in \mathcal{W}^0(X_0, X_1)$, $q^0 \in \mathcal{W}^0(X_1, X_0)$ two closed morphisms. We say that q^1 and q^0 are *homotopy inverse* if for any transverse set $\{T, X_0, X_1\}$ and $i = 0, 1$ the map $\tilde{\mu}^2(q^i, \tilde{\mu}^2(q^{1-i}, -)) = q^i \circ (q^{1-i} \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(T, X_i)}$ and the map $\tilde{\mu}^2(\tilde{\mu}^2(-, q^i), q^{1-i}) = (- \circ q^i) \circ q^{i-1}$ is homotopic to $\text{id}_{\mathcal{W}(X_i, T)}$.

A degree zero closed morphism is called a *homotopy equivalence* if it admits a homotopy inverse. Two objects $X, X' \in \text{Ob } \mathcal{W}$ are called *homotopy equivalent*

if for some $n \geq 0$ there exist a chain of homotopy equivalences

$$X = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n = X'.$$

Note that, in the notation of Definition 1.28, the morphisms $\tilde{\mu}^2(q^0, q^1)$ and $\tilde{\mu}^2(q^1, q^0)$ are usually not defined since (X_0, X_1, X_0) and (X_1, X_0, X_1) are usually not transverse sequences. Standard homological algebra implies that homotopy equivalences are quasi-isomorphisms.

The following definition is a natural adaptation of [31, Definition 10] from quasi-isomorphisms to homotopy equivalences.

Definition 1.29. We say that the A_∞ -pre-category $\mathcal{W}$ has enough homotopy equivalences if transverse sequences in $\mathcal{W}$ form transverse sets and for any object $X \in \text{Ob } \mathcal{W}$ and a finite set of finite transverse sets $\{S_i\}_{i \in I}$ there exists an object X' and a homotopy equivalence $q: X \rightarrow X'$ such that the set $S_i \sqcup \{X'\}$ is transverse for each $i \in I$.

Having enough homotopy equivalences is a substitute for being unital for A_∞ -pre-categories and makes homotopy equivalences satisfy the expected properties, see §3.1.

Note that having enough homotopy equivalences is a property of an A_∞ -pre-category rather than an additional structure. Also observe that, in contrast with the homotopy unit framework of Fukaya–Oh–Ohta–Ono [23, §3.3] we do not require any higher coherences from homotopy equivalences. This works well in our setting since our A_∞ -pre-categories are triangulated by construction.

The following characterisation of homotopy equivalences in strictly unital A_∞ -categories indicates that homotopy equivalences are the right notion of invertible morphism in an A_∞ -pre-category.

Recall that an A_∞ -category $\mathcal{C}$ is called *strictly unital* if for each $X \in \text{Ob } \mathcal{C}$ there exists a closed morphism $1_X \in \mathcal{C}^0(X, X)$ such that $1_X \circ a = a$ for any $a \in \mathcal{C}(Y, X)$, $a \circ 1_X = a$ for any $a \in \mathcal{C}(X, Y)$, and $\tilde{\mu}^n(\dots, 1_X, \dots) = 0$ for $n \geq 3$. Note that strict units 1_X , $X \in \text{Ob } \mathcal{C}$ are unique if they exist.

Let $\mathcal{C}$ be a strictly unital A_∞ -category. Observe that strict units in $\mathcal{C}$ are homotopy equivalences and that they certify that $\mathcal{C}$ has enough homotopy equivalences. Moreover, a closed morphism $q^1 \in \mathcal{C}^0(X_0, X_1)$ in a strictly unital A_∞ -category $\mathcal{C}$ is a homotopy equivalence if and only if there exists a closed morphism $q^0 \in \mathcal{C}^0(X_1, X_0)$ such that $q^1 \circ q^0$ is cohomologous to 1_{X_1} and $q^0 \circ q^1$ is cohomologous to 1_{X_0} , see Lemma 3.5.

1.4.2. A_∞ -functors. Let $\mathcal{W}$ and $\mathcal{V}$ be two A_∞ -pre-categories. An A_∞ -functor $\mathcal{F}: \mathcal{W} \rightarrow \mathcal{V}$ is the following data.

- A map $\mathcal{F}^0: \text{Ob } \mathcal{W} \rightarrow \text{Ob } \mathcal{V}$ such that for any transverse sequence $(X_0, \dots, X_n)$ in $\mathcal{W}$ the sequence $(\mathcal{F}^0(X_0), \dots, \mathcal{F}^0(X_n))$ is transverse in $\mathcal{V}$.

- For each $n \geq 1$ and a transverse sequence $(X_0, \dots, X_n)$ in $\mathcal{W}$ an R -linear map

$$\mathcal{F}^n: \mathcal{W}(X_{n-1}, X_n) \otimes \dots \otimes \mathcal{W}(X_0, X_1) \rightarrow \mathcal{V}(\mathcal{F}^0(X_0), \mathcal{F}^0(X_n))$$

of degree $1 - n$ satisfying the A_n -functor-relation

$$(1.27) \quad \sum_{n \geq j \geq i \geq 0} (-1)^\dagger \mathcal{F}^{n-j+1+i} (a^n, \dots, a^{j+1}, \tilde{\mu}_{\mathcal{W}}^{j-i}(a^j, \dots, a^{i+1}), a^i, \dots, a^1) = \\ \sum_{r \geq 0} \sum_{n=i_r \geq \dots \geq i_0=0} \tilde{\mu}_{\mathcal{V}}^r (\mathcal{F}^{n-i_{r-1}}(a^n, \dots, a^{i_{r-1}+1}), \dots, \mathcal{F}^{i_1}(a^{i_1}, \dots, a^1)).$$

where $\dagger = |a^i| + \dots + |a^1| - i$ for any $n \geq 1$ and any chain of homogeneous morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$ in $\mathcal{W}$.

The A_∞ -functor $\mathcal{F}$ is called *strict* if $\mathcal{F}^n = 0$ for $n \geq 2$. For strict A_∞ -functors the A_∞ -relations (1.27) boils down to

$$\mathcal{F}^1 (\tilde{\mu}_{\mathcal{W}}^n(a^n, \dots, a^1)) = \tilde{\mu}_{\mathcal{V}}^n (\mathcal{F}^1(a^n), \dots, \mathcal{F}^1(a^1))$$

for any $n \geq 1$ and any chain of morphisms $X_0 \xrightarrow{a^1} X_1 \xrightarrow{a^2} \dots \xrightarrow{a^n} X_n$ in $\mathcal{W}$.

A strict A_∞ -functor $\mathcal{F}$ is called an *isomorphism* if $\mathcal{F}^0$ is a bijection and $\mathcal{F}^1$ is an isomorphism of graded R -modules on each morphism space in $\mathcal{W}$. Note that this is equivalent to $\mathcal{F}$ having an inverse *strict* A_∞ -functor.

If $\mathcal{C}$ and $\mathcal{D}$ are two strictly unital A_∞ -categories, then an A_∞ -functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is called *strictly unital* if $\mathcal{F}^1(1_X) = 1_{\mathcal{F}^0(X)}$ and $\mathcal{F}^n(\dots, 1_X, \dots) = 0$ for $n \geq 2$ for any $X \in \text{Ob } \mathcal{C}$. A strictly unital A_∞ -functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is called a *homotopy equivalence* if

- For each $X_0, X_1 \in \text{Ob } \mathcal{C}$ the map $\mathcal{F}^1: \mathcal{W}(X_0, X_1) \rightarrow \mathcal{W}(\mathcal{F}^0(X_0), \mathcal{F}^0(X_1))$ is a homotopy equivalence; and
- each $Y \in \text{Ob } \mathcal{D}$ is homotopy equivalent to $\mathcal{F}^0(X)$ for some $X \in \text{Ob } \mathcal{C}$.

Two strictly unital A_∞ -categories $\mathcal{C}$ and $\mathcal{D}$ are called *homotopy equivalent* if there exists a finite sequence of A_∞ -pre-categories $\mathcal{C}_0, \dots, \mathcal{C}_n$, $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{D}_n = \mathcal{D}$ such that for any $i = 1, \dots, n$ there exists a homotopy equivalence from $\mathcal{C}_{i-1}$ to $\mathcal{C}_i$ or vice versa.

1.4.3. Triangulated A_∞ -pre-categories. Let $\mathcal{W}$ be an A_∞ -pre-category. The triangulated envelope $\text{Tw } \mathcal{W}$, whose objects are twisted complexes of objects from $\mathcal{W}$, is defined similarly to the case of A_∞ -categories, see §3.3.1. The result is a ($\mathbb{Z}$ -graded R -linear) A_∞ -pre-category $\text{Tw } \mathcal{W}$. Informally, $\text{Tw } \mathcal{W}$ is obtained from $\mathcal{W}$ by adding integer shifts of objects and iterated cones of closed morphisms. There is a natural strict A_∞ -functor $\mathcal{W} \hookrightarrow \text{Tw } \mathcal{W}$ that is injective on objects and is an isomorphism of graded R -modules on morphisms.

We use a strong version of being triangulated for A_∞ -pre-categories that states that there exists a strict totalisation functor from $\mathrm{Tw} \mathcal{W}$ to $\mathcal{W}$. Namely, an A_∞ -pre-category $\mathcal{W}$ is called *triangulated* if there exists a strict A_∞ -functor $\mathrm{Tot}: \mathrm{Tw} \mathcal{W} \rightarrow \mathcal{W}$ such that

- (1) $\mathrm{Tot}^1: \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathrm{Tot}^0(\mathbf{X}^0), \mathrm{Tot}^0(\mathbf{X}^1))$ is an isomorphism of graded modules for each transverse pair $(\mathbf{X}^0, \mathbf{X}^1)$ in $\mathrm{Tw} \mathcal{W}$;
- (2) $\mathrm{Tot}^0(X) = X$ for each $X \in \mathrm{Ob} \mathcal{W}$;
- (3) Tot^1 is identity on $\mathrm{Tw} \mathcal{W}(X^0, X^1) = \mathcal{W}(X^0, X^1)$ for each transverse pair (X^0, X^1) in $\mathcal{W}$.

Since shifts of objects and cones of closed morphisms are tautologically defined in $\mathrm{Tw} \mathcal{W}$, we obtain shifts and cones in a triangulated A_∞ -pre-category $\mathcal{W}$ by post-composing with the totalisation functor. It is instructive to think that $\mathcal{W}$ is triangulated if it has shifts of objects and cones of closed morphisms, though we do not define what it means intrinsically on $\mathcal{W}$.

If $\mathcal{W}$ is triangulated and has enough homotopy equivalences, then $\mathrm{Tw} \mathcal{W}$ also has enough homotopy equivalences and any twisted complex is homotopy equivalent to an object of $\mathcal{W}$, see Lemma 3.9.

1.4.4. Remarks on Theorem 1. We have explained all the words in Theorem 1 and now make some remarks on the proof.

Let $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ be the relative wrapped Fukaya category of a weighted surface $(\mathcal{S}, w)$. That $\mathcal{W}$ is a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category is immediate from the construction. That $\mathcal{W}$ is independent of the Liouville structure follows from the convexity of the space of Liouville structures, see Proposition 4.4.

To show that $\mathcal{W}$ has enough homotopy equivalences we apply the usual method of perturbing the underlying graded Lagrangian slightly inside the space of Lagrangians. If $\mathbb{L}_0$ is a Lagrangian brane supported on the Lagrangian L_0 and L_1 is a perturbation of L_0 , then the brane structure of $\mathbb{L}_0$ transports to Lagrangian brane $\mathbb{L}_1$ on L_1 and the homotopy inverse morphisms between $\mathbb{L}_0$ and $\mathbb{L}_1$ can be constructed explicitly, see §4.4.

That $\mathcal{W}$ is triangulated should be expected: adding a multiple of π to the grading realises the shift of objects, and a closed morphism a of degree 1 between two Lagrangian branes $\mathbb{L}_0 = (\mathcal{L}_0, \delta_0)$ and $\mathbb{L}_1 = (\mathcal{L}_1, \delta_1)$ gives a bounding cochain δ_a in the A_∞ -algebra $(\mathcal{A}_{\mathbb{L}}, \mu_{\mathbb{L}})$ where $\mathbb{L} = (\mathcal{L}, \delta_0 + \delta_1)$ is a composite Lagrangian brane of $\mathbb{L}_0$ and $\mathbb{L}_1$. The Lagrangian brane $(\mathcal{L}, \delta_0 + \delta_1 + \delta_a)$ then realises the cone of a , see §4.5.

Kontsevich and Soibelman [31, Conjecture 4] conjectured that the quasi-equivalence classes of R -linear A_∞ -pre-categories are in bijection with quasi-equivalence classes of strictly unital R -linear A_∞ -categories. This was proven when R is a field by Efimov [17], but remains open for arbitrary commutative

rings, which is the setting we are in. We therefore resort to a folklore way to construct an A_∞ -category spanned by a collection of objects in an A_∞ -pre-category that works well for our geometric setup.

The key ingredient in this construction is that for any Lagrangian brane $\mathbb{L}^0$ we can construct a transverse set $\{\mathbb{L}^j\}_{j=0}^\infty$ and homotopy equivalences $e^{j_0, j_1} \in \mathcal{W}(\mathbb{L}^{j_0}, \mathbb{L}^{j_1})$ for each $j_0 \neq j_1$ that behave like strict units, see §3.2.1 for the precise definition. This system of strict units is obtained by taking small Lagrangian perturbations of the underlying Lagrangian of $\mathbb{L}^0$, see §4.6.1.

To prove that the homotopy class of $\mathcal{W}^{\mathbf{L}}$ does not depend on any choices, we use that the group of Hamiltonian diffeomorphisms $\text{Ham}(\mathcal{S})$ acts on $\mathcal{W}$ by isomorphisms. See §4.2.3 for the description of the action and §4.6.4 for the application of the action to the invariance.

1.4.5. Examples for Theorem 1. We finish the two examples started in the introduction.

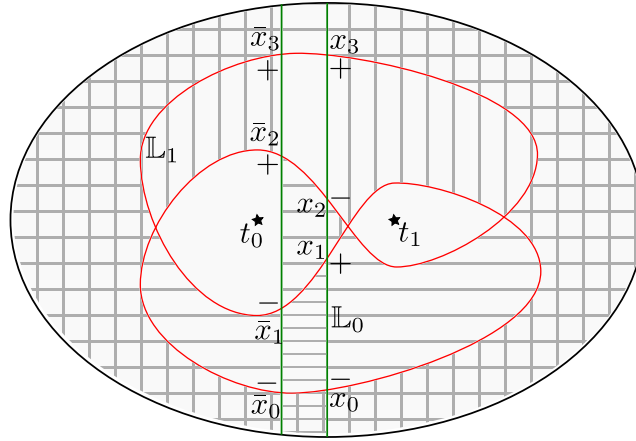


FIGURE 9. Morphisms between $\mathbb{L}_1$ and two copies of $\mathbb{L}_0$.

Two immersed polygons are indicated by vertically and horizontally striped areas respectively. The area with denser horizontal strips is covered by the corresponding polygon twice.

Example 1.30. Let S be a disc, $D = \{0, 1\}$, $\Lambda = \emptyset$, $R = \mathbb{C}[t_0, t_1]$, and $w: D \rightarrow R$ given by $w(0) = t_0$, $w(1) = t_1$, as in Fig. 1(b). We compute the A_∞ -endomorphism algebra $\text{End}_{\mathcal{W}}(\mathbb{L}_0) = \mathcal{E}$ where $\mathbb{L}_0$ is supported on an arc separating the two points of D , the local system is R , and the bounding cochain is zero.

The only non-zero A_∞ -operations in $\mathcal{E}$ are $\tilde{\mu}^1$ and $\tilde{\mu}^2$, we compute them in terms of $d(a) = (-1)^{|a|}\tilde{\mu}^1(a)$ and $b \cdot a = (-1)^{|a|}\tilde{\mu}^2(b, a)$ to obtain a dg-algebra. As a cochain complex over R we have

$$\mathcal{E} = \frac{R\langle y_0, y_1 \rangle}{y_0^2, y_1^2}, \quad |y_0| = |y_1| = -1, \quad dy_1 = -t_1, \quad dy_2 = t_2$$

and the graded linear map

$$\mathcal{E} \rightarrow \mathcal{E}, \quad y_0 \dots y_0 \mapsto -y_0 \dots y_0, \quad y_0 \dots y_1 \mapsto y_0 \dots y_1, \quad y_1 \dots \mapsto y_1 \dots$$

conjugates $\cdot$ to become the standard product on $\frac{R\langle y_0, y_1 \rangle}{y_0^2, y_1^2}$ hence we have identified

$$\mathcal{E} = \frac{\mathbb{C}[t_0, t_1]\langle y_0, y_1 \rangle}{y_0^2, y_1^2}, \quad |y_0| = |y_1| = -1, \quad dy_0 = t_0, \quad dy_1 = t_1$$

as dg-algebras over $\mathbb{C}[t_0, t_1]$.

It is easy to check that

$$(\mathbb{C}[z], dz = 0, |z| = -2) \rightarrow \mathcal{E}, \quad z \mapsto y_0 y_1 + y_1 y_0$$

is a quasi-isomorphism.

On the other hand, $(\mathbb{C}[z], dz = 0, |z| = -2)$ is the Adams–Hilton model for S^3 , i.e. it is quasi-isomorphic to the chains on the loop space $C_{-*}\Omega S^3$. Abouzaid [2] proved that the wrapped Fukaya category $\mathcal{W}(T^*S^3)$ is quasi-equivalent to the category of twisted complexes over $C_{-*}\Omega S^3$ with the cotangent fibre corresponding to $C_{-*}\Omega S^3$ and the zero section to $C_{-*}\Omega S^3[3] \xrightarrow{\sim} C_{-*}\Omega S^3$.

We have therefore established a quasi-equivalence between $\mathcal{W}(T^*S^3)$ and the subcategory of $\tilde{\mathcal{W}}$ generated by $\mathbb{L}_0$. To prove that under this equivalence the zero section corresponds to $\mathbb{L}_1$, it is enough to prove that $\mathbb{L}_1$ is quasi-isomorphic to the twisted complex $\mathbb{L}_0[3] \xrightarrow{y_0 y_1 + y_1 y_0} \mathbb{L}_0$ in $\tilde{\mathcal{W}}$. This can be shown by a direct computation as follows. In the dg-sign conventions the morphism spaces between $\mathbb{L}_0$ and $\mathbb{L}_1$ can be identified with

$$\mathcal{W}(\mathbb{L}_0, \mathbb{L}_1) = Rx_2 \oplus Rx_0 \oplus Rx_3 \oplus Rx_1, \quad |x_2| = -2, \quad |x_0| = |x_3| = -1, \quad |x_1| = 0;$$

$$dx_2 = t_0 x_0 - t_1 x_3, \quad dx_0 = t_1 x_1, \quad dx_3 = t_0 x_1, \quad dx_1 = 0;$$

and

$$\mathcal{W}(\mathbb{L}_1, \mathbb{L}_0) = R\bar{x}_1 \oplus R\bar{x}_0 \oplus R\bar{x}_3 \oplus R\bar{x}_2, \quad |\bar{x}_1| = 1, \quad |\bar{x}_0| = |\bar{x}_3| = 2, \quad |\bar{x}_2| = 3;$$

$$d\bar{x}_1 = t_0 \bar{x}_3 + t_1 \bar{x}_0, \quad d\bar{x}_0 = t_0 \bar{x}_2, \quad d\bar{x}_3 = -t_1 \bar{x}_2, \quad d\bar{x}_2 = 0;$$

see Fig. 9. In this notation

$$\begin{array}{ccc} & \mathbb{L}_1 & \\ \swarrow \bar{x}_2 & & \\ \mathbb{L}_0[3] & \xrightarrow{y_0 y_1 + y_1 y_0} & \mathbb{L}_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{L}_0[3] & \xrightarrow{y_0 y_1 + y_1 y_0} & \mathbb{L}_0 \\ & \searrow x_1 & \\ & \mathbb{L}_1 & \end{array}$$

are the desired quasi-inverse morphisms as witnessed by the two immersed polygons in Fig. 9.

Example 1.31. Let $(\mathcal{S}, w)$ be the weighted surface given by an annulus S , $|\Lambda| = 1$, $D = \{q\}$, radial line field η , $R = k[y]$, and $w(q) = y$, as in Fig. 2(b). We compute the full A_∞ -subcategory of $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_w(\mathcal{S})$ on objects $\mathbb{L}_0$ and $\mathbb{L}_1$, where $\mathbb{L}_i$ are the Lagrangian branes depicted in Fig. 2(b) with the grading chosen such that $|b| = 1$. The only non-zero A_∞ -operations are $\tilde{\mu}^1$ and $\tilde{\mu}^2$, as in Example 1.30 we use the dg-algebra sign conventions. After a diagonal change of basis we can identify

$$\begin{aligned}\tilde{\mathcal{W}}(\mathbb{L}_0, \mathbb{L}_0) &= \frac{k[y]\langle a \rangle}{a^2}, \quad |a| = -1, \quad da = y; \\ \tilde{\mathcal{W}}(\mathbb{L}_1, \mathbb{L}_1) &= k[y]\langle x \rangle, \quad |x| = 0, \quad dx = 0\end{aligned}$$

as dg-algebras and

$$\begin{aligned}\tilde{\mathcal{W}}(\mathbb{L}_1, \mathbb{L}_0) &= 0; \\ \tilde{\mathcal{W}}(\mathbb{L}_0, \mathbb{L}_1) &= k[y]b \oplus k[y]ba, \quad |b| = 1, \quad db = 0, \quad d(ba) = -yb\end{aligned}$$

as dg-modules with the natural $\mathcal{W}(\mathbb{L}_1, \mathbb{L}_1) - \mathcal{W}(\mathbb{L}_0, \mathbb{L}_0)$ bimodule structure on $\tilde{\mathcal{W}}(\mathbb{L}_0, \mathbb{L}_1)$.

The cohomology category is then given by

$$H\tilde{\mathcal{W}}(\mathbb{L}_0, \mathbb{L}_0) = k, \quad H\tilde{\mathcal{W}}(\mathbb{L}_0, \mathbb{L}_1) = k[-1], \quad H\tilde{\mathcal{W}}(\mathbb{L}_1, \mathbb{L}_1) = k[x, y], \quad H\tilde{\mathcal{W}}(\mathbb{L}_1, \mathbb{L}_0) = 0$$

and is intrinsically formal for degree reasons.

Recall the projection $\pi: X^B \xrightarrow{\pi} \mathbb{A}_y^1$ from Fig. 2(a) that represents the blow-up of the affine plane as a relative curve. By Orlov's blow-up formula [41] we have a semi-orthogonal decomposition $D^b \text{Coh } X^B = \langle i_* \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{X^B} \rangle$ and

$$\begin{aligned}\text{Ext}^*(i_* \mathcal{O}_{\mathbb{P}^1}(-1), i_* \mathcal{O}_{\mathbb{P}^1}(-1)) &= k, \quad \text{Ext}^*(i_* \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{X^B}) = k[-1], \\ \text{Ext}^*(\mathcal{O}_{X^B}, \mathcal{O}_{X^B}) &= k[x, y]\end{aligned}$$

by adjunction. Therefore, $D^b \text{Coh } X^B$ is quasi-isomorphic to a subcategory of $\tilde{\mathcal{W}}_w(\mathcal{S})$ with $i_* \mathcal{O}_{\mathbb{P}^1}(-1)$ corresponding to $\mathbb{L}_0$ and $\mathcal{O}_{X^B}$ corresponding to $\mathbb{L}_1$.

1.5. Setup for Theorem 2

In this section we first give the necessary definitions for Theorem 2 in §§1.5.1 to 1.5.3, and then remark on the proof of Theorem 2 in §1.5.4.

1.5.1. Legendrian isotopy. Let $\mathcal{S} = (S; \Lambda, D; \lambda)$ be a Liouville marked surface. Recall that a Lagrangian $L = (I, \gamma, P)$ gives a Legendrian $\gamma^+ = (\gamma, P): I \rightarrow S^+$ in the contactisation $\mathcal{S}^+ = (S^+, \lambda^+)$ defined in (1.2). We now define a larger class of Legendrians in the contactisation by removing the general position requirements (3) and (4) on γ .

Given a point $p \in \partial S^+ = (\partial S \setminus \Lambda) \times \mathbb{R}_z$ let y_p be the maximal Reeb chord $y_p: [0, l) \rightarrow \partial S \setminus \Lambda$ with $y_p(0) = \pi(p)$ (here $l \in (0, +\infty]$) and $y_p^+: [0, l) \rightarrow \partial S^+$ be its Legendrian lift starting at p , i.e. $y_p^+(0) = p$, $\pi \circ y_p^+ = y_p$, and $(y_p^+)^* \lambda^+ = 0$.

Definition 1.32. A *Legendrian* L^+ in the contactisation $\mathcal{S}^+$ is the data of

- a 1-dimensional compact manifold I ;
- an embedding $\gamma^+ = (\gamma, P): I \rightarrow S^+ = S \setminus (\Lambda \cup D) \times \mathbb{R}$

such that

- (1) $(\gamma^+)^{-1}(\partial S^+) = \partial I$;
- (2) $(\gamma^+)^* \lambda^+ = 0$;
- (3) for any $t \in \partial I$ the curve $y_{\gamma^+(t)}^+: [0, l)_\tau \rightarrow \partial S^+$ does not intersect $\gamma(I)$ except at $\tau = 0$.

Note that γ^+ being an embedding is a reformulation for condition (6) from Definition 1.2 that allows γ to have worse than transverse double point singularities and that (3) from Definition 1.32 is a reformulation of (7) from Definition 1.2. In particular, given a Lagrangian $L = (I, \gamma, P)$ we get a Legendrian $L^+ = (I, \gamma^+ = (\gamma, P))$, called the *Legendrian lift* of L .

For a compact 1-dimensional manifold I denote by $\text{Lag}_I(\mathcal{S})$ (resp. $\text{Leg}_I(\mathcal{S}^+)$) the space of Lagrangians in $\mathcal{S}$ (resp. Legendrians in $\mathcal{S}^+$) with the C^∞ -topology. Taking the Legendrian lift defines an open embedding $\text{Lag}(\mathcal{S}) \hookrightarrow \text{Leg}(\mathcal{S}^+)$.

Recall from Definition 1.7 that the grading is defined in terms of γ and the definition works for any immersed curve γ , hence we get the notion of *graded Legendrian*. We also get the corresponding open embeddings $\text{Lag}_I^{\text{gr}}(\mathcal{S}) \hookrightarrow \text{Leg}_I^{\text{gr}}(\mathcal{S}^+)$ of the space of graded Lagrangians to the space of graded Legendrians.

We say that two (graded) Lagrangians are *Lagrangian isotopic* if they lie in the same component of $\text{Lag}(\mathcal{S})$ (resp. $\text{Lag}_I^{\text{gr}}(\mathcal{S})$). We say that two (graded) Legendrians are *Legendrian isotopic* if they lie in the same component of $\text{Leg}(\mathcal{S}^+)$ (resp. $\text{Leg}_I^{\text{gr}}(\mathcal{S}^+)$). Two (graded) Lagrangians are Legendrian isotopic if their Legendrian lifts are.

Lagrangian isotopy can be checked combinatorially and Legendrian isotopy is a concatenation of Lagrangian isotopies and Legendrian Reidemeister moves. These are standard results for closed Lagrangians and Legendrians, see Lemmas 2.12 and 2.13 for the extension to our setting.

1.5.2. A_∞ -morphisms. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two structured R -modules (see Definition 1.13). A *structured morphism* $\mathcal{F}: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ is a collection of linear maps $\mathcal{F}^n: \mathcal{A}_0^{\otimes n} \rightarrow \mathcal{A}_1$ of degree $1 - n$ for each $n \geq 0$ such that only finitely

many entries

$$\mathcal{F}^{z^0; z^n, \dots, z^1} : \mathcal{A}_0(z^n) \otimes \dots \otimes \mathcal{A}_0(z^1) \rightarrow \mathcal{A}_1(z^0)$$

are non-zero.

If $\mathcal{A}_0 \xrightarrow{\mathcal{F}_1} \mathcal{A}_1 \xrightarrow{\mathcal{F}_2} \mathcal{A}_2$ are structured morphisms, then their *composition* $\mathcal{F}_2 \circ \mathcal{F}_1 : \mathcal{A}_0 \rightarrow \mathcal{A}_2$ is the structured morphism given by

$$(1.28) \quad (\mathcal{F}_2 \circ \mathcal{F}_1)^{z^0; z^n, \dots, z^1} = \sum_{\substack{r \geq 0 \\ n=i_r \geq \dots \geq i_0=0 \\ z_1^r, \dots, z_1^1 \in Z_1}} \mathcal{F}_2^{z^0; z_1^r, \dots, z_1^1} \circ \left(\mathcal{F}_1^{z_1^r; z^n, \dots, z^{i_{r-1}+1}} \otimes \dots \otimes \mathcal{F}_1^{z_1^1; z^{i_1}, \dots, z^1} \right).$$

for $z^0 \in Z_2$ and $z^n, \dots, z^1 \in Z_0$. This composition is associative and the identity formal morphism for a graded module $\mathcal{A}$ is $\mathcal{I}d : \mathcal{A} \rightarrow \mathcal{A}$ given by $\mathcal{I}d^1 = \text{id}_{\mathcal{A}}$ and $\mathcal{I}d^n = 0$ for $n \neq 1$.

Now let $(\mathcal{A}_0, \mu_0)$ and $(\mathcal{A}_1, \mu_1)$ be two structured A_∞ -algebras. An A_∞ -*morphism* $\mathcal{F} : (\mathcal{A}_0, \mu_0) \rightarrow (\mathcal{A}_1, \mu_1)$ is a structured morphism $\mathcal{F} : \mathcal{A}_0 \rightarrow \mathcal{A}_1$ such that the A_n -morphism-relation

$$(1.29) \quad \sum_{n \geq j \geq i \geq 0} \sum_{z \in Z_0} (-1)^\dagger \mathcal{F}^{z^0; z^n, \dots, z^{j+1}, z, z^i, \dots, z^1} \circ \left(\text{id} \otimes \mu_0^{z; z^j, \dots, z^{i+1}} \otimes \text{id} \right) = \sum_{\substack{r \geq 0 \\ n=i_r \geq \dots \geq i_0=0 \\ z_1^r, \dots, z_1^1 \in Z_1}} \mu_1^{z^0; z_1^r, \dots, z_1^1} \circ \left(\mathcal{F}^{z_1^r; z^n, \dots, z^{i_{r-1}+1}} \otimes \dots \otimes \mathcal{F}^{z_1^1; z^{i_1}, \dots, z^1} \right)$$

where $\dagger = |z^i| + \dots + |z^1| - i$ holds for any $n \geq 0$, $z^0 \in Z_1$, and $z^n, \dots, z^1 \in Z_0$. The sums in (1.29) are finite since $\mathcal{A}_1$ and $\mathcal{F}$ are structured.

The composition of two A_∞ -morphisms is an A_∞ -morphism, the identity formal morphism $\mathcal{I}d_{\mathcal{A}}$ is an A_∞ -morphism for any structured A_∞ -operation μ on $\mathcal{A}$, and if an A_∞ -morphism is invertible as a structured morphism, then the inverse is an A_∞ -morphism. These properties can be shown by direct computation or derived from more familiar properties of chain maps via the bar construction, see Remark 1.33 below.

If δ is a bounding cochain in $\mathcal{A}_0$ (see §1.2.4) and $\mathcal{F} : \mathcal{A}_0 \rightarrow \mathcal{A}_1$ is an A_∞ -morphism, then

$$(1.30) \quad \mathcal{F}_*(\delta) = \sum_{n \geq 0} \mathcal{F}^n(\underbrace{\delta, \dots, \delta}_n)$$

is a bounding cochain in $\mathcal{A}_1$. This sum is finite since $\mathcal{F}$ is structured.

Remark 1.33. To make some sense of (1.14) and (1.29) one can apply the bar construction to a structured A_∞ -algebra $\mathcal{A}$ and assemble a coderivation d on the bar coalgebra $B\mathcal{A}$ from the A_∞ -operations. The A_∞ -relations (1.14) are then equivalent to d being a differential. A structured morphism of structured

modules induces a coalgebra homomorphism on the bar coalgebra and the A_∞ -morphism relations (1.29) are equivalent to this homomorphism being a chain map. See [23, §3.2] for details.

An element $\delta \in \mathcal{A}^1$ gives a group-like element $e^\delta = 1 + \delta + \delta \otimes \delta + \dots$ in the completion of $B\mathcal{A}$ with respect to the action filtration. The Maurer-Cartan equation (1.22) is equivalent to e^δ being closed. See [23, §3.6] for details.

1.5.3. A_∞ -homotopy. Let $\mathcal{F}_0, \mathcal{F}_1: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ be two A_∞ -morphisms between structured A_∞ -algebras. An A_∞ -homotopy $\mathcal{H}: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ from $\mathcal{F}_0$ to $\mathcal{F}_1$ is a collection of linear maps $\mathcal{H}^n: \mathcal{A}_0^{\otimes n} \rightarrow \mathcal{A}_1$ of degree $-n$ for each $n \geq 0$ such that $\mathcal{H}: \mathcal{A}_0 \rightarrow \mathcal{A}_1[-1]$ is a structured morphism and the A_n -homotopy-relation

$$\begin{aligned}
 & \mathcal{F}_1^{z^0; z^n, \dots, z^1} - \mathcal{F}_0^{z^0; z^n, \dots, z^1} = \\
 & \sum_{n \geq j \geq i \geq 0} (-1)^\dagger \times \left(\sum_{z \in Z_0} \mathcal{H}^{z^0; z^n, \dots, z^{j+1}, z, z^i, \dots, z^1} \circ \left(\text{id} \otimes \mu_0^{z; z^j, \dots, z^{i+1}} \otimes \text{id} \right) \right. \\
 (1.31) \quad & + \sum_{\substack{r, s \geq 0 \\ i=i_r \geq \dots \geq i_0=0 \\ n=k_s \geq \dots \geq k_0=j}} \sum_{\substack{z \in Z_1 \\ z_0^r, \dots, z_0^1 \in Z_1 \\ z_1^s, \dots, z_1^1 \in Z_1}} \mu_1^{z^0; z_1^s, \dots, z_1^1, z, z_0^r, \dots, z_0^1} \circ \left(\mathcal{F}_1^{z_1^s, z^n, \dots, z^{k_{s-1}+1}} \otimes \dots \right. \\
 & \left. \left. \otimes \mathcal{F}_1^{z_1^1; z^{k_1}, \dots, z^{j+1}} \otimes \mathcal{H}^{z; z^j, \dots, z^{i+1}} \otimes \mathcal{F}_0^{z_0^r; z^i, \dots, z^{i_{r-1}+1}} \otimes \dots \otimes \mathcal{F}_0^{z_0^1; z^{i_1}, \dots, z^1} \right) \right)
 \end{aligned}$$

holds for all $n \geq 0$, $z^0 \in Z_1$, and $z^n, \dots, z^1 \in Z_0$. The sums in (1.31) are finite since everything is structured.

Remark 1.34. We sketch the conceptual explanation for (1.31) from [23, §4.2]. First, given a structured A_∞ -algebra $\mathcal{A}$ one defines a structured A_∞ -algebra

$$(\mathcal{A}^{[0,1]} = \mathcal{A} \oplus \mathcal{A}[-1] \oplus \mathcal{A}, \mu^{[0,1]}),$$

which is a ‘model for $\mathcal{A} \times [0, 1]$ ’. This comes with an inclusion A_∞ -morphism $\mathcal{I}: \mathcal{A} \rightarrow \mathcal{A}^{[0,1]}$ with the only non-zero term given by $\mathcal{I}^1(a) = (a, 0, a)$ for $a \in \mathcal{A}$ and two evaluation A_∞ -morphisms $\mathcal{E}_0, \mathcal{E}_1: \mathcal{A}^{[0,1]} \rightarrow \mathcal{A}$ with the non-zero terms given by $\mathcal{E}_0^1(a, b, c) = a$ and $\mathcal{E}_1^1(a, b, c) = c$.

Now, two A_∞ -morphisms $\mathcal{F}_0, \mathcal{F}_1: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ are homotopic if there exists an A_∞ -morphism $\tilde{\mathcal{H}}: \mathcal{A}_0 \rightarrow \mathcal{A}_1^{[0,1]}$ such that $\mathcal{E}_0 \circ \tilde{\mathcal{H}} = \mathcal{F}_0$ and $\mathcal{E}_1 \circ \tilde{\mathcal{H}} = \mathcal{F}_1$. (1.31) is the unpacked version of this with $\mathcal{H}$ being (up to a sign) the part of $\tilde{\mathcal{H}}$ going from $\mathcal{A}_0$ to $\mathcal{A}_1[-1]$.

If there is a homotopy from $\mathcal{F}_0$ to $\mathcal{F}_1$ and from $\mathcal{F}_1$ to $\mathcal{F}_0$, then we say that $\mathcal{F}_0$ and $\mathcal{F}_1$ are homotopic and write $\mathcal{F}_0 \sim \mathcal{F}_1$. An A_∞ -morphism $\mathcal{F}_1: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ is called a *homotopy equivalence* if there exists an A_∞ -morphism $\mathcal{F}_0: \mathcal{A}_1 \rightarrow \mathcal{A}_0$ such that

$$\mathcal{F}_0 \circ \mathcal{F}_1 \sim \text{Id}_{\mathcal{A}_0} \text{ and } \mathcal{F}_1 \circ \mathcal{F}_0 \sim \text{Id}_{\mathcal{A}_1}.$$

A non-trivial theorem [23, Theorem 4.2.34] on the equivalence of any two models of $\mathcal{A} \times [0, 1]$ shows that in the filtered setting the existence of homotopy from $\mathcal{F}_0$ to $\mathcal{F}_1$ implies the existence of homotopy from $\mathcal{F}_1$ to $\mathcal{F}_0$. We do not prove a similar statement in our setting since the only homotopy equivalences we need are stabilisations (see §5.2.2) and for them the homotopies in both directions are explicit.

1.5.4. Remarks on Theorem 2. We have defined all the words in Theorem 2 and now proceed with some remarks on the proof.

A generic Legendrian isotopy can be decomposed into a finite sequence of Lagrangian isotopies and Legendrian Reidemeister moves, see Lemma 2.13. This observation reduces the proof of Theorem 2 to combinatorial considerations.

Lagrangian isotopy induces an isomorphism of obstruction algebras, see Lemma 4.3. This isomorphism, however, does not preserve the action of double points and Reeb chords, this is why the action function in Definition 1.13 is not part of the data and there are no action requirements on structured morphisms of structured A_∞ -algebras.

A Legendrian Reidemeister move induces a homotopy equivalence of obstruction algebras, see Proposition 5.5. This homotopy equivalence can be chosen to respect a given bounding cochain, which is essential for the proof that the corresponding Lagrangian branes are homotopy equivalent in §5.4.

CHAPTER 2

Deformations

2.1. Deformations of the surface

In this section we study graded Liouville marked structures and their deformations on a given surface.

In §2.1.1 we describe the space of graded Liouville marked structures on a given surface. In §2.1.2 we recall that Hamiltonian diffeomorphisms form the identity component of exact symplectomorphisms. In §2.1.3 we prove that the diffeomorphism group of a marked surface retracts onto the symplectomorphism group of a Liouville marked surface supported on it and that any two Liouville structures on the same marked surface are exact symplectomorphic. In §2.1.4 we define graded diffeomorphisms and extend the above retraction statement to the graded setting.

2.1.1. Graded Liouville marked structures on a given surface.

Fix a compact connected oriented 2-dimensional manifold S with non-empty boundary. Recall that S is determined up to an orientation-preserving diffeomorphism by its genus $g(S)$ and the number $b(S)$ of connected components of ∂S . We describe the homotopy type of the space of graded Liouville marked surface structures on S , see Definitions 1.1 and 1.7.

For a non-negative integer n and a function σ from the set of connected components of ∂S to $\mathbb{Z}_{\geq 0}$ let

$$(2.1) \quad \mathcal{M}(S; \sigma, n) = \left\{ \begin{array}{l} \Lambda \subset \partial S \quad | \Lambda \cap C| = \sigma(C) \text{ for all } C \in \pi_0 \partial S \\ D \subset \text{Int } S \quad \text{and } |D| = n \end{array} \right\}$$

be the set parametrising the choice of a marked surface structure $(S; \Lambda, D)$ on S .

For finite sets $\Lambda \subset \partial S$ and $D \subset \text{Int } S$ define

$$(2.2) \quad \mathcal{LS}(S; \Lambda, D) = \left\{ \lambda \in \Omega^1(S) : \begin{array}{l} \lambda \text{ is a Liouville structure on } (S; \Lambda, D) \\ \text{and } d\lambda \text{ induces the orientation of } S \end{array} \right\}.$$

Recall that the choice of grading on S is parametrised by $C^\infty(S, \mathbb{P}TS)$.

PROPOSITION 2.1. *Let S be a compact connected oriented 2-dimensional manifold with non-empty boundary. Then*

- (1) $\mathcal{M}(S; \sigma, n)$ is a smooth aspherical manifold of dimension $2n + \sum_{C \in \pi_0 \partial S} \sigma(C)$ for any $\sigma: \pi_0 \partial S \rightarrow \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$.
- (2) $\mathcal{LS}(S; \Lambda, D)$ is a non-empty convex subset of $\Omega^1(S)$ for any finite sets $\Lambda \subset \partial S$ and $D \subset \text{Int } S$.
- (3) $\pi_0 C^\infty(S, \mathbb{P}TS)$ is a torsor over $H^1(S; \mathbb{Z})$ and each component of $C^\infty(S, \mathbb{P}TS)$ is a $K(\mathbb{Z}, 1)$ space. For a line field η on S the generator of $\pi_1(C^\infty(S, \mathbb{P}TS), \eta) = \mathbb{Z}$ is given by a counter-clockwise rotation of η by π in each tangent space $T_p S$, $p \in S$.

In particular, the space of graded marked surface structures on S with $|D| = n$ and $|\Lambda \cap C| = \sigma(C)$ for all $C \in \pi_0 \partial S$ has the homotopy type of $S^1 \times H^1(S; \mathbb{Z}) \times \mathcal{M}_{\sigma, n}(S)$.

PROOF. We prove (1). $\mathcal{M} = \mathcal{M}(S; \sigma, n)$ is the orbit space of the trivial-stabiliser locus of the action of the product of symmetric groups

$$S_n \times \prod_{C \in \pi_0 \partial S} S_{\sigma(C)} \quad \text{on} \quad (\text{Int } S)^n \times \prod_{C \in \pi_0 \partial S} C^{\sigma(C)}$$

where each symmetric group acts on the corresponding factor by permuting entries. Since the trivial-stabiliser locus (i.e. the complement of the big diagonal) is open, $\mathcal{M}$ is a smooth manifold of dimension $2n + \sum_{C \in \pi_0 \partial S} \sigma(C)$. $\mathcal{M}$ is connected since both the complement of finitely many embedded closed intervals in S and the configuration space of a fixed number of (unordered) points in S^1 are connected. That $\pi_k(\mathcal{M}) = 0$ for $k \geq 2$ follows by induction on $n + \sum_{C \in \pi_0 \partial S} \sigma(C)$ from the homotopy sequence of fibration and the fact that the complement of finitely many points in S (resp. in a component of ∂S) is aspherical (resp. a disjoint union of contractible spaces).

To prove (2) we first show $\mathcal{LS} = \mathcal{LS}(S; \Lambda, D)$ is non-empty. Recall that S admits a Stein structure, i.e. a complex structure J together with a Morse subharmonic function $\phi: S \rightarrow \mathbb{R}$ such that ∂S is a regular level set of ϕ where ϕ attains its maximum. Moreover, up to reparametrisation of $\mathbb{R}$ we can choose ϕ to be arbitrary as long as it does not have critical points of index 2. In particular, we can ensure that ϕ attains a non-degenerate local minimum at each point of D , see Cieliebak–Eliashberg [16, Chapter 8] for details. The 1-form $\lambda' = -d\phi \circ J$ is almost what we need, except the behaviour of λ' along the boundary. By pushing S inside its Liouville completion and rescaling we can ensure that $\int_C \lambda' = 1$ for all components C of ∂S . Next, for each $C \in \pi_0 \partial S$ fix an orientation-preserving diffeomorphism

$$\tau_C: C \rightarrow \mathbb{R}/\mathbb{Z} \text{ such that } \lambda|_C = \tau_C^*(dt)$$

and pick a function $g_C: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ such that $g'_C(t) \geq -1$ for all $t \in \mathbb{R}/\mathbb{Z}$ and the solution set of $g'_C(t) = -1$ is $\tau_C(C \cap \Lambda)$ (this can be done since we can

ensure $\int_{\mathbb{R}/\mathbb{Z}} g'_C(t)dt = 0$). Extend the collection of functions $\{g_C \circ \tau_C\}_{C \in \pi_0 \partial S}$ to a function $g: S \rightarrow \mathbb{R}$ supported near ∂S such that $(\lambda' + dg)(p) = 0$ for each $p \in \Lambda$ (this can be done since $(\lambda' + dg)|_{\partial S}(\Lambda) = 0$ and we can prescribe the derivative of g along ∂S in the normal direction). We obtain the form $\lambda = \lambda' + dg$ that lies in $\mathcal{LS}$.

Now show that $\mathcal{LS}$ is convex. Observe that $\lambda \in \Omega(S)$ belongs to $\mathcal{LS}$ if and only if

- (1) λ vanishes on $\Lambda \cup D$;
- (2) $\int_C \lambda = 1$ for each $C \in \pi_0 \partial S$;
- (3) $d\lambda$ is a volume form on S inducing the orientation of S ;
- (4) $\lambda|_{\partial S \setminus \Lambda}$ is a volume form on $\partial S \setminus \Lambda$ inducing the orientation of ∂S .

(1) is linear in λ , (2) is affine in λ , and (3), (4) are convex in λ , hence the result.

To show (3) note that TS is a trivial bundle, so S admits a line field. Fix a line field η_0 and a Riemannian metric on S . The space of line fields $C^\infty(S, \mathbb{P}TS)$ is then identified with the group $C^\infty(S, \mathbb{R}/\pi\mathbb{Z})$ by sending a line field η to the angle from η_0 to η . Therefore, $\pi_0 C^\infty(S, \mathbb{P}TS)$ is identified with $\pi_0 C^\infty(S, \mathbb{R}/\pi\mathbb{Z})$ which is isomorphic to $H^1(S; \mathbb{Z})$ since $\mathbb{R}/\pi\mathbb{Z}$ is a $K(\mathbb{Z}, 1)$ space. Picking η_0 in a given path-connected component of $C^\infty(S, \mathbb{P}TS)$ identifies this component with the space of null-homotopic maps from S to $\mathbb{R}/\pi\mathbb{Z}$, and the latter is a $K(\mathbb{Z}, 1)$ space since its universal cover $C^\infty(S, \mathbb{R})$ is contractible. See Lekili–Polishchuk [35, §1.1] for details. $\square$

2.1.2. Hamiltonian diffeomorphisms. Fix a Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda)$. Recall from (1.1) that $\text{Diff}(\mathcal{S})$ is the group of diffeomorphisms of S that preserve Λ and D setwise and $\text{Symp}_{\text{ex}}(\mathcal{S})$ is its subgroup consisting of exact symplectomorphisms. We describe the identity component $\text{Symp}_{\text{ex}}(\mathcal{S})_0$ of $\text{Symp}_{\text{ex}}(\mathcal{S})$.

Let $H = \{H_s: S \rightarrow \mathbb{R}\}_{s \in [0,1]}$ be a smooth family such that dH_s vanishes on $\Lambda \cup D$ and $H_s|_{\partial S}$ is locally constant. Then, since $d\lambda$ is non-degenerate, the equation $d\lambda(X_s, -) = dH_s$ defines a family of vector fields

$$[0, 1] \rightarrow C^\infty(S, TS), s \mapsto X_s^H.$$

As X_s^H vanishes on $\Lambda \cup D$ and is tangent to ∂S , it integrates to a family

$$[0, 1] \rightarrow \text{Symp}_{\text{ex}}(\mathcal{S}), s \mapsto \Phi_s^H.$$

We therefore obtain a map

$$C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R}) \rightarrow \text{Symp}_{\text{ex}}(\mathcal{S}), H = \{H_s\}_{s \in [0,1]} \mapsto \Phi_1^H$$

where $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ denotes the space of smooth families of functions $\{H_s\}_{s \in [0,1]}$ described above. The image of this map is a connected subgroup $\text{Ham}(\mathcal{S})$ of $\text{Symp}_{\text{ex}}(\mathcal{S})$ consisting of *Hamiltonian diffeomorphisms* of $\mathcal{S}$.

Lemma 2.2. $\text{Ham}(\mathcal{S}) = \text{Symp}_{\text{ex}}(\mathcal{S})_0$.

PROOF. $\text{Ham}(\mathcal{S})$ is connected hence we only need to prove that the right hand side is contained in the left hand side. Let $\Phi_s \in \text{Symp}_{\text{ex}}(\mathcal{S}), s \in [0, 1]$ be a family of symplectomorphisms generated by a family of vector fields $X_s \in C^\infty(S, TS), s \in [0, 1]$. Since a function on S is determined up to a constant (on each component of S) by its differential, we get a family $g_s \in C^\infty(S, \mathbb{R}), s \in [0, 1]$ such that

$$\Phi_s^* \lambda - \lambda = dg_s \text{ for } s \in [0, 1].$$

Differentiating with respect to s and using Cartan's magic formula we obtain

$$\Phi_s^* (d(\lambda(X_s)) + d\lambda(X_s, -)) = d \left(\frac{dg_s}{ds} \right).$$

Therefore,

$$d\lambda(X_s, -) = dH_s \text{ where } H_s = \frac{dg_s}{ds} \circ \Phi_s^{-1} - \lambda(X_s).$$

Since Φ_s is a diffeomorphism of S fixing $\Lambda \cup D$ pointwise, X_s vanishes on $\Lambda \cup D$ and is tangent to ∂S . Therefore, dH_s vanishes on $\Lambda \cup D$ and H_s is locally constant on ∂S . That is, $H = \{H_s\}_{s \in [0,1]} \in C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ and hence Φ_s is a path in $\text{Ham}(\mathcal{S})$. $\square$

2.1.3. Deforming diffeomorphisms to exact symplectomorphisms.

The purpose of proving Lemma 2.2 was mostly to warm up before the following statement.

PROPOSITION 2.3. *Let $\mathcal{S}_i = (S_i; \Lambda_i, D_i; \lambda_i), i = 0, 1$ be two Liouville marked surfaces. Then $\text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$ is a strong deformation retract of $\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$.*

$\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ in Proposition 2.3 is the space of orientation-preserving diffeomorphisms φ from S_0 to S_1 such that $\varphi(D_0) = D_1$ and $\varphi(\Lambda_0) = \Lambda_1$ with the C^∞ -topology. $\text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$ is the subspace of $\text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ consisting of exact symplectomorphisms, i.e. such that $\varphi^* \lambda_1 - \lambda_0 = dg$ for some $g \in C^\infty(S; \mathbb{R})$.

Applying Proposition 2.3 to $\mathcal{S}_0 = \mathcal{S}_1$ we get

Corollary 2.4. *If $\mathcal{S}$ is a Liouville marked surface, then $\text{Symp}_{\text{ex}}(\mathcal{S})$ is a strong deformation retract of $\text{Diff}(\mathcal{S})$.*

Applying Proposition 2.3 to Liouville structures on the same marked surface we get

Corollary 2.5. *If λ_0 and λ_1 are Liouville structures on a marked surface $(S; \Lambda, D)$, then there exists a smooth family $\varphi_s \in \text{Diff}(S; \Lambda, D)$ such that $\varphi_0 = \text{id}_S$ and $\varphi_1 \in \text{Symp}_{\text{ex}}((S; \Lambda, D; \lambda_0), (S; \Lambda, D; \lambda_1))$.*

Remark 2.6. A related result by Bourgeois–Ekholm–Eliashberg [12, Lemma 1.1] is that a symplectomorphism between Liouville manifolds with cylindrical ends can be deformed to an exact symplectomorphism.

Note that for a Liouville domain M the natural map $\pi_0 \text{Symp}_{\text{ex}}(M) \rightarrow \pi_0 \text{Symp}(M)$ is not necessarily surjective. For instance, on the cylinder $M = [-1, 3]_r \times (\mathbb{R}/\mathbb{Z})_\theta$ with the Liouville form $\lambda = r d\theta$ the symplectomorphism $(r, \theta) \mapsto (2 - r, -\theta)$ swaps the components of the boundary, but there is no exact symplectomorphism φ swapping the components since we would have

$$0 = \int_{\{3\} \times \mathbb{R}/\mathbb{Z}} (\varphi^* \lambda - \lambda) = \int_{-\{1\} \times \mathbb{R}/\mathbb{Z}} -d\theta - \int_{\{3\} \times \mathbb{R}/\mathbb{Z}} 3d\theta = -2.$$

On the other hand, a slight adaptation of our proof yields that for any Liouville domain M the group $\text{Symp}_{\text{ex}}(M)$ is a strong deformation retract of the group

$$\{\varphi \in \text{Symp}(M) : (\varphi^* \lambda - \lambda)|_{\partial M} \text{ is exact}\}$$

of symplectomorphisms exact on the boundary.

To prove Proposition 2.3 we use a Moser argument and the following lemma.

Lemma 2.7. *Let $(S; \Lambda, D)$ be a marked surface. Then there exists a continuous (in the C^∞ -topology) map $\mu \mapsto g_\mu$ from*

$$(2.3) \quad \tilde{\Omega}^1(S) = \left\{ \mu \in \Omega^1(S) : \int_C \mu = 0 \text{ for all } C \in \pi_0 \partial S \right\}$$

to $C^\infty(S, \mathbb{R})$ such that $\mu|_{\partial S} = dg_\mu|_{\partial S}$ for any $\mu \in \tilde{\Omega}^1(S)$, $\mu(p) = d_p g_\mu(p)$ for any $p \in \Lambda \cup D$, and $dg_\mu = \mu$ whenever μ is exact.

PROOF OF PROPOSITION 2.3. For any $\varphi \in \text{Diff}(\mathcal{S}_0, \mathcal{S}_1)$ we have $\varphi^* \lambda_1 - \lambda_0 \in \tilde{\Omega}^1(S)$ and hence a function $g_{\varphi^* \lambda_1 - \lambda_0}$ provided by Lemma 2.7 that depends on φ continuously. The equation

$$\varphi^* \lambda_1 - \lambda_0 + d\tilde{\lambda}_s(X_s, -) = dg_{\varphi^* \lambda_1 - \lambda_0} \text{ where } \tilde{\lambda}_s = s\varphi^* \lambda_1 + (1 - s)\lambda_0$$

defines a family of vector fields

$$[0, 1] \rightarrow C^\infty(S, TS), \quad s \mapsto X_s^\varphi = X_s$$

since $d\tilde{\lambda}_s$ is non-degenerate. As $d\tilde{\lambda}_s$ is skew-symmetric and $d\tilde{\lambda}_s(X_s, -)|_{\partial S}$ vanishes, X_s is tangent to ∂S . Since $\varphi^* \lambda_1 - \lambda_0$ and hence $dg_{\varphi^* \lambda_1 - \lambda_0}$ vanish on $\Lambda \cup D$, so does X_s . Therefore, X_s integrates to a family of diffeomorphisms

$$[0, 1] \rightarrow \text{Diff}(\mathcal{S}_0), \quad s \mapsto \Phi_s^\varphi = \Phi_s; \quad \Phi_0 = \text{id}_{\mathcal{S}_0}.$$

Integrating

$$\frac{d}{ds}\Phi_s^*\tilde{\lambda}_s = \Phi_s^*\left(\frac{d}{ds}\tilde{\lambda}_s + d(\tilde{\lambda}_s(X_s)) + d\tilde{\lambda}_s(X_s, -)\right) = \Phi_s^*(d(\tilde{\lambda}_s(X_s)) + dg_{\varphi^*\lambda_1-\lambda_0})$$

we obtain

$$(\varphi \circ \Phi_1)^*\lambda_1 - \lambda_0 = \Phi_1^*\tilde{\lambda}_1 - \lambda_0 = dh \text{ where } h = \int_0^1 (\tilde{\lambda}_s(X_s) + g_{\varphi^*\lambda_1-\lambda_0}) \circ \Phi_s ds.$$

Therefore, $\varphi \circ \Phi_1 \in \text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$. If $\varphi \in \text{Symp}_{\text{ex}}(\mathcal{S}_0, \mathcal{S}_1)$, then $\Phi_s^\varphi = \text{id}_{\mathcal{S}_0}$ since in that case $dg_{\varphi^*\lambda_1-\lambda_0} = \varphi^*\lambda_1 - \lambda_0$. Thus the map

$$\text{Diff}(\mathcal{S}_0, \mathcal{S}_1) \times [0, 1] \rightarrow \text{Diff}(\mathcal{S}_0, \mathcal{S}_1), \quad (\varphi, s) \mapsto \varphi \circ \Phi_s^\varphi$$

defines the desired strong deformation retraction. $\square$

PROOF OF LEMMA 2.7. We perform the construction in three steps. The first step takes care of exactness using the Helmholtz decomposition, the second of the behaviour on $\Lambda \cup D$ and ∂S using a partition of unity argument, and the third combines the two.

Step 1. Fix a Riemannian metric on S and recall that the Helmholtz decomposition on S yields for any $\mu \in \Omega^1(S)$ an L^2 -orthogonal splitting $\mu = \mu_{\text{ex}} + \mu_{\text{co}}$ where μ_{ex} is exact and μ_{co} is co-closed with vanishing normal component, see e.g. Schwarz [43, p. 87]. Since $g \in C^\infty(S, \mathbb{R})$ is determined by its differential up to a constant we obtain a continuous linear map

$$\Omega^1(S) \rightarrow C^\infty(S, \mathbb{R}), \quad \mu \mapsto g_\mu^0$$

such that $\mu = dg_\mu^0$ if and only if μ is exact.

Step 2. Note that the integral condition in (2.3) is equivalent to $\mu|_{\partial S}$ being exact. Therefore, the behaviour of $\mu \in \tilde{\Omega}^1(S)$ on $\Lambda \cup D$ and ∂S can be recorded by a continuous linear map

$$\mu \mapsto ((\mu(q))_{q \in D}, (\mu(p))_{p \in \Lambda}, f_\mu)$$

from $\tilde{\Omega}^1(S)$ to

$$\mathcal{T} = \left\{ ((\eta_q)_{q \in D}, (\eta_p)_{p \in \Lambda}, f) : \begin{array}{l} d_p f = \eta_p|_{T_p \partial S} \text{ for all } p \in \Lambda \end{array} \right\} \subset \prod_{q \in D} T_q^* S \times \prod_{p \in \Lambda} T_p^* S \times C^\infty(\partial S, \mathbb{R})$$

satisfying $\mu|_{\partial S} = df_\mu$.

A standard partition of unity argument gives a continuous linear map

$$\mathcal{T} \rightarrow C^\infty(S, \mathbb{R}), \quad \mathbf{t} \mapsto g_{\mathbf{t}}$$

such that $g_{\mathbf{t}}|_{\partial S} = f$ and $d_p g_{\mathbf{t}} = \eta_p$ for all $p \in \Lambda \cup D$ for any $\mathbf{t} = ((\eta_q)_{q \in D}, (\eta_p)_{p \in \Lambda}, f)$.

Composing the two maps above we get

$$\tilde{\Omega}^1(S) \rightarrow C^\infty(S; \mathbb{R}), \quad \mu \mapsto g_\mu^1$$

satisfying $\mu|_{\partial S} = dg_\mu^1|_{\partial S}$ and $\mu(p) = d_p g_\mu^1$ for any $p \in \Lambda \cup D$.

Step 3. Define the continuous linear map

$$\tilde{\Omega}^1(S) \rightarrow C^\infty(S, \mathbb{R}), \quad \mu \mapsto g_\mu = g_\mu^0 + g_{\mu - dg_\mu^0}^1.$$

Then

$$d_p g_\mu = d_p g_\mu^0 + (\mu - dg_\mu^0) = \mu \text{ for any } p \in \Lambda \cup D$$

and $dg_\mu|_{\partial S} = dg_\mu^0|_{\partial S} + (\mu - dg_\mu^0)|_{\partial S} = \mu|_{\partial S}$. Moreover, if μ is exact, then $dg_\mu^0 = \mu$ and hence $dg_\mu = \mu$. Thus $\mu \mapsto g_\mu$ satisfies all the requirements and we are done. $\square$

2.1.4. Graded diffeomorphisms. We recall the well-known way to incorporate the grading into diffeomorphisms and symplectomorphisms, see Seidel [44, §2.b] for a more general discussion.

Recall that the group of diffeomorphisms of a surface S acts on the space $C^\infty(S, \mathbb{P}TS)$ of line fields by $(\varphi_* \eta)(p) = d_{\varphi^{-1}(p)} \varphi(\eta(\varphi^{-1}(p)))$.

Fix a graded Liouville marked surfaces $\mathcal{S} = (S; \Lambda, D; \lambda; \eta)$. A diffeomorphism $\varphi \in \text{Diff}(\mathcal{S})$ is called *gradable* if $\varphi_* \eta$ and η lie in the same component of $C^\infty(S, \mathbb{P}TS)$. Let φ_s , $s \in [0, 1]$ be a family in $\text{Diff}(\mathcal{S})$ and $\tilde{\varphi}_0$ a path from η to $(\varphi_0)_* \eta$ in $C^\infty(S, \mathbb{P}TS)$. Then we can get a family of paths $\tilde{\varphi}_s$ from η to $(\varphi_s)_* \eta$ by setting, for example,

$$(2.4) \quad \tilde{\varphi}_s(t) = \begin{cases} \tilde{\varphi}_0(2t), & \text{if } 0 \leq t \leq 1/2; \\ (\varphi_0)_* \eta, & \text{if } 1/2 \leq t \leq 1 - s/2; \\ (\varphi_{2t+s-2})_* \eta, & \text{if } 1 - s/2 \leq t \leq 1. \end{cases}$$

This implies that in a given component of $\text{Diff}(\mathcal{S})$ either all diffeomorphisms are gradable or none is.

Let $\text{Diff}^{\text{gr}}(\mathcal{S}) = \text{Diff}^{\text{gr}}(S; \Lambda, D; \eta)$ be the group of pairs $(\varphi, \tilde{\varphi})$ where $\varphi \in \text{Diff}(\mathcal{S})$ is gradable and $\tilde{\varphi}$ is a homotopy class of paths from η to $\varphi_* \eta$, i.e. an element of $\Pi_1 C^\infty(S, \mathbb{P}TS)(\eta, \varphi_* \eta)$. The composition is given by

$$(\varphi, \tilde{\varphi}) \circ (\psi, \tilde{\psi}) = (\varphi \circ \psi, \varphi_* \tilde{\psi} \cdot \tilde{\varphi})$$

where $\cdot$ denotes concatenation. Denote by $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S})$ the subgroup of $\text{Diff}^{\text{gr}}(\mathcal{S})$ consisting of the pairs $(\varphi, \tilde{\varphi})$ with $\varphi \in \text{Symp}_{\text{ex}}(\mathcal{S})$.

Each component of S can be treated separately, so we assume S is connected. Proposition 2.1 (3) yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}) & \longrightarrow & \mathrm{Diff}(\mathcal{S}) \longrightarrow H^1(S; \mathbb{Z}) \\
& & \parallel & & \uparrow & & \uparrow & & \parallel \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Symp}_{\mathrm{ex}}^{\mathrm{gr}}(\mathcal{S}) & \longrightarrow & \mathrm{Symp}_{\mathrm{ex}}(\mathcal{S}) \longrightarrow H^1(S; \mathbb{Z})
\end{array}$$

where all maps except the ones to $H^1(S; \mathbb{Z})$ are group homomorphisms and $H^1(S; \mathbb{Z})$ is considered as a pointed set. The construction in (2.4) implies that the projection map $(\varphi, \tilde{\varphi}) \mapsto \varphi$ sends each component of $\mathrm{Diff}^{\mathrm{gr}}(\mathcal{S})$ (resp. $\mathrm{Symp}_{\mathrm{ex}}^{\mathrm{gr}}(\mathcal{S})$) homeomorphically onto a component of $\mathrm{Diff}(\mathcal{S})$ (resp. $\mathrm{Symp}_{\mathrm{ex}}(\mathcal{S})$). In particular, the identity component $\mathrm{Diff}(\mathcal{S})_0$ (resp. $\mathrm{Symp}_{\mathrm{ex}}(\mathcal{S})_0 = \mathrm{Ham}(\mathcal{S})$) can be identified with the identity component $\mathrm{Diff}^{\mathrm{gr}}(\mathcal{S})_0$ (resp. $\mathrm{Symp}_{\mathrm{ex}}^{\mathrm{gr}}(\mathcal{S})_0$).

If λ' is another Liouville structure on the marked surface $(S; \Lambda, D)$ and $\mathcal{S}'$ is the corresponding graded Liouville marked surface, then $\mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}) = \mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}')$ and we have a subspace

$$\mathrm{Symp}_{\mathrm{ex}}^{\mathrm{gr}}(\mathcal{S}, \mathcal{S}') = \mathrm{Symp}_{\mathrm{ex}}(\mathcal{S}, \mathcal{S}') \cap \mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}).$$

Proposition 2.3 and the above discussion yield a strong deformation retraction for graded diffeomorphisms.

Corollary 2.8. *$\mathrm{Symp}_{\mathrm{ex}}^{\mathrm{gr}}(\mathcal{S}, \mathcal{S}')$ is a strong deformation retract of $\mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}) = \mathrm{Diff}^{\mathrm{gr}}(\mathcal{S}')$.*

2.2. Deformations of curves

In this section we study Lagrangians and their deformations in a fixed Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda)$. In §2.2.1 we characterise immersed curves in $\mathcal{S}$ that can be isotoped to an exact curve. In §2.2.2 we describe the Lagrangian structures on a given exact curve. In §2.2.3 we give a topological characterisation of Lagrangian isotopy. In §2.2.4 we define Legendrian Reidemeister moves. In §2.2.5 we prove that a Legendrian isotopy decomposes into a finite sequence of Lagrangian isotopies and Legendrian Reidemeister moves. In §2.2.6 we describe gradings on a given immersed curve.

2.2.1. Exact curves isotopic to a given curve. Let I be a compact 1-dimensional manifold and $\gamma: I \rightarrow S \setminus (\Lambda \cup D)$ a generic enough properly immersed curve, i.e. γ satisfies conditions (1)–(4) from Definition 1.2. Let $\mathrm{Imm}_I(\mathcal{S})$ be the space of all such immersions with C^∞ -topology. We say that two curves in $\mathrm{Imm}_I(\mathcal{S})$ are *isotopic* if they lie in the same component of $\mathrm{Imm}_I(\mathcal{S})$. We give a criterion for the existence of an exact (i.e. such that

$\gamma^*\lambda$ is exact) immersion $\gamma' \in \text{Imm}_I(\mathcal{S})$ isotopic to a given $\gamma \in \text{Imm}_I(\mathcal{S})$, cf. [15, Lemma 11.4].

Observe that the components of $\text{Imm}_I(\mathcal{S})$ are the orbits of the action of $\text{Diff}(\mathcal{S})_0 \times \text{Diff}(I)_0$ on $\text{Imm}_I(\mathcal{S})$ by $(\varphi, \psi)_*\gamma = \varphi \circ \gamma \circ \psi^{-1}$. Indeed, the orbits lie in the same component and if we have an isotopy $\gamma_s \in \text{Imm}_I(\mathcal{S})$, $s \in [0, 1]$, then we can first make the double points fixed throughout the isotopy by an isotopy $\varphi_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$, $\varphi_0 = \text{id}_S$, then make the preimage of each double point in I fixed throughout the isotopy by an isotopy $\psi_s \in \text{Diff}(I)$, $s \in [0, 1]$, $\psi_0 = \text{id}_I$, and then extend the time-dependent vector field $\left. \frac{d\gamma_s(t)}{ds} \right|_{s=s_0} \in C^\infty(I, \gamma^*TS)$, $s_0 \in [0, 1]$ to a time-dependent vector field on S that integrates to an isotopy $\tilde{\varphi}_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$, $\tilde{\varphi}_0 = \text{id}_S$.

Note that if $\gamma \in \text{Imm}_I(\mathcal{S})$ is exact, then the restriction of γ to any union of connected components of I is exact and that we can add a number of embedded segments to γ and it will remain exact. It therefore suffices to study the case when $I = I_0 \sqcup I_1 \sqcup I_2$ where I_0 is a disjoint union of circles, I_1 and I_2 are disjoint unions of segments, and $\gamma|_{I_2}$ is an embedded curve such that $S \setminus \text{Im } \gamma$ is contractible. We call such γ *totally separating*. Observe that in a given connected component $\mathcal{C}$ of $\text{Imm}_I(\mathcal{S})$ either all curves are totally separating or none are. Note that I_2 has $l = 2g + b - 1$ components where g is the genus and b is the number of boundary components of S . In particular, $I_2 = \emptyset$ if and only if S is a disc.

Let $\mathcal{R}$ be the set of regions cut out by γ from S . Let $\Sigma: \mathbb{R}^{\mathcal{R}} \rightarrow \mathbb{R}$ be the linear map $(\alpha_R)_{R \in \mathcal{R}} \mapsto \sum_{R \in \mathcal{R}} \alpha_R$. Note that if (I, γ, P) is a Lagrangian, then

$$\Sigma \left(\int_R d\lambda \right)_{R \in \mathcal{R}} = b$$

by the Stokes theorem and condition (4) in Definition 1.1.

For each $t_+ \in \partial I_2$ let $y_{t_+}: [0, 1] \rightarrow \partial S$ be a path on the boundary that starts at $\gamma(t_+)$ and ends at $\gamma(t_-)$ where $t_- \in \partial I_2$ is the point following t_+ in the order of the orientation of ∂S (y_{t_+} can pass through Λ and t_- can be equal to t_+). Let Y_2 be the set of such paths, it has order $2l$. For $y \in Y_2$ denote by $y_\pm \in \partial I$ the points satisfying $\gamma(y_+) = y(0)$ and $\gamma(y_-) = y(1)$ as in §1.1.3. If $(I_2, \gamma|_{I_2}, P_2)$ is a Lagrangian, then we can define the actions $A(y) = P_2(y_+) - P_2(y_-) - \int_0^1 y^* \lambda$ for $y \in Y_2$.

Let $B: \mathbb{R}^{Y_2} \rightarrow \mathbb{R}^{\pi_0 \partial S}$ be the linear map

$$(2.5) \quad (A_y)_{y \in Y_2} \mapsto \left(\sum_{y \in Y_2: \text{Im } y \subset C} A_y \right)_{C \in \pi_0 \partial S}.$$

Note that if $(I_2, \gamma|_{I_2}, P_2)$ is a Lagrangian, then $B(A(y))_{y \in Y_2} = (-1, \dots, -1)$ by condition (4) in Definition 1.1. If $Y_2 = \emptyset$ we define $B: \{0\} \rightarrow \mathbb{R}$ by $B(0) = -1$.

Fix an orientation of I . Then we have a linear map $E: \mathbb{R}^{\mathcal{R}} \oplus \mathbb{R}^{Y_2} \rightarrow \mathbb{R}^{\pi_0 I_0}$ that, given the areas $(\alpha_R)_{R \in \mathcal{R}}$ of the regions and the actions $(A_y)_{y \in Y_2}$ of the paths computes the integrals $(\int_C \gamma^* \lambda)_{C \in \pi_0 I_0}$ by the Stokes theorem. Note that the definitions of Σ , B , and E only depends on the component of γ in $\text{Imm}_I(\mathcal{S})$.

Lemma 2.9. *Let $\mathcal{C}$ be a component of $\text{Imm}_I(\mathcal{S})$ consisting of totally separating immersions. Then there exists an exact immersion in $\mathcal{C}$ if and only if there exist $\alpha_R > 0$, $R \in \mathcal{R}$ and $A_y \in \mathbb{R}$, $y \in Y_2$ such that*

$$\Sigma(\alpha_R)_{R \in \mathcal{R}} = b, \quad B(A_y)_{y \in Y_2} = (-1, \dots, -1), \quad \text{and} \quad E((\alpha_R)_{R \in \mathcal{R}}, (A_y)_{y \in Y_2}) = 0.$$

PROOF. The ‘only if’ part is clear from the discussion above. Assume we have $(\alpha_R)_{R \in \mathcal{R}}$ and $(A_y)_{y \in Y_2}$ as in the statement. We want to find an immersion $\gamma \in \mathcal{C}$ and a function $P: I \rightarrow \mathbb{R}$ such that $\gamma^* \lambda + dP = 0$.

Pick an immersion $\gamma_0 \in \mathcal{C}$. By the condition on B we can pick $2l$ numbers $P(t)$, $t \in \partial I_2$ such that $A(y) = P(y_+) - P(y_-) - \int_0^1 y^* \lambda = A_y$ for each $y \in Y_2$. Recall that $\gamma_0|_{I_2}$ cuts S to form one polygon. By deforming the interiors of two sides of this polygon corresponding to a component C of I we can achieve any value of $\int_C \gamma^* \lambda$. Since this deformation can be realised by an isotopy in $\text{Diff}(\mathcal{S})$, we can obtain a new immersion $\gamma_1 \in \mathcal{C}$ such that $P(t_1) - P(t_0) + \int_{t_0}^{t_1} \gamma_1^* \lambda = 0$ for each component $C \in \pi_0 I_2$ where $\partial C = \{t_0, t_1\}$. Hence we extend P to I_2 to satisfy $(\gamma^* \lambda + dP)|_{I_2} = 0$.

We now proceed by removing the regions R from $\mathcal{R}$ one by one after ensuring that $\int_R d\lambda = \alpha_R$. We start with the polygon $\mathcal{T}_1 = (T, D)$ cut out by $\gamma_1|_{I_2}$ from S , the curve $\tilde{\gamma}_1 = \gamma_1|_{I \setminus I_2}$, and $\mathcal{R}_1 = \mathcal{R}$. At each step we have a polygon $\mathcal{T}_k = (T_k \subset T, D_k = D \cap T_k)$ and a curve $\tilde{\gamma}_k$ cutting T_k into regions $\mathcal{R}_k \subset \mathcal{R}$ such that

$$\sum_{R \in \mathcal{R}_k} \int_R d\lambda = \sum_{R \in \mathcal{R}_k} \alpha_R.$$

We take a region $R \in \mathcal{R}_k$ adjacent to ∂T_k such that $T_k \setminus R$ is connected. If $\partial R \subset \partial T_k$, then $R = T_k$, $\int_R d\lambda = \alpha_R$ and we are done. Otherwise, we can find an isotopy

$$\varphi_s \in \text{Diff}(\mathcal{T}) = \{\varphi \in \text{Diff}(T): \varphi|_{\partial S} = \text{id}, \varphi(D_k) = D_k\}, \quad s \in [0, 1], \quad \varphi_0 = \text{id}_T$$

such that $\int_{\varphi_1(R)} d\lambda = \alpha_R$. We set T_{k+1} to be the closure (in T_k) of $T_k \setminus \varphi_1(R)$ and $\tilde{\gamma}_{k+1}$ to be the part of $\varphi_1 \circ \tilde{\gamma}_k$ that lies inside T_{k+1} .

After the procedure above we obtain an immersion $\gamma \in \mathcal{C}$ such that $\int_R d\lambda = \alpha_R$ for each $R \in \mathcal{R}$ and $A(y) = A_y$ for each $y \in Y_2$. The vanishing of E now implies that γ is exact and we are done. $\square$

2.2.2. Lagrangian structures on a given exact curve. Let $\gamma \in \text{Imm}_I(\mathcal{S})$ be an exact immersed curve. We study Lagrangians in $\mathcal{S}$ with underlying curve γ .

Let $L = (I, \gamma, P)$ be such Lagrangian. Recall that $Y = Y_L$ is the set of all Reeb chords in $\partial S \setminus \Lambda$ between the ends of L . For $B \in \mathbb{R}$ define

$$(2.6) \quad Y^{>B} = \{y \in Y : A(y) > B\} \text{ and } Y^{<B} = \{y \in Y : A(y) < B\}.$$

We make the following simple observation

Lemma 2.10. *Let B be a real number. Then $Y^{>B}$ is a finite set and $Y^{<B}$ is finite if and only if for any $t \in \partial I$ there is a stop on the component C of ∂S with $\gamma(t) \in C$.*

PROOF. ∂I is finite and for fixed $y_+, y_- \in \partial I$ the set of Reeb chords from $\gamma(y_+)$ to $\gamma(y_-)$ either contains at most one element or is equal to $\{yz^k \mid k \in \mathbb{Z}_{\geq 0}\}$ where y is the shortest Reeb chord from $\gamma(y_+)$ to $\gamma(y_-)$ and z is the Reeb chord from $\gamma(y_+)$ to itself that goes around the component of ∂S once. As $A(yz^{k+1}) - A(yz^k) = -l_z < 0$, only finitely many of these chords will have action larger than B . On the other hand, if $\gamma(t) \in C$ for some $t \in \partial I$ and a component $C \in \pi_0 \partial S$ with $C \cap \Lambda = \emptyset$, then $A(z^k) < B$ for $k \gg 0$ where z is the Reeb chord from $\gamma(t)$ to itself going around C once. $\square$

Given L we obtain Reeb signs at each double point of γ (recall that this is the same information as specifying under- and over-crossings) and the set $Y_L^{>0}$ of Reeb chords of positive action. Recall that this is the only information on γ we need to compute the obstruction algebra $\mathcal{A}_L$ for a Lagrangian brane supported on L .

Let $\gamma \in \text{Imm}_I(\mathcal{S})$ be a totally separating exact curve with Reeb signs at each double point $x \in X$ and a finite set $Y^{>0} \subset Y$ of Reeb chords between the ends of γ specified. We give a criterion for when there exists a Lagrangian supported on γ inducing this information, cf. [15, Proposition 11.2].

Let Y_{short} be the set of paths $y: [0, 1] \rightarrow \partial S$ that follow the orientation of ∂S , start at $\gamma(y_+)$, end at $\gamma(y_-)$ for some $y_{\pm} \in \partial I$, and do not have the ends of γ in the interior. The number of paths in Y_{short} is twice the number of segment connected components of I .

Recall that $\mathcal{R}$ is the set of regions cut out by γ from S . We have a linear map $\text{Area}: \mathbb{R}^{X \cup Y_{\text{short}}} \rightarrow \mathbb{R}^{\mathcal{R}}$ that computes the area of the region R by the Stokes theorem (1.9). We also have the map $F: \mathbb{R}^{Y_{\text{short}}} \rightarrow \mathbb{R}^Y$ computing the action of each $y \in Y$ from the actions on Y_{short} by the additivity with respect to concatenation and the map $\tilde{B}: \mathbb{R}^{Y_{\text{short}}} \rightarrow H^1(\gamma(I_2) \cup \partial S; \mathbb{R})$ that, given a closed curve σ in $\gamma(I_2) \cup \partial S$ thought of as a 1-cycle, computes $\sum_{y \in Y_{\text{short}}} \deg_{\sigma}(y)$ where $\deg_{\sigma}(y)$ is the signed number of times σ covers y .

Lemma 2.11. *Let $\gamma \in \text{Imm}_I(\mathcal{S})$ be a totally separating curve with Reeb signs at each double point $x \in X$ and a finite set $Y^{>0} \subset Y$ of Reeb chords between the ends of γ specified. Then there exists a Lagrangian $L = (I, \gamma, P)$ inducing the same Reeb signs and with $Y_L^{>0} = Y^{>0}$ if and only if there exists $A_x > 0$, $x \in X$ and $A_y \in \mathbb{R}$, $y \in Y_{\text{short}}$ such that*

- (1) $(F(A_y)_{y \in Y_{\text{short}}})_y > 0$ for $y \in Y^{>0}$;
- (2) $(F(A_y)_{y \in Y_{\text{short}}})_y < 0$ for $y \in Y \setminus Y^{>0}$;
- (3) $\text{Area}((A_x)_{x \in X}, (A_y)_{y \in Y_{\text{short}}}) = (\int_{\mathbb{R}} d\lambda)_{\mathbb{R} \in \mathbb{R}}$;
- (4) $\tilde{B}(A_y)_{y \in Y_{\text{short}}} = \lambda|_{\gamma(I_2) \cup \partial S}$ in $H^1(\gamma(I_2) \cup \partial S; \mathbb{R})$.

Note that it is enough to check condition (2) on the shortest Reeb chord $yz^k \in Y \setminus Y^{>0}$ from the infinite set $\{yz^k \mid k \geq 0\}$ where z goes around a component of ∂S once. Hence there are finitely many conditions to check in Lemma 2.11.

PROOF. The ‘only if’ part is clear. To prove the ‘if’ part we construct a function $P: I \rightarrow \mathbb{R}$ such that $\gamma^*\lambda + dP = 0$, $A(x) = P(x_+) - P(x_-) = A_x$ for each $x \in X$, and $A(y) = P(y_+) - P(y_-) - \int_0^1 y^*\lambda = A_y$ for each $y \in Y_{\text{short}}$. Then $L = (I, \gamma, P)$ is a Lagrangian since $A(z) \neq 0$ for each $z \in X \cup Y$, it induces the the correct Reeb signs since $A_x > 0$ for $x \in X$ and $Y_L^{>0} = Y^{>0}$ by (1) and (2).

Define $P(t) = 0$ for some $t \in I$ and spread using $\gamma^*\lambda + dP = 0$ and A_z for $z \in X \cup Y_{\text{short}}$ to all I (if $\text{Im } \gamma \cup \partial S$ is not connected, then do this separately for each component). No conflict arises by conditions (3) and (4) since any closed curve in $\text{Im } \gamma \cup \partial S$ is a linear combination (as a 1-chain) of boundaries of the regions from Y and closed curves in $\gamma(I_2) \cup \partial S$. $\square$

2.2.3. Lagrangian isotopy. Recall that two Lagrangians with the same domain I are Lagrangian isotopic if they lie in the same component of $\text{Lag}_I(\mathcal{S})$, see §1.5.1.

The group $\text{Symp}_{\text{ex}}(\mathcal{S})$ acts on $\text{Lag}_I(\mathcal{S})$ by

$$\varphi_*(I, \gamma, P) = (I, \varphi \circ \gamma, P - \gamma^*g)$$

where $g \in C^\infty(S, \mathbb{R})$ is such that $\varphi^*\lambda - \lambda = dg$ and $g(q_0) = 0$ for some fixed basepoint $q_0 \in S$. Note that φ_*L is Lagrangian isotopic to L for $\varphi \in \text{Ham}(\mathcal{S})$.

We give a topological characterisation of Lagrangian isotopy.

Lemma 2.12. *Let $L_i = (I, \gamma_i, P_i)$, $i = 0, 1$ be two Lagrangians in the Liouville marked surface $\mathcal{S}$ and such that γ_0 and γ_1 lie in the same component of $\text{Imm}_I(S)$. Suppose that the Reeb signs at the corresponding double points of L_0 and L_1 agree and the actions of the corresponding Reeb chords between the ends of L_0 and between the ends of L_1 have the same sign.*

Then L_1 is Lagrangian isotopic to L_0 .

PROOF. The isotopy of the domain induces a Lagrangian isotopy so we can assume that there exists an isotopy $\varphi_s \in \text{Diff}(\mathcal{S})$, $s \in [0, 1]$ such that $\varphi_1 \circ \gamma_0 = \gamma_1$. To prove the lemma it suffices to find a path φ_s , $s \in [1, 2]$ in $\text{Diff}(\mathcal{S})$ such that $\varphi_2 \in \text{Symp}_{\text{ex}}(\mathcal{S})_0$ and the Lagrangian $L_2 = (\varphi_2)_* L_0$ is Lagrangian isotopic to L_1 . Indeed, Corollary 2.4 then implies that there exists a path ψ_s , $s \in [0, 1]$ in $\text{Symp}_{\text{ex}}(\mathcal{S})$ such that $\psi_0 = \text{id}_S$ and $\psi_1 = \varphi_2$ hence an isotopy from L_0 to L_1 can be obtained as the concatenation of $s \mapsto (\psi_s)_* L_0$ and the isotopy from L_2 to L_1 .

We now find the desired path φ_s , $s \in [1, 2]$ in $\text{Diff}(\mathcal{S})$. Consider the equation

$$(\varphi_1^{-1})^* \lambda - \lambda + d\lambda_s(X_s, -) = dg_{(\varphi_1^{-1})^* \lambda - \lambda} \text{ where } s \in [0, 1].$$

where $\lambda_s = s(\varphi_1^{-1})^* \lambda + (1-s)\lambda$ and $g_{(\varphi_1^{-1})^* \lambda - \lambda}$ is provided by Lemma 2.7. As in the proof of Proposition 2.3, this equation determines a time-dependent vector field X_s on S which integrates to a family of diffeomorphisms

$$[0, 1] \rightarrow \text{Diff}(\mathcal{S}), \quad s \mapsto \Phi_s, \quad \Phi_0 = \text{id}_S.$$

Cartan's magic formula yields

$$s(\varphi_1^{-1})^* \lambda + (1-s)\lambda = (\Phi_s^{-1})^* \lambda + dh_s \text{ where } h_s = sg_{(\varphi_1^{-1})^* \lambda - \lambda} + \int_0^s \lambda_t(X_t) dt$$

We define the Lagrangian isotopy $L_u = (I, \gamma_u, P_u)$, $u \in [1, 2]$ by

$$\gamma_{s+1} = \Phi_s^{-1} \circ \gamma_1, \quad P_{s+1} = sP_0 + (1-s)P_1 + \gamma_1^* h_s.$$

A straightforward calculation shows that $\gamma_{s+1}^* \lambda + dP_{s+1} = 0$. For any self-intersection point x of γ_{s+1} we have $\gamma_{s+1}(x_+) = x = \gamma_{s+1}(x_-)$ for some $x_+, x_- \in I$ such that $P_i(x_+) > P_i(x_-)$ for $i = 0, 1$ by assumption of the lemma. Therefore,

$$\begin{aligned} P_{s+1}(x_+) &= sP_0(x_+) + (1-s)P_1(x_+) + h_s(x) > \\ &sP_0(x_-) + (1-s)P_1(x_-) + h_s(x) = P_{s+1}(x_-). \end{aligned}$$

For any Reeb chord $y: [0, l] \rightarrow \partial S \setminus \Lambda$ connecting $\gamma_{s+1}(y_+) = y(0)$ and $\gamma_{s+1}(y_-) = y(l)$ we have that $P_0(y_+) - l - P_0(y_-)$ and $P_1(y_+) - l - P_1(y_-)$ are non-zero and have the same sign by assumption, therefore, a computation similar to the above shows that $P_{s+1}(y_+) - l - P_{s+1}(y_-)$ is non-zero. Hence L_u , $u \in [1, 2]$ is indeed a Lagrangian isotopy.

We set $\varphi_{s+1} = \Phi_s^{-1} \circ \varphi_1$, $s \in [0, 1]$. Then

$$\varphi_2^* \lambda - \lambda = \varphi_1^* ((\Phi_1^{-1})^* \lambda - (\varphi_1^{-1})^* \lambda) = d(-\varphi_1^* h_1) \text{ hence } \varphi_2 \in \text{Symp}_{\text{ex}}(\mathcal{S})_0$$

and

$$(\varphi_2)_*L_0 = (I, \Phi_1^{-1} \circ \varphi_1 \circ \gamma_0, P_0 - \gamma_0^*(-\varphi_1^*h_1)) = (I, \Phi_1^{-1} \circ \gamma_1, P_0 + \gamma_1^*h_1) = L_2,$$

as required. $\square$

2.2.4. Legendrian Reidemeister moves. Recall from §1.5.1 that two Lagrangian $L_0, L_1 \in \text{Lag}_I(\mathcal{S})$ are Legendrian isotopic if their Legendrian lifts lie in the same component of $\text{Leg}_I(\mathcal{S})$.

We describe a number of local models for Legendrian isotopies. Let $L = (I, \gamma, P)$ be a Lagrangian in $\mathcal{S}$ and U a contractible open subset of $S \setminus (\Lambda \cup D)$. Suppose that there is a diffeomorphism from U to an open subset of the plane (or the half-plane if $\partial S \cap U \neq \emptyset$) that maps L to either L_0 or L_1 in one of the pictures in Fig. 10. A *Legendrian Reidemeister move* is one of the following operations that replaces L with a Lagrangian L' Legendrian isotopic to L . In each move L is changed only on a subsegment of I , the move happens in U , and the grading is transported by the obvious isotopy. The result of a move below is a Lagrangian which is Legendrian isotopic to L as long as the darker shaded area is equal to the lighter shaded area. All areas are computed with respect to the push-forward of $d\lambda$ from S . We also list sufficient conditions to perform each move and state what assumptions can be made on the result of the move, cf. Kálmán [28, Theorem 4.1].

Move II: passing from L_0 to L_1 in Fig. 10(a). Can be done if $P(t_0) - \int_0^1 \sigma^* \lambda > P(t_1)$. The total area of the regions H and E (which is equal to the total area of A and C) can be made arbitrarily small.

Move II⁻¹: passing from L_1 to L_0 in Fig. 10(a). Can be done if the area of the bigon H is smaller than the area of any other region cut out from S by L_1 . The area of the region E can be made arbitrarily small.

Move II_{>0} (resp. *II_{<0}*): passing from L_0 to L_1 in Fig. 10(b) (resp. Fig. 10(c)). Can always be done. The total area of the regions D and F (which is equal to the area of B) can be made arbitrarily small.

Move II_{>0}⁻¹ (resp. *II_{<0}⁻¹*): passing from L_1 to L_0 in Fig. 10(b) (resp. Fig. 10(c)). Can be done if the area of the region F is smaller than the area of any other region cut out from S by L_1 . The area of the region D can be made arbitrarily small.

Move III_a (resp. *III_b*): passing from L_0 to L_1 in Fig. 10(d) (resp. Fig. 10(e)). Can be done if the area of the triangle H is smaller than the area of any other region cut out from S by L_0 and that the action of any self-intersection point or Reeb chord $z \in Z_{L_0}$. The total area of the regions G , I , and K can be made arbitrarily small.

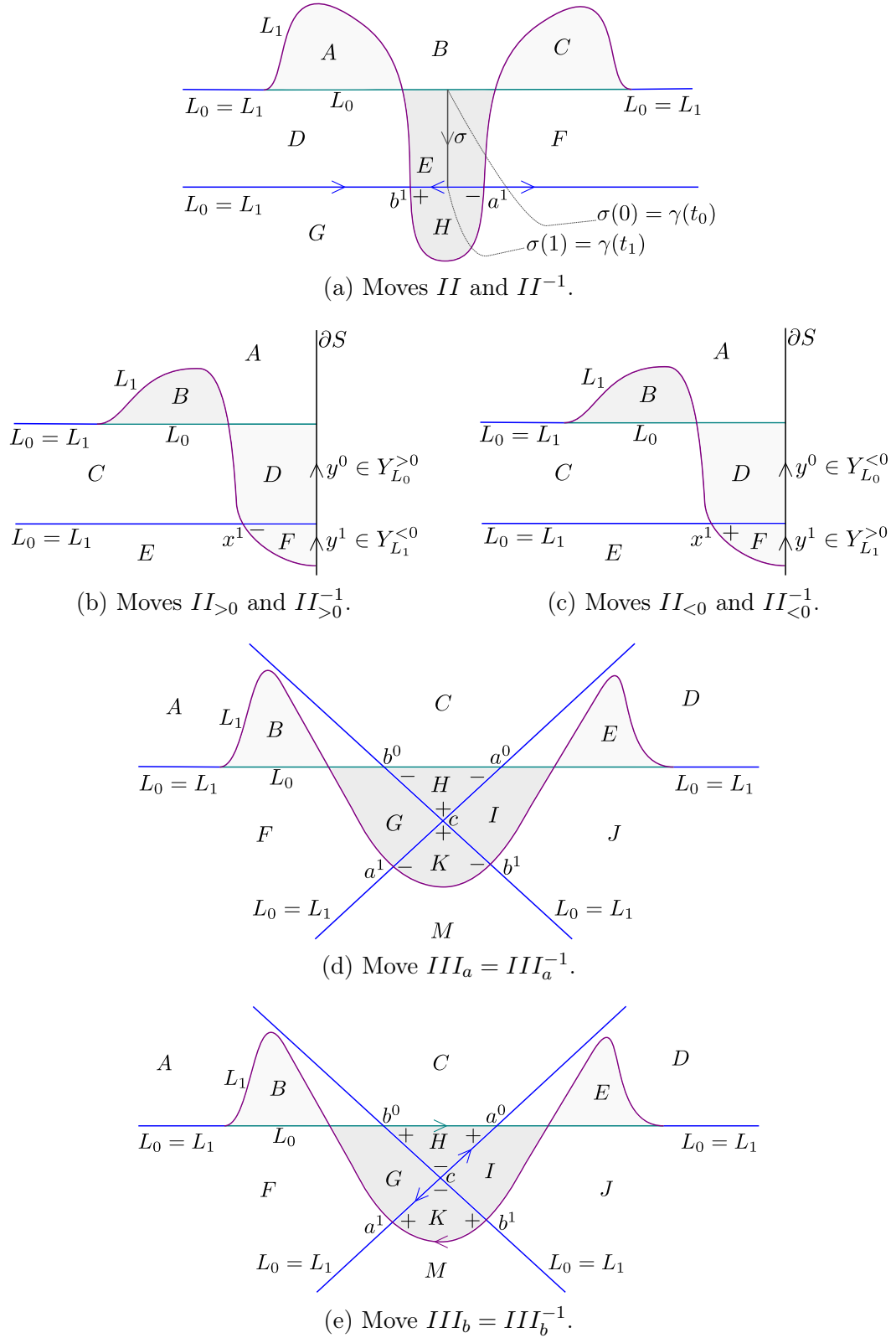


FIGURE 10. Legendrian Reidemeister moves.

Note that the condition for Move II is not actually a constraint, but a way to orient the picture: for a generic choice of σ we have $P(t_0) - \int_0^1 \sigma^* \lambda \neq P(t_1)$

and we can rotate L_0 in Fig. 10(a) by π around the center of the picture to satisfy the condition.

Our sufficient conditions for the moves are not necessary. For example, to perform move II^{-1} it is enough for the total area of the regions cut out by L_1 and containing the regions A , C , D , and F from Fig. 10(a) to be larger than the area of H . However, we only perform the Legendrian Reidemeister moves if the sufficient conditions listed above are satisfied.

2.2.5. Decomposition of Legendrian isotopy. The following lemma quantifies the difference between Lagrangian and Legendrian isotopies and will allow us to reduce proofs of invariance of algebraic structures under Legendrian isotopies to combinatorics.

Lemma 2.13. *Let L and L' be two Lagrangians in the Liouville marked surface S . Then L and L' are Legendrian isotopic if and only if there is a sequence of Lagrangians $L = L_0, L_1, \dots, L_n = L'$ such that L_{i-1} and L_i are either Lagrangian isotopic or differ by a Legendrian Reidemeister move for any $i = 1, \dots, n$. Moreover, the Legendrian Reidemeister moves in the sequence can be chosen to satisfy the sufficient conditions and assumptions for them listed in §2.2.4.*

PROOF. The ‘if’ part is clear as the Lagrangians before and after a Legendrian Reidemeister move are Legendrian isotopic. We now prove the ‘only if’ part.

For a generic Legendrian isotopy $\{L_s^+\}_{s \in [0,1]}$ between L and L' there exists $0 < s_1 < \dots < s_m < 1$ such that L_s^+ is a Legendrian lift of a Lagrangian L_s for $s \in [0, 1] \setminus \{s_1, \dots, s_m\}$ and for $i = 1, \dots, m$ the projection of $L_{s_i}^+$ to S has either a unique triple point or a simple self-tangency.

Fix $i = 1, \dots, m$. From the above it follows that for small $\varepsilon > 0$ we can perform a Legendrian Reidemeister move on $L_{s_i - \varepsilon}$ to obtain a Lagrangian L'_i such that L'_i and $L_{s_i + \varepsilon}$ satisfy the assumptions of Lemma 2.12, hence L'_i and $L_{s_i + \varepsilon}$ are Lagrangian isotopic. Namely, if L_{s_i} had a triple point, then we do either move $IIIa$ or move $IIIb$; if L_{s_i} has a simple self-tangency at an interior point, then we do a move II or II^{-1} ; and if L_{s_i} has a simple self-tangency at a boundary point, then we do a move $II_{>0}$, $II_{>0}^{-1}$, $II_{<0}$, or $II_{<0}^{-1}$.

Choosing ε small enough we ensure that the conditions for the move from §2.2.4 are satisfied for $L_{s_i - \varepsilon}$. Indeed, the area that needs to be smaller tends to 0 as $s \rightarrow s_i - 0$, while the areas and actions that need to be larger are bounded from below for L_s with $s \in [s_i - \varepsilon, s_i + \varepsilon]$.

Therefore, choosing one ε satisfying all the conditions above, we get the required sequence $(L, L_{s_1 - \varepsilon}, L'_1, L_{s_1 + \varepsilon}, L_{s_2 - \varepsilon}, \dots, L_{s_m + \varepsilon}, L')$. $\square$

We prove that the move $II_{<0}$ (resp. $II_{<0}^{-1}$) can be replaced by first an application of move II (resp. $II_{>0}$) and then of move $II_{>0}^{-1}$ (resp. II^{-1}).

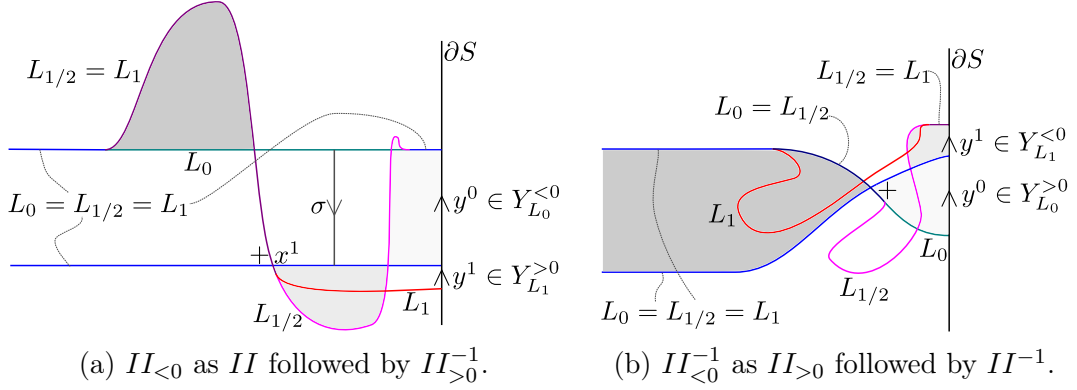


FIGURE 11. Redundancy of Legendrian Reidemeister moves $II_{<0}$ and $II_{<0}^{-1}$.

Lemma 2.14. *In Lemma 2.13 the Legendrian Reidemeister moves $II_{<0}$ and $II_{<0}^{-1}$ are not needed.*

In view of Lemma 2.14 we will not discuss the moves $II_{<0}^{\pm 1}$ in the rest of the paper.

PROOF. Suppose L_1 can be obtained from L_0 by a move $II_{<0}$, as in Fig. 11(a). As $A(y^0) < 0$, if the path σ is close enough to y^0 , then the condition to perform move II is satisfied. We perform move II to get $L_{1/2}$ in Fig. 11(a) ensuring that the lighter shaded area is much smaller than the areas of all other regions cut out by $L_{1/2}$, except the medium shaded area. By increasing the medium shaded area (and therefore increasing the darker shaded area) we can achieve that the medium shaded area is larger than the lighter shaded area. Therefore, the condition to perform the move $II_{>0}^{-1}$ on $L_{1/2}$ is satisfied, and by performing this move we obtain L_1 .

Suppose L_1 can be obtained from L_0 by a move $II_{<0}^{-1}$, as in Fig. 11(b) and the sufficient condition for the move is satisfied, in particular, the darker shaded area is larger than the lighter shaded area. Perform the move $II_{>0}$ on L_0 to obtain $L_{1/2}$ in Fig. 11(b) in such a way that the area of the medium shaded region is small enough for the area of the bigon bounded by $L_{1/2}$ be less than the areas of all other regions cut out by $L_{1/2}$, except the medium shaded area. Increasing the medium shaded area (and therefore decreasing the action of y^1) we make the area of the bigon bounded by $L_{1/2}$ smaller than the areas of all other regions cut out by $L_{1/2}$. Therefore, the sufficient condition to perform the move II^{-1} on $L_{1/2}$ is satisfied, and performing this move we obtain L_1 . $\square$

2.2.6. Gradings. Let η be a line field on S and $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow S \setminus (\Lambda \cup D)$ a properly immersed curve. The tangent line $\dot{\gamma}$ is a section of $\gamma^*\mathbb{P}TS$ and η gives a trivialisation of $\mathbb{P}TS$, hence we obtain the *winding number* $w_\gamma(\eta) \in \mathbb{Z}$ as the degree of $\dot{\gamma}$. We have the following easy lemma characterising gradings on a given curve.

Lemma 2.15. *Let $\gamma: I \rightarrow S \setminus (\Lambda \cup D)$ be a properly immersed curved in a graded surface $(S; \Lambda, D; \eta)$. Then*

- (1) *γ admits a grading $\text{gr} \in \Pi_1 C^\infty(I, \gamma^*\mathbb{P}TS)(\gamma^*\eta, \dot{\gamma})$ if and only if the winding number $w_{\gamma|_C}(\eta)$ vanishes for each circle component C of I .*
- (2) *If γ admits a grading, then the set of gradings on γ is a torsor over $\mathbb{Z}^{\pi_0 I}$.*

PROOF. (1) is clear since given a path gr_t from $\gamma^*\eta(t)$ to $\dot{\gamma}(t)$ for some $t \in I$ there is a homotopically unique way to extend it to a path $i^*\text{gr}$ from $i^*\gamma^*\eta$ to $i^*\dot{\gamma}$ for any immersion $i: [0, 1] \rightarrow C$ with $i(0) = t$. If $i: [0, 1] \rightarrow C$ is a parametrisation of a circle component C of I with $i(0) = i(1) = t$, then the winding number $w_{\gamma|_C}(\eta)$ is equal to

$$i^*\text{gr}(0)^{-1} \cdot i^*\text{gr}(1) \in \pi_1(\mathbb{P}T_{\gamma(t)}S, \eta(\gamma(t))) = \mathbb{Z}.$$

(2) follows from the above since

$$\pi_1(C^\infty(C, \gamma^*\mathbb{P}TS), \gamma^*\eta) = \pi_1(\mathbb{P}T_{\gamma(t)}S, \eta(\gamma(t))) = \mathbb{Z}$$

for any connected component C of I and $t \in C$. $\square$

Lemma 2.15 implies that grading is discrete data defined for any properly immersed curve in $S \setminus (\Lambda \cup D)$ hence it is transported along isotopies of such curves.

The group $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S})$ acts on the space of graded Lagrangian $\text{Lag}_I^{\text{gr}}(\mathcal{S})$ by

$$(\varphi, \tilde{\varphi})_*(L, \text{gr}) = (\varphi_*L, \varphi_*(\text{gr}) \cdot \tilde{\varphi})$$

for $(\varphi, \tilde{\varphi}) \in \text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S})$ and a graded Lagrangian (L, gr) . The action on $\text{Leg}_I^{\text{gr}}(\mathcal{S})$ is defined analogously. Because of the identification $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S})_0 = \text{Ham}(\mathcal{S})$ in §2.1.4 we also have the action of Hamiltonian diffeomorphisms on graded Lagrangians and Legendrians.

2.3. Polygons and wrapping

Let $\mathcal{S} = (S; \Lambda, D; \lambda; \eta)$ be a Liouville graded marked surface. In this section we prove that successive application of move $II_{>0}$ can make any Lagrangian *sufficiently self-wrapped* meaning that there are no Reeb chords of positive action between the ends of L , i.e. $Y_L^{>0} = \emptyset$.

In §2.3.1 we give the general action and degree properties of polygons bounded by a Lagrangian. In §2.3.2 we prove that if a Lagrangian L_1 differs

from L_0 by a move $II_{>0}$, then there are natural bijections between the sets Z_{L_0} and Z_{L_1} and between sets of polygons bounded by L_0 and L_1 . In §2.3.3 we explain how to get rid of the Reeb chords of positive action by a repetitive application of move $II_{>0}$.

2.3.1. Action and degree properties of polygons. Let $L = (I, \gamma, P, \text{gr})$ be a graded Lagrangian in $\mathcal{S}$. Recall from §1.1.3 the set Z_L of double points of L and Reeb chords of positive action between the ends of L . Recall that in Definition 1.5 we defined the sets of polygons $\Delta(L; z^0; z^n, \dots, z^1)$ that are bounded by L and have a ‘positive corner’ at z^0 and ‘negative corners’ at $z^1, \dots, z^n$, and that the elements of these sets are actually polygons except in the case (c) from Definition 1.5, when z^0, z^1, z^2 are Reeb chords and $z^0 = z^2 z^1$.

Lemma 2.16.

- (1) *If the case (c) in Definition 1.5 holds, then cases (a) and (b) do not hold.*
 (2) *Suppose that $\Delta^k(L; z^0; z^n, \dots, z^1) \neq \emptyset$ for some $k \geq 0$, $z^0, z^1, \dots, z^n \in Z$. Then*

$$0 < \sum_{i=1}^n A(z^i) \leq A(z^0) \quad \text{and} \quad |z^0| = 2 - n + k + \sum_{i=1}^n |z^i|.$$

Moreover, the inequality on actions is strict unless we are in the case (c) in Definition 1.5.

- (3) *The set*

$$\bigcup_{\substack{n \geq 0, \\ z^0, z^1, \dots, z^n \in Z}} \Delta(L; z^0; z^n, \dots, z^1)$$

is finite.

PROOF. Suppose the case (c) in Definition 1.5 holds, i.e. $z^0 = z^2 z^1$ for $z^0, z^1, z^2 \in Y^{>0}$. Then z^1 and z^2 are sub-chords of z^0 . But if (a) (resp. (b)) holds, then z^0 is a sub-chord of z^1 (resp. z^2). This proves (1). Also note that in this case $|z^0| = |z^1| + |z^2|$ and $A(z^0) = A(z^1) + A(z^2)$, so (2) holds.

We prove (2) in the remaining cases. Pick a polygon $\mathbf{u} = (\Pi, u) \in \Delta^k(L; z^0; z^n, \dots, z^1)$. From (1.9) and additivity of action with respect to the concatenation of Reeb chords we get

$$(2.7) \quad 0 < \int_{\Pi} u^* d\lambda = A(z^0) - \sum_{i=1}^n A(z^i) < A(z^0).$$

This proves the inequalities for the action in (2). The formula for the degree in (2) follows from the following observation.

Let $\mathbf{u} = (\Pi, u)$ be any polygon bounded by L . Since the winding number of $\partial\Pi(u)$ with respect to $u^*\eta$ (or any other line field on Π) is 2, we obtain

$$(2.8) \quad \sum_{w \in W^+(\mathbf{u})} |u(w)| + \sum_{w \in W^-(\mathbf{u})} (1 - |u(w)|) + \sum_{\substack{w \in W(\mathbf{u}): \\ u(w) \in Y}} (1 - |u(w)|) = 2 + \text{vdim } \mathbf{u}.$$

Regrouping the terms for $\mathbf{u} \in \Delta^k(L; z^0; z^n, \dots, z^1)$ we obtain the desired formula in (2).

To show (3), as Z is finite, (2.7) implies that it suffices to prove that for any $C > 0$ there are only finitely many (equivalence classes of) polygons bounded by L of area not exceeding C . This is indeed the case since each polygon bounded by L is obtained by gluing copies of the regions into which L divides S along the sides of these regions (a side is a smooth segments of the boundary of the region between two points from X or between a point from X and $\gamma(\partial I)$). There are finitely many such regions, each of them has positive area and finitely many sides, so there are only finitely many ways to glue a number of their copies to get the total area of no more than C . $\square$

2.3.2. Move $II_{>0}$. Let L_0 and L_1 be graded Lagrangians in $\mathcal{S}$. Suppose that L_1 is obtained from L_0 by the Legendrian Reidemeister move $II_{>0}$, as in Fig. 10(b). We have $X_{L_1} = X_{L_0} \cup \{x^1\}$. For the unique $t \in \partial I$ such that $\gamma_0(t) \neq \gamma_1(t)$ let $r_t = y^0 y^1$, otherwise let r_t be the constant path at $\gamma_0(t) = \gamma_1(t)$. Define the map $\psi: X_{L_0} \cup Y_{L_0} \rightarrow X_{L_1} \cup Y_{L_1}$ by

$$(2.9) \quad \psi(z) = \begin{cases} z, & \text{if } z \in X_{L_0}; \\ x^0, & \text{if } z = y^0; \\ z', & \text{such that } z r_{z_-} = r_{z_+} z' \text{ if } z \in Y_{L_0} \setminus \{y^0\}. \end{cases}$$

In other words, on $Y_{L_0} \setminus \{y^0\}$ the map ψ is defined by continuously deforming the Reeb chord by the isotopy from L_0 to L_1 .

Lemma 2.17.

- (1) ψ is a bijection from $X_{L_0} \cup Y_{L_0}$ to $X_{L_1} \cup Y_{L_1} \setminus \{y^1\}$;
- (2) ψ preserves the degree;
- (3) for $z \in X_{L_0} \cup Y_{L_0}$ we have

$$A(\psi(z)) = \begin{cases} A(z) & \text{if } z \neq y^0; \\ A(y^0) - \text{Area}(\text{region } F) & \text{if } z = y^0. \end{cases}$$

PROOF. (1) is clear from the construction. To prove (2) note that the degree is a discrete invariant, so it is constant in continuous families, therefore

the only thing to check is that $|x^0| = |y^0|$, which follows from the compatibility between the definitions of the degree for self-intersection points and Reeb chords in §1.1.3.

We prove (3). If $z \in X_{L_0}$, then $\psi(z) = z$ and there is nothing to prove. If $z \in Y_{L_0} \setminus \{y^0\}$, then

$$A(\psi(z)) = (P_{L_0}(z_+) - l_{r_{z_+}}) - (P_{L_0}(z_-) - l_{r_{z_-}}) - (l_z + l_{r_{z_-}} - l_{r_{z_+}}) = A(z).$$

For $z = y^0$ the formula follows from (1.9) applied to region D and the fact that $\text{Area}(B) = \text{Area}(D) + \text{Area}(F)$. $\square$

Next we analyse the effect of going from L_0 to L_1 on polygons. Define the map

$$(2.10) \quad \psi_*: \Delta^0(L_0; z^0; z^n, \dots, z^1) \rightarrow \Delta^0(L_1; \psi(z^0); \psi(z^n), \dots, \psi(z^1))$$

as follows.

Construction 2.18. Suppose we are in the case (c) of Definition 1.5, i.e. $n = 2$, $z^0, z^1, z^2 \in Y^{>0}$, $z^0 = z^2 z^1$, and $\Delta(L_0; z^0; z^2, z^1) = \{*\}$. Note that $z^0 \neq y^0$ since y^0 doesn't have ends of L_0 in its interior.

- (a) If $z^1 = y^0$, then $\psi(z^0) = z^0$, $\psi(z^1) = x^1$, and $\psi(z^2) = z^1 y^0 y^1$. We set $\psi_*(*) \in \Delta^0(L'; z^0; \psi(z^2), x^1)$ to be the polygon corresponding to the shaded area in Fig. 12(a).
- (b) If $z^2 = y^0$, then $y^0 y^1 \psi(z^0) = z_0$, $\psi(z^1) = z^1$, and $\psi(z^2) = x^1$. We set $\psi_*(*) \in \Delta^0(L'; \psi(z^0); x^1, z^1)$ to be the polygon corresponding to the shaded area in Fig. 12(b).
- In the rest of the cases $\psi(z^0) = \psi(z^2)\psi(z^1)$ and we set $\psi_*(*)$ to be the unique element of $\Delta^0(L'; \psi(z^0); \psi(z^2), \psi(z^1))$.

Now suppose we are not in the case (c) of Definition 1.5 and $\mathbf{u} = (\Pi, u)$ is a polygon bounded by L in $\Delta^0(L; z^0; z^n, \dots, z^1)$ with $W(\mathbf{u}) = \{w^0, \dots, w^m\}$ ($m \in \{n-1, n\}$). We do the following modifications to $\mathbf{u}$ near $W(\mathbf{u})$ to obtain the polygon $\psi_*(\mathbf{u})$ in $\Delta^0(L'; \psi(z^0); \psi(z^n), \dots, \psi(z^1))$.

- If $u(w^i) \in X_{L_0}$ or $u(w^i) \in Y_{L_0}$ and $u(w^i)_\pm \neq y^0_+$ for some $i \in \{0, \dots, m\}$, then there are no modifications near w^i .
- (c) If $z^0 = y^0$ and $\mathbf{u}$ is in the case (a) from Definition 1.5, then modify $\mathbf{u}$ by going from the vertically stripped area to the horizontally stripped area in Fig. 12(c).
- (d) If $z^0 = y^0$ and $\mathbf{u}$ is in the case (b) from Definition 1.5, then modify $\mathbf{u}$ by going from vertically stripped area to the horizontally stripped area in Fig. 12(d).
- (e) If $z^i = y^0$ for some $i \in \{1, \dots, n\}$, then then modify $\mathbf{u}$ by going from vertically stripped area to the horizontally stripped area in Fig. 12(e).

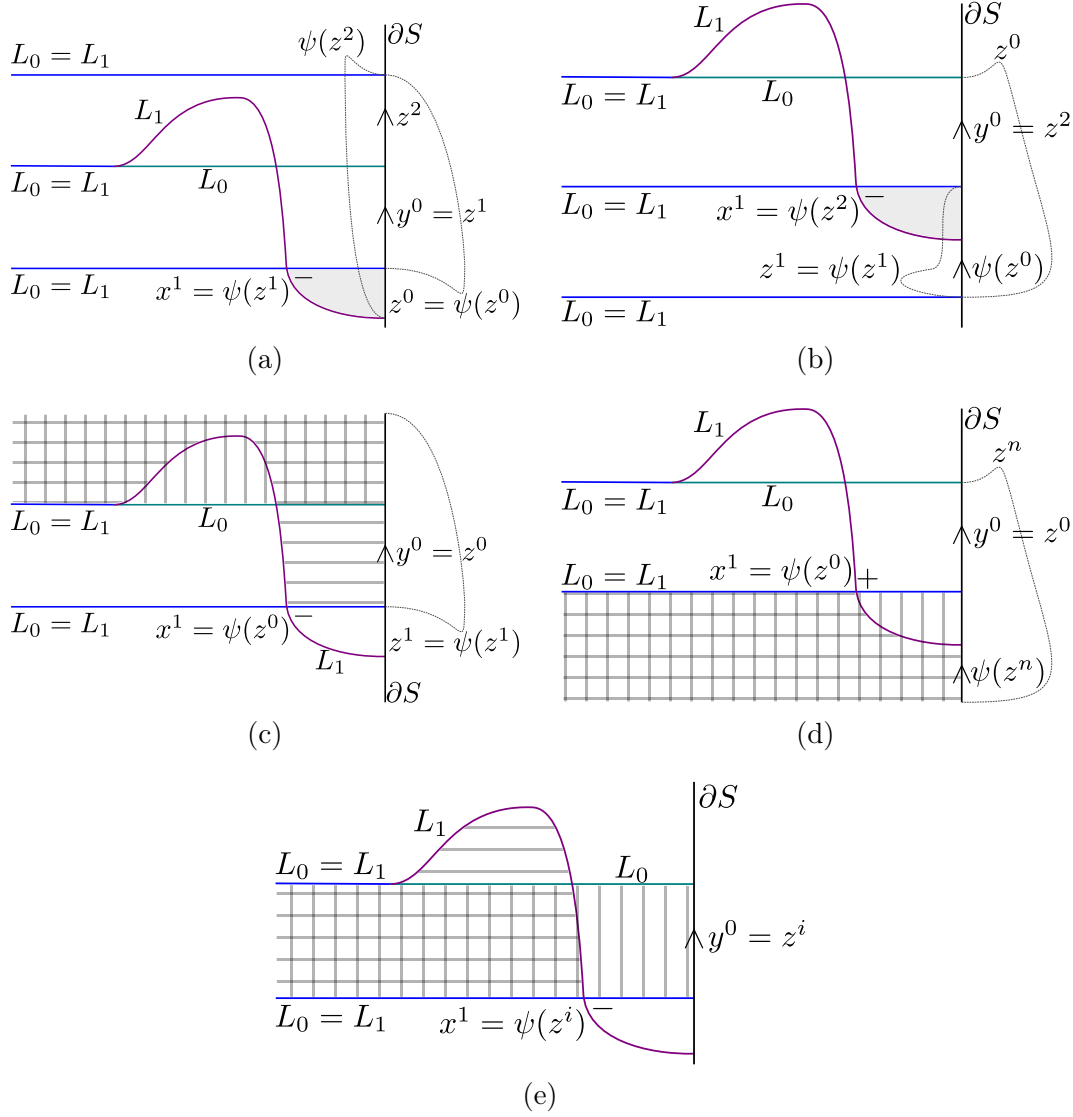


FIGURE 12. Effect of Legendrian Reidemeister move $II_{>0}$ on polygons.

- (f) If $u(w^i) \in Y \setminus \{y^0\}$ and $u(w^i)_+ = y_+^0$ or $u(w^i)_- = y_+^0$ for some $i \in \{0, \dots, m\}$, then continuously deform $\mathbf{u}$ near w^i to a polygon that sends w^i to $\psi(u(w^i))$.

Lemma 2.19. *For any $n \geq 0$ and $z^0, \dots, z^n \in Z_{L_0}$ the map ψ_* in (2.10) is a bijection*

PROOF. All the modifications of Construction 2.18 are clearly invertible, so the only thing to check is that all elements of $\Delta^0(L_1; \psi(z^0); \psi(z^n), \dots, \psi(z^1))$ are in the image of ψ_* .

Fix $z^0, \dots, z^n \in Z_{L_1}$. If $n = 2$, $z^0, z^1, z^2 \in Y_{L_1}^{>0}$ and $\psi(z^0) = \psi(z^2)\psi(z^1)$, then $z^0 = z^2z^1$ and we are done. Otherwise, consider a polygon $\mathbf{u} = (\Pi, u)$ in $\Delta^0(L_1; \psi(z^0); \psi(z^n), \dots, \psi(z^1))$.

If the part of L_1 that differs from L_0 (the purple curve in Fig. 10(b)) does not intersect $u(\partial\Pi)$, then $\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)$ and $\psi(\mathbf{u}) = \mathbf{u}$. If it

does, then consider a connected component $C \subset \partial\Pi$ of the preimage of this intersection under u . C is diffeomorphic to $(0, 1]$ and the point corresponding to 1 is mapped to either x^1 or $\gamma_1(y_-^1)$. If the former is the case, then $\mathbf{u}$ in the neighbourhood of C is the result of Construction 2.18 a), b) c), d) e) depending on which quadrant of x^1 is covered by $\mathbf{u}$. If the latter is the case, then $\mathbf{u}$ in the neighbourhood of C is the result of Construction 2.18 f). So $\mathbf{u}$ is the result of Construction 2.18 near any point from $\partial\Pi$ hence is in the image of ψ_* . $\square$

2.3.3. Sufficiently self-wrapped Lagrangians. There are quite a few differently looking polygons in the moduli spaces of polygons bounded by a Lagrangian L from Definition 1.5, see Fig. 6. To avoid considering too many cases we prove the following.

PROPOSITION 2.20. *Let L be a Lagrangian in a graded marked surface $\mathcal{S}$. Then for any neighbourhood U of ∂S and any $\varepsilon > 0$ there exists a sufficiently self-wrapped Lagrangian L^w such that*

- (1) L^w is Legendrian isotopic to L by an isotopy that is constant outside U ;
- (2) There is a bijection $\psi: X_L \cup Y_L^{>0} \rightarrow X_{L^w}$ that preserves the degree, changes the action by no more than ε , and for any $n \geq 0$ and $z^0, \dots, z^n \in X_L \cup Y_L^{>0}$ induces bijections

$$\psi_*: \Delta^0(L; z^0, z^n, \dots, z^1) \rightarrow \Delta^0(L^w; \psi(z^0), \psi(z^n), \dots, \psi(z^1))$$

such that $\mathbf{u}$ and $\psi_*(\mathbf{u})$ coincide outside U .

PROOF. This follows by induction on $|Y_L^{>0}|$ from the fact that the Legendrian Reidemeister move $II_{>0}$ can always be done. Indeed, if $Y_L^{>0} \neq \emptyset$ and y^0 is the shortest Reeb chord in $Y_L^{>0}$ then all its proper sub-chords must have negative action, but as the action is additive with respect to concatenation, y^0 does not have proper sub-chords hence no ends of L in the interior. Therefore, L in the neighbourhood of y^0 is diffeomorphic to L_0 in Fig. 10(b). We then perform the move $II_{>0}$ on L making sure it is supported inside U and that $\text{Area}(F) < \varepsilon$. We therefore obtain a Lagrangian L' that is Legendrian isotopic to L by an isotopy constant outside U and with $|Y_{L'}^{>0}| = |Y_L^{>0}| - 1$. By Lemmas 2.17 and 2.19 Lagrangian L' satisfies (2) of the claim. $\square$

CHAPTER 3

A_∞ -machinery

3.1. Homotopy equivalence in A_∞ -pre-categories

Recall the definitions of A_∞ -pre-categories from §1.3.2, of homotopy equivalences in them from Definition 1.28, and of A_∞ -pre-categories with enough homotopy equivalences from Definition 1.29.

In this section we prove a number of lemmas about A_∞ -pre-categories with enough homotopy equivalences. In §3.1.1 we prove the 2-out-of-6 property for homotopy equivalences and deduce from it that transverse homotopy equivalent objects have a homotopy equivalence between them. In §3.1.2 we characterise exact morphisms of degree zero. In §3.1.3 we prove that homotopy equivalence can be tested pointwise. In §3.1.4 we characterise homotopy equivalences in strictly unital A_∞ -categories.

3.1.1. We prove that composition of homotopy equivalences is a homotopy equivalence whenever it is defined and that homotopy equivalences satisfy the 2-out-of-6 property. Note that one does not follow from the other since we do not have identity morphisms, but the 2-out-of-3 property follows from these.

Lemma 3.1. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences. Then*

- (1) *Composition of homotopy equivalences is a homotopy equivalence: if $\{X_0, X_1, X_2\}$ is a transverse set and $q^1 \in \mathcal{W}(X_0, X_1)$, $q^2 \in \mathcal{W}(X_1, X_2)$ are homotopy equivalences, then $q^2 \circ q^1$ is a homotopy equivalence.*
- (2) *Homotopy equivalences satisfy the 2-out-of-6 property: for a transverse set $\{X_0, X_1, X_2, X_3\}$ and closed morphisms $q^1 \in \mathcal{W}^0(X_0, X_1)$, $q^2 \in \mathcal{W}^0(X_1, X_2)$, $q^3 \in \mathcal{W}^0(X_2, X_3)$ if $q^2 \circ q^1$ and $q^3 \circ q^2$ are homotopy equivalences, then so are q^1, q^2, q^3 .*

As can be seen from the proof below, having enough homotopy equivalences is crucial for any homotopy argument in A_∞ -pre-categories.

PROOF. We prove (1). Denote $q^0 = q^2 \circ q^1$. First note that $|q^0| = |q^1| + |q^2| + (2 - 2) = 0$ and $\tilde{\mu}^1(q^0) = 0$ by the A_2 -relation. Let $p^1 \in \mathcal{W}(X_1, X_0)$ be a homotopy inverse to q^1 and $p^2 \in \mathcal{W}(X_2, X_1)$ a homotopy inverse to q^2 . We show that $p^0 = p^1 \circ p^2$ is homotopy inverse to q^0 . Let $T \in \text{Ob } \mathcal{W}$ be such that

$\{T, X_0, X_1\}$ is a transverse set. Since $\mathcal{W}$ has enough homotopy equivalences, there is an object T' and homotopy inverse morphisms $e \in \mathcal{W}(T', T)$, $e' \in \mathcal{W}(T, T')$ such that the sets $\{T', T, X_0, X_2\}$ and $\{T', X_0, X_1, X_2\}$ are transverse.

The A_3 -relation implies that the following diagram is homotopy commutative.

$$\begin{array}{ccccc}
 \mathcal{W}(T, X_0) & \xrightarrow{q^0 \circ -} & \mathcal{W}(T, X_2) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T, X_0) \\
 \downarrow - \circ e & & \downarrow - \circ e \quad \uparrow - \circ e' & & \uparrow - \circ e' \\
 \mathcal{W}(T', X_0) & \xrightarrow{q^0 \circ -} & \mathcal{W}(T', X_2) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T', X_0) \\
 \searrow q^1 \circ - & & \nearrow q^2 \circ - & & \nearrow p^1 \circ - \\
 & & \mathcal{W}(T', X_1) & & \mathcal{W}(T', X_1)
 \end{array}$$

Using that $(- \circ e) \circ e'$, $p^2 \circ (q^2 \circ -)$, $p^1 \circ (q^1 \circ -)$ are all homotopic to identity we conclude that so is $p^0 \circ (q^0 \circ -)$. The other three maps from Definition 1.28 are homotopic to identity for similar reasons.

We prove (2). Let $p^{20} \in \mathcal{W}(X_2, X_0)$ be a homotopy inverse to $q^{02} = q^2 \circ q^1$ and $p^{31} \in \mathcal{W}(X_3, X_1)$ a homotopy inverse to $q^{13} = q^3 \circ q^2$. We show that $p_0^2 = q^1 \circ p^{20}$ and $p_3^2 = p^{31} \circ q^3$ are homotopy inverse to q^2 . Let $T \in \text{Ob } \mathcal{W}$ be such that $\{T, X_1, X_2\}$ is a transverse set. Since $\mathcal{W}$ has enough homotopy equivalences, there is an object T' and homotopy inverse morphisms $e \in \mathcal{W}(T', T)$, $e' \in \mathcal{W}(T, T')$ such that the sets $\{T', T, X_1, X_2\}$ and $\{T', X_0, X_1, X_2, X_3\}$ are transverse.

The A_3 -relation implies that the following diagram is homotopy commutative.

$$\begin{array}{ccccc}
 \mathcal{W}(X_2, T) & \xrightarrow{- \circ q^2} & \mathcal{W}(X_1, T) & \xrightarrow{- \circ p_0^2} & \mathcal{W}(X_2, T) \\
 \downarrow e' \circ - & & \downarrow e' \circ - \quad \uparrow e \circ - & & \uparrow e \circ - \\
 \mathcal{W}(X_2, T') & \xrightarrow{- \circ q^2} & \mathcal{W}(X_1, T') & \xrightarrow{- \circ p_0^2} & \mathcal{W}(X_2, T') \\
 \searrow - \circ q^{02} & & \downarrow - \circ q^1 & & \nearrow - \circ p^{02} \\
 & & \mathcal{W}(X_0, T') & &
 \end{array}$$

Therefore, $(- \circ q^2) \circ p_0^2$ is homotopic to $\text{id}_{\mathcal{W}(X_2, T)}$. A similar diagram implies that $(- \circ p_3^2) \circ q^2$ is homotopic to $\text{id}_{\mathcal{W}(X_1, T)}$. Hence $- \circ p_0^2$ and $- \circ p_3^2$ are homotopic as morphisms $\mathcal{W}(X_1, T) \rightarrow \mathcal{W}(X_2, T)$, $(- \circ q^2) \circ p_3^2$ is homotopic to $\text{id}_{\mathcal{W}(X_2, T)}$, and $(- \circ p_0^2) \circ q^2$ is homotopic to $\text{id}_{\mathcal{W}(X_1, T)}$. The other four maps from Definition 1.28 applied to (q^2, p_0^2) and (q^2, p_3^2) are homotopic to identity for similar reasons hence p_0^2 and p_3^2 are homotopy inverse to q^2 .

A similar argument shows that $p^{20} \circ q^2$ is homotopy inverse to q^1 and $q^2 \circ p^{31}$ is homotopy inverse to q^3 . $\square$

Lemma 3.2. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences. Then for any pair of homotopy equivalent objects X_0, X_1 such that $\{X_0, X_1\}$ is a transverse set there exists a homotopy equivalence $q: X_0 \rightarrow X_1$.*

Note that Lemma 3.2 implies that for any homotopy equivalent objects X_0, X_1 there exists an object X_2 and homotopy equivalences $X_0 \rightarrow X_2$ and $X_2 \rightarrow X_1$.

PROOF. Since X_0 and X_1 are homotopy equivalent, there exist a sequence $X_0 = Y_0, Y_1, \dots, Y_n = X_1$ of objects in $\mathcal{W}$ and homotopy equivalences $q^i \in \mathcal{W}(X_{i-1}, X_i)$ for $i = 1, \dots, n$. We need to construct a homotopy equivalence $q \in \mathcal{W}(Y_0, Y_n)$ provided that Y_0 and Y_n are transverse. As $\mathcal{W}$ has enough homotopy equivalences, we can take an object Y' and homotopy inverse morphisms $q'_0 \in \mathcal{W}(Y', Y_0)$ and $p'_0 \in \mathcal{W}(Y_0, Y')$ such that the sets $\{Y', Y_0, Y_n\}$ and $\{Y', Y_{i-1}, Y_i\}$, $i = 1, \dots, n$ are transverse. Using Lemma 3.1 (1) we obtain homotopy equivalences $q'_i \in \mathcal{W}(Y', Y_i)$ by $q'_i = q_i \circ q'_{i-1}$ for $i = 1, \dots, n$. Setting $q = q'_n \circ p'_0$ gives the desired homotopy equivalence from $Y_0 = X_0$ to $Y_n = X_1$. $\square$

3.1.2. We give criteria for a degree zero closed morphism to be exact.

Lemma 3.3. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences and $q \in \mathcal{W}^0(X_0, X_1)$ a closed morphism. Then the following are equivalent.*

- (1) q is exact;
- (2) for any transverse set $\{T, X_0, X_1\}$ the map $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ is null-homotopic;
- (3) there exists a homotopy equivalence $e_0 \in \mathcal{W}(X'_0, X_0)$ such that the set $\{X'_0, X_0, X_1\}$ is transverse and $q \circ e_0 \in \mathcal{W}(X'_0, X_1)$ is exact;
- (4) for any transverse set $\{X_0, X_1, T\}$ the map $\mathcal{W}(X_1, T) \xrightarrow{- \circ q} \mathcal{W}(X_0, T)$ is null-homotopic;
- (5) there exists a homotopy equivalence $e'_1 \in \mathcal{W}(X_1, X'_1)$ such that the set $\{X_0, X_1, X'_1\}$ is transverse and $e'_1 \circ q \in \mathcal{W}(X_0, X'_1)$ is exact.

PROOF. We show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. The argument for $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ is similar.

Assume $q = \tilde{\mu}^1(a)$ for some $a \in \mathcal{W}^{-1}(X_0, X_1)$. Let $\{T, X_0, X_1\}$ be a transverse set. The A_2 -relation then implies that

$$q \circ b = \tilde{\mu}^1(\tilde{\mu}^2(a, b)) + \tilde{\mu}^2(a, \tilde{\mu}^1(b)) \text{ for any homogeneous } b \in \mathcal{W}(T, X_0).$$

Hence $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ is indeed null-homotopic.

That (2) implies (3) is clear since we can take X'_0 for T and $\tilde{\mu}^1(e_0) = 0$.

Now assume (3). As e_0 is a homotopy equivalence, $\mathcal{W}(X_0, X_1) \xrightarrow{-\circ e_0} \mathcal{W}(X'_0, X_1)$ is a homotopy equivalence. Since $q \circ e_0 \in \mathcal{W}(X'_0, X_1)$ is exact, so is $q \in \mathcal{W}(X_0, X_1)$. $\square$

3.1.3. We prove that homotopy equivalence can be tested pointwise.

Lemma 3.4. *Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences, and $X_0 \xrightarrow{q} X_1$ a closed morphism of degree zero in $\mathcal{W}$. Suppose that for any transverse set $\{T, X_0, X_1\}$ the maps $\mathcal{W}(T, X_0) \xrightarrow{q \circ -} \mathcal{W}(T, X_1)$ and $\mathcal{W}(X_1, T) \xrightarrow{- \circ q} \mathcal{W}(X_0, T)$ are homotopy equivalences.*

Then q is a homotopy equivalence in $\mathcal{W}$.

PROOF. It suffices to construct two degree zero closed morphisms $p^0, p^1 \in \mathcal{W}(X_1, X_0)$ such that for any transverse set $\{T, X_0, X_1\}$ the maps $p^0 \circ (q \circ -)$, $(- \circ p^0) \circ q$, $q \circ (p^1 \circ -)$, and $(- \circ q) \circ p^1$ are homotopic to identity. Then the maps $p^0 \circ -$ and $p^1 \circ -$ are homotopic hence by Lemma 3.3 p^0 is cohomologous to p^1 and they are both homotopy inverse to q .

To construct p^0 pick homotopy inverse morphisms $e \in \mathcal{W}(X'_0, X_0)$ and $e' \in \mathcal{W}(X_0, X'_0)$ such that $\{X'_0, X_0, X_1\}$ is a transverse set. Let $\mathcal{W}(X_0, X'_0) \xrightarrow{\alpha} \mathcal{W}(X_1, X'_0)$ be a homotopy inverse to $- \circ q$. Set $p = e \circ \alpha(e') \in \mathcal{W}(X_1, X_0)$. Clearly, p is closed of degree zero. Let $\{T, X_0, X_1\}$ be a transverse set. As in the proof of Lemma 3.1 we can assume that $\{T, X_0, X_1, X'_0\}$ is transverse. Note that $\alpha(e') \circ q$ is cohomologous to e' since α is homotopy inverse to $- \circ q$. The A_3 -relation and Lemma 3.3 (1) now imply that the diagram

$$\begin{array}{ccccc} \mathcal{W}(T, X_0) & \xrightarrow{q \circ -} & \mathcal{W}(T, X_1) & \xrightarrow{p^0 \circ -} & \mathcal{W}(T, X_0) \\ & \searrow & \searrow \alpha(e') \circ - & \nearrow & \nearrow e \circ - \\ & & \mathcal{W}(T, X'_0) & & \end{array}$$

is homotopy commutative. As e is homotopy inverse to e' we conclude that $p^0 \circ (q \circ -)$ is homotopic to identity. A similar argument shows that $(- \circ p^0) \circ q$ is homotopic to identity.

To construct p^1 pick homotopy inverse morphisms $e \in \mathcal{W}(X'_1, X_1)$ and $e' \in \mathcal{W}(X_1, X'_1)$ such that $\{X'_1, X_0, X_1\}$ is a transverse set. Let $\mathcal{W}(X'_1, X_1) \xrightarrow{\alpha} \mathcal{W}(X'_1, X_0)$ be a homotopy inverse to $q \circ -$. Set $p^1 = \alpha(e) \circ e' \in \mathcal{W}(X_1, X_0)$. Clearly, p^1 is closed of degree zero. Let $\{T, X_0, X_1\}$ be a transverse set. As in the proof of Lemma 3.1 we can assume that $\{T, X_0, X_1, X'_1\}$ is transverse. Note that $q \circ \alpha(e)$ is cohomologous to e since α is homotopy inverse to $q \circ -$.

The A_3 -relation and Lemma 3.3 (1) now imply that the diagram

$$\begin{array}{ccccc}
 \mathcal{W}(T, X_1) & \xrightarrow{p^1 \circ -} & \mathcal{W}(T, X_0) & \xrightarrow{q^0 \circ -} & \mathcal{W}(T, X_1) \\
 & \searrow e' \circ - & \nearrow \alpha(e) \circ - & & \nearrow e \circ - \\
 & & \mathcal{W}(T, X'_1) & &
 \end{array}$$

is homotopy commutative. As e is homotopy inverse to e' we conclude that $q^0 \circ (p^1 \circ -)$ is homotopic to identity. A similar argument shows that $(- \circ q^0) \circ p^1$ is homotopic to identity.

We have therefore constructed the desired $p^0, p^1 \in \mathcal{W}(X_1, X_0)$. $\square$

3.1.4. We characterise homotopy equivalences in a strictly unital A_∞ -category $\mathcal{C}$.

Lemma 3.5. *Let $q^1 \in \mathcal{C}(X_0, X_1)$ and $q^0 \in \mathcal{C}(X_1, X_0)$ be two closed morphisms of degree 0. Then q^0 and q^1 are homotopy inverse if and only if $q^1 \circ q^0 \in \mathcal{C}(X_1, X_1)$ is cohomologous to 1_{X_1} and $q^0 \circ q^1 \in \mathcal{C}(X_0, X_0)$ is cohomologous to 1_{X_0} .*

PROOF. The ‘if’ part follows from Lemmas 3.1 and 3.3: $q^1 \circ 1_{X_0} = q^1 = 1_{X_1} \circ q^1$ is homotopy inverse to q^0 since strict units are homotopy inverse to themselves.

To show the ‘only if’ part, due to the symmetry between q^0 and q^1 , it suffices to show that $q^0 \circ (q^1 \circ -)$ is homotopy equivalent to $\text{id}_{\mathcal{W}(T, X_0)}$ and $(- \circ q^0) \circ q^1$ is homotopy equivalent to $\text{id}_{\mathcal{W}(X_0, T)}$ for any $T \in \text{Ob } \mathcal{C}$ provided that $q^0 \circ q^1$ is cohomologous to 1_{X_0} .

To see that this is the case fix $T \in \text{Ob } \mathcal{C}$ and observe that the following diagram

$$\begin{array}{ccc}
 \mathcal{C}(T, X_0) & \begin{array}{c} \xrightarrow{1_{X_0} \circ -} \\ \xrightarrow{(q^0 \circ q^1) \circ -} \end{array} & \mathcal{C}(T, X_0) \\
 & \searrow q^1 \circ - \quad \nearrow q^0 \circ - & \\
 & \mathcal{C}(T, X_1) &
 \end{array}$$

is homotopy commutative by the A_3 -relation and Lemma 3.3 (1). Since $1_{X_0} \circ - = \text{id}_{\mathcal{W}(T, X_0)}$, we conclude that $q^0 \circ (q^1 \circ -)$ is homotopy equivalent to $\text{id}_{\mathcal{W}(T, X_0)}$ as required. A similar diagram shows the other homotopy equivalence. $\square$

3.2. From A_∞ -pre-categories to A_∞ -categories

Let $\mathcal{W}$ be an A_∞ -pre-category with enough homotopy equivalences.

The key property needed for the A_∞ -category spanned by a collection of objects $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ to be computable is that $\mathbf{X}^0$ admits a system of strict

units $\mathbf{X}^\bullet$, which we define in §3.2.1. We then construct the A_∞ -category $\mathcal{W}_{\mathbf{X}^\bullet}$ with $\text{Ob } \mathcal{W}_{\mathbf{X}^\bullet} = \mathbf{X}^0$ in §3.2.2 and check that it is a strictly unital A_∞ -category in §3.2.3. In §3.2.4 we give simple-minded conditions for the constructions above to give a strictly unital A_∞ -category $\tilde{\mathcal{W}}$ with $\text{Ob } \tilde{\mathcal{W}} = \text{Ob } \mathcal{W}$ such that two object of $\mathcal{W}$ are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\tilde{\mathcal{W}}$.

3.2.1. System of strict units. A *system of strict units* on a collection $\{X_i^0\}_{i \in I}$ of objects in $\mathcal{W}$ is the data of

- a transverse set $\{X_i^j\}_{j=0, \dots, \infty}$ in $\mathcal{W}$ for each $i \in I$;
- homotopy equivalences $e_i^{j_0, j_1} \in \mathcal{W}(X_i^{j_0}, X_i^{j_1})$ for each $i \in I$ and $j_0 \neq j_1$

such that

- (1) $e_i^{j_1, j_2} \circ e_i^{j_0, j_1} = e_i^{j_0, j_2}$ for any $i \in I$ and pairwise distinct $j_0, j_1, j_2 \geq 0$;
- (2) for any $i_0, i_1 \in I$ there exists $N > 0$ such that for any $j_0, j_1, j'_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \min\{j_1, j'_1\} < \max\{j_1, j'_1\}$ the set $\{X_{i_0}^{j_0}, X_{i_1}^{j_1}, X_{i_1}^{j'_1}\}$ is transverse and the map

$$e_{i_1}^{j'_1, j_1} \circ (e_{i_1}^{j_1, j'_1} \circ -) \text{ is the identity on } \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1});$$

- (3) for any $i_0, i_1 \in I$ there exists $N > 0$ such that for any $j_0, j'_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $\min\{j_0, j'_0\} < \max\{j_0, j'_0\} < j_1$ the set $\{X_{i_0}^{j_0}, X_{i_0}^{j'_0}, X_{i_1}^{j_1}\}$ is transverse and the map

$$(- \circ e_{i_0}^{j'_0, j_0}) \circ e_{i_0}^{j_0, j'_0} \text{ is the identity on } \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1});$$

- (4) for any $n \geq 3$, $0 \leq l < n$, and $i_0, \dots, i_{l-2}, i_{l-1} = i = i_l, i_{l+1}, \dots, i_n \in I$ there exists $N > 0$ such that for any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_{l-2} < \min\{j_{l-1}, j_l\} < \max\{j_{l-1}, j_l\} < j_{l+1} < \dots < j_n$ the set $\{X_{i_0}^{j_0}, \dots, X_{i_n}^{j_n}\}$ is transverse and the map

$$\begin{aligned} & \mathcal{W}(X_{i_{n-1}}^{j_{n-1}}, X_{i_n}^{j_n}) \dots \mathcal{W}(X_i^{j_l}, X_{i_{l+1}}^{j_{l+1}}) \otimes \mathcal{W}(X_{i_{l-2}}^{j_{l-2}}, X_i^{j_{l-1}}) \dots \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1}) \\ & \quad \downarrow \tilde{\mu}^n(\dots, e_i^{j_{l-1}, j_l}, \dots) \\ & \mathcal{W}(X_{i_0}^{j_0}, X_{i_n}^{j_n}) \end{aligned}$$

vanishes.

3.2.2. A_∞ -category spanned by a collection of objects. Let

$$\mathbf{X}^\bullet = \left\{ \left(\{X_i^j\}_{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$$

be a system of strict units on a collection $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ of objects in $\mathcal{W}$. We define an A_∞ -category $\mathcal{W}_{\mathbf{X}^\bullet}$ spanned by $\mathbf{X}^0$ as follows.

Objects. $\text{Ob } \mathcal{W}_{\mathbf{X}^\bullet} = \mathbf{X}^0$.

Morphisms. Fix $X_{i_0}^0, X_{i_1}^0 \in \mathbf{X}^0$. The morphism space is given by

$$\mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) = \operatorname{colim} \mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1})$$

where the colimit is taken with respect to the directed system

$$e_{i_1}^{j_1, j'_1} \circ - : \mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1}) \rightarrow \mathcal{W}(X_{i_0}^0, X_{i_1}^{j'_1})$$

for $0 \ll j_1 < j'_1$. Properties (1) and (4) in §3.2.1 and the A_3 -relation imply that the above is indeed a directed system. Property (2) implies that the natural map $\mathcal{W}(X_{i_0}^0, X_{i_1}^{j_1}) \rightarrow \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0)$ is an isomorphism of graded R -modules for $j_1 \gg 0$. Combining this with Property (3), for some $N > 0$ we obtain isomorphisms

$$\varphi_{i_0, i_1}^{j_0, j_1} : \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) \rightarrow \mathcal{W}(X_{i_0}^{j_0}, X_{i_1}^{j_1})$$

for any $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$.

A_∞ -operations. For $X_{i_0}^0, \dots, X_{i_n}^0 \in \mathbf{X}^0$ the A_∞ -operation

$$\tilde{\mu}_{\mathcal{W}_{\mathbf{X}^\bullet}}^n : \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_{n-1}}^0, X_{i_n}^0) \otimes \dots \otimes \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_1}^0) \rightarrow \mathcal{W}_{\mathbf{X}^\bullet}(X_{i_0}^0, X_{i_n}^0)$$

is determined by

$$\varphi_{i_0, i_n}^{j_0, j_n} \circ \tilde{\mu}_{\mathcal{W}_{\mathbf{X}^\bullet}}^n = \tilde{\mu}_{\mathcal{W}}^n \circ \left(\varphi_{i_{n-1}, i_n}^{j_{n-1}, j_n} \otimes \dots \otimes \varphi_{i_0, i_1}^{j_0, j_1} \right)$$

for sufficiently large $N > 0$ and any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$. We check that this definition makes sense in Lemma 3.6.

3.2.3. We check that the definition of an A_∞ -category spanned by a collection of objects in §3.2.2 makes sense and prove that homotopy equivalence in $\mathcal{W}$ and $\mathcal{W}_{\mathbf{X}^\bullet}$ is the same.

Lemma 3.6. *Let $\mathbf{X}^\bullet = \{(\{X_i^j\}_{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1})\}_{i \in I}$ be a system of strict units on a collection of objects $\mathbf{X}^0 = \{X_i^0\}_{i \in I}$ in an A_∞ -pre-category $\mathcal{W}$ with enough homotopy equivalences.*

Then $\mathcal{W}_{\mathbf{X}^\bullet}$ is a strictly unital A_∞ -category with the strict unit $e_i \in \mathcal{W}_{\mathbf{X}^\bullet}(X_i^0, X_i^0)$ defined by $\varphi_{i, i}^{j_0, j_1}(e_i) = e_i^{j_0, j_1}$ for sufficiently large $N > 0$ and $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$.

Moreover, $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\mathcal{W}_{\mathbf{X}^\bullet}$.

PROOF. That $\mathcal{W}_{\mathbf{X}^\bullet}$ is a strictly unital A_∞ -category follows from the statement below.

CLAIM. *Let $n \geq 1$ and $i_0, \dots, i_n \in I$. Then there exists $N > 0$ such that for any $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$ the following holds.*

(1) *if $j'_0 \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_0 < j'_0 < j_1$, then*

$$\tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_2, a_1 \circ e_{i_0}^{j'_0, j_0}) = \tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_1) \circ e_{i_0}^{j'_0, j_0}$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;
 (2) if $j'_l \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_l < j'_l < j_{l+1}$, then

$$\tilde{\mu}_{\mathcal{W}}^n \left(a_n, \dots, a_{l+2}, a_{l+1} \circ e_{i_l}^{j'_l, j_l}, e_{i_l}^{j_l, j'_l} \circ a_l, a_{l-1}, \dots, a_1 \right) = \tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_1)$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;
 (3) if $j'_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ is such that $j_n < j'_n$, then

$$\tilde{\mu}_{\mathcal{W}}^n \left(e_{i_n}^{j_n, j'_n} \circ a_n, a_{n-1}, \dots, a_1 \right) = e_{i_n}^{j_n, j'_n} \circ \tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_1)$$

for any sequence of homogeneous morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ in $\mathcal{W}$;

The Claim implies that for $n \geq 1$ and $i_0, \dots, i_n \in I$ there exists $N > 0$ such that

$$(\varphi_{i_0, i_n}^{j_0, j_n})^{-1} \circ \tilde{\mu}_{\mathcal{W}}^n \circ \left(\varphi_{i_{n-1}, i_n}^{j_{n-1}, j_n} \otimes \dots \otimes \varphi_{i_0, i_1}^{j_0, j_1} \right)$$

does not depend on the choice of $j_0, \dots, j_n \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < \dots < j_n$. Indeed, if we have two such choices $j'_0 < \dots < j'_n$ and $j''_0 < \dots < j''_n$, then they give the same result as $j_0 < \dots < j_n$ for $j_0 > \max\{j'_n, j''_n, N\}$. Therefore, $\tilde{\mu}_{\mathcal{W}_{\mathbf{x}^\bullet}}$ is well-defined and satisfies the A_∞ -relations.

Property (1) in §3.2.1 and the definition of φ imply that $(\varphi_{i,i}^{j_0, j_1})^{-1} (e_i^{j_0, j_1})$ does not depend on the choice of $j_0, j_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1$ for sufficiently large $N > 0$. Since $e_i^{j_0, j_1}$ is a homotopy equivalence, e_i is a closed morphism of degree zero. Properties (2) and (3) imply that e_i satisfies the part of the definition of strict unit involving $\circ$, and Property (4) implies that e_i satisfies the remaining part of the definition, involving $\tilde{\mu}^n$ for $n \geq 3$.

Suppose $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$. Then by Lemma 3.2 there exists a homotopy equivalence $X_{i_0}^{j_0} \xrightarrow{q^{j_0, j_1}} X_{i_1}^{j_1}$ in $\mathcal{W}$ for $0 \ll j_0 < j_1$ hence $q = (\varphi_{i_0, i_1}^{j_0, j_1})^{-1} (q^{j_0, j_1})$ is a homotopy equivalence in $\mathcal{W}_{\mathbf{x}^\bullet}$ by Lemma 3.4 since $q \circ -$ (resp. $- \circ q$) on $\mathcal{W}_{\mathbf{x}^\bullet}(X_{i_2}^0, X_{i_0}^0)$ (resp. on $\mathcal{W}_{\mathbf{x}^\bullet}(X_{i_1}^0, X_{i_2}^0)$) can be computed as $\varphi_{i_0, i_1}^{j_0, j_1}(q) \circ -$ (resp. $- \circ \varphi_{i_0, i_1}^{j_0, j_1}(q)$) on $\mathcal{W}(X_{i_2}^{j_2}, X_{i_0}^{j_0})$ (resp. $\mathcal{W}(X_{i_1}^{j_1}, X_{i_2}^{j_2})$) for $0 \ll j_2 < j_0 < j_1$ (resp. $0 \ll j_0 < j_1 < j_2$).

Now suppose that $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}_{\mathbf{x}^\bullet}$. Then by Lemma 3.5 there exist $j_0 < j_1 < j'_0 < j'_1$ and closed morphisms of degree zero $X_{i_0}^{j_0} \xrightarrow{q^{j_0, j_1}} X_{i_1}^{j_1} \xrightarrow{q^{j_1, j'_0}} X_{i_0}^{j'_0} \xrightarrow{q^{j'_0, j'_1}} X_{i_1}^{j'_1}$ such that $q^{j_1, j'_0} \circ q^{j_0, j_1}$ is cohomologous to $e_{i_0}^{j_0, j'_0}$ and $q^{j'_0, j'_1} \circ q^{j_1, j'_0}$ is cohomologous to $e_{i_1}^{j_1, j'_1}$. By Lemma 3.3 (1) and Lemma 3.1 q^{j_0, j_1} is a homotopy equivalence and hence $X_{i_0}^0$ and $X_{i_1}^0$ are homotopy equivalent in $\mathcal{W}$. $\square$

PROOF OF CLAIM. Take $N > 0$ such that Properties (2) and (3) in §3.2.1 are satisfied for any $i, i' \in \{i_0, \dots, i_n\}$ and Property (4) is satisfied for any tuple $(i'_0, \dots, i'_{n+2})$ where $i'_0, \dots, i'_{n+2} \in \{i_0, \dots, i_n\}$.

(1) of the Claim is equivalent to

$$\tilde{\mu}^n(a_n, \dots, a_2, \tilde{\mu}^2(a_1, e_{i_0}^{j'_0, j_0})) + (-1)^{0-1} \tilde{\mu}^2(\tilde{\mu}^n(a_n, \dots, a_1), e_{i_0}^{j'_0, j_0}) = 0,$$

which is the A_{n+1} -relation since $\tilde{\mu}^1(e_{i_0}^{j'_0, j_0}) = 0$ and all other terms vanish by Property (4).

(3) of the Claim is equivalent to

$$\begin{aligned} & (-1)^{|a_n|} \tilde{\mu}^n(\tilde{\mu}^2(e_{i_n}^{j_n, j'_n}, a_n), a_{n-1}, \dots, a_1) + \\ & (-1)^{|a_n| + \dots + |a_1| - n - 1} \tilde{\mu}^2(e_{i_n}^{j_n, j'_n}, \tilde{\mu}^n(a_n, \dots, a_1)) = 0, \end{aligned}$$

which is a multiple of the A_{n+1} -relation since $\tilde{\mu}^1(e_{i_0}^{j'_0, j_0}) = 0$ and all other terms vanish by Property (4).

To prove (2) of the Claim consider the A_{n+1} -relation for the sequence of morphisms $X_{i_0}^{j_0} \xrightarrow{a_1} \dots \xrightarrow{a_l} X_{i_l}^{j_l} \xrightarrow{e_{i_l}^{j_l, j'_l}} X_{i_l}^{j'_l} \xrightarrow{\tilde{\mu}^2(a_{l+1}, e_{i_l}^{j'_l, j_l})} X_{i_{l+1}}^{j_{l+1}} \xrightarrow{a_{l+2}} \dots \xrightarrow{a_n} X_{i_n}^{j_n}$ and use that $e_{i_l}^{j_l, j'_l}$ is closed and Property (4) to get

$$\begin{aligned} & (-1)^{|a_{l-1}| + \dots + |a_1| + l} \tilde{\mu}^n(a_n, \dots, a_{l+2}, \tilde{\mu}^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), \tilde{\mu}^2(e_{i_l}^{j_l, j'_l}, a_l), a_{l-1}, \dots, a_1) + \\ & (-1)^{|a_l| + \dots + |a_1| + l - 1} \tilde{\mu}^n(a_n, \dots, a_{l+2}, \tilde{\mu}^2(\tilde{\mu}^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), e_{i_l}^{j_l, j'_l}), a_l, \dots, a_1) = 0. \end{aligned}$$

Using Property (3) we obtain

$$\begin{aligned} & \tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_{l+2}, a_{l+1} \circ e_{i_l}^{j'_l, j_l}, e_{i_l}^{j_l, j'_l} \circ a_l, a_{l-1}, \dots, a_1) = \\ & (-1)^{|a_l|} \tilde{\mu}^n(a_n, \dots, a_{l+2}, \tilde{\mu}^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), \tilde{\mu}^2(e_{i_l}^{j_l, j'_l}, a_l), a_{l-1}, \dots, a_1) = \\ & \tilde{\mu}^n(a_n, \dots, a_{l+2}, \tilde{\mu}^2(\tilde{\mu}^2(a_{l+1}, e_{i_l}^{j'_l, j_l}), e_{i_l}^{j_l, j'_l}), a_l, \dots, a_1) = \tilde{\mu}_{\mathcal{W}}^n(a_n, \dots, a_1) \end{aligned}$$

as required. $\square$

3.2.4. Invariance. Let

$$\mathbf{X}^\bullet = \{(\{X_i^j\}^{j=0, \dots, \infty}, \{e_i^{j_0, j_1}\}^{j_0 \neq j_1})\}_{i \in I}$$

be a system of strict units in $\mathcal{W}$ and $\mathcal{I}$ an isomorphism of $\mathcal{W}$. Then

$$\mathcal{I}_* \mathbf{X}^\bullet = \{(\{\mathcal{I}^0 X_i^j\}^{j=0, \dots, \infty}, \{\mathcal{I}^1 e_i^{j_0, j_1}\}^{j_0 \neq j_1})\}_{i \in I}$$

is also a system of strict units in $\mathcal{W}$ and $\mathcal{I}$ induces an isomorphism of strictly unital A_∞ -categories $\mathcal{W}_{\mathbf{X}^\bullet} \rightarrow \mathcal{W}_{\mathcal{I}_* \mathbf{X}^\bullet}$.

Definition 3.7. A collection $\mathcal{SU}$ of systems of strict units in $\mathcal{W}$ is called *saturated* if

- (1) for any system of strict units $\mathbf{X}^\bullet \in \mathcal{SU}$ and any $X \in \text{Ob } \mathcal{W}$ there exists an object X_i^0 of $\mathbf{X}^\bullet$ homotopy equivalent to X ;

- (2) for any two systems of strict units $\mathbf{X}_0^\bullet, \mathbf{X}_1^\bullet \in \mathcal{SU}$ there exists an isomorphism $\mathcal{I}$ of $\mathcal{W}$ such that $\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet \in \mathcal{SU}$.

Given a system of strict units $\mathbf{X}^\bullet \in \mathcal{SU}$ on the collection $\mathbf{X}^0$ of objects in $\mathcal{W}$ we define the strictly unital A_∞ -category $\tilde{\mathcal{W}}_{\mathbf{X}^\bullet}$ as follows. First, for each $Y \in \text{Ob } \mathcal{W}$ pick $X_Y^0 \in \mathbf{X}^0$ such that Y and X_Y^0 are homotopy equivalent in $\mathcal{W}$. Then set $\tilde{\mathcal{W}}_{\mathbf{X}^\bullet}$ to be the full A_∞ -subcategory of the strictly unital A_∞ -category $\mathcal{W}_{\mathbf{X}^\bullet}$ spanned by $\{X_Y^0\}_{Y \in \text{Ob } \mathcal{W}}$.

Lemma 3.8. *Let $\mathcal{SU}$ be a saturated collection of systems of strict units in $\mathcal{W}$. Then $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_{\mathbf{X}^\bullet}$ is independent of any choices up to homotopy equivalence of strictly unital A_∞ -categories.*

PROOF. Consider two systems of strict units $\mathbf{X}_0^\bullet, \mathbf{X}_1^\bullet \in \mathcal{SU}$ and let $\tilde{\mathcal{W}}_{\mathbf{X}_0^\bullet}$ and $\tilde{\mathcal{W}}_{\mathbf{X}_1^\bullet}$ be the corresponding strictly unital A_∞ -categories. Pick an isomorphism $\mathcal{I}$ of $\mathcal{W}$ such that $\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet \in \mathcal{SU}$. Consider the full A_∞ -subcategory $\mathcal{V}$ of the strictly unital A_∞ -category $\mathcal{W}_{\mathcal{I}_* \mathbf{X}_0^\bullet \sqcup \mathbf{X}_1^\bullet}$ spanned by $\{\mathcal{I}^0 X_{Y,0}^0\}_{Y \in \text{Ob } \mathcal{W}} \sqcup \{X_{Y,1}^0\}_{Y \in \text{Ob } \mathcal{W}}$. Then we have a strict inclusion $\tilde{\mathcal{W}}_{\mathbf{X}_1^\bullet} \hookrightarrow \mathcal{V}$ and $\mathcal{I}$ induces a strict inclusion $\tilde{\mathcal{W}}_{\mathbf{X}_0^\bullet} \hookrightarrow \mathcal{V}$, with both inclusions being isomorphisms of graded R -modules on morphism spaces. By Lemma 3.6 for any object X in $\mathcal{V}$ there are objects $Y_0, Y_1 \in \text{Ob } \mathcal{W}$ such that $\mathcal{I}^0 X_{Y_0,0}^0$ and $X_{Y_1,1}^0$ are homotopy equivalent to X in $\mathcal{V}$. Thus the inclusions above are homotopy equivalences of strictly unital A_∞ -categories (see §1.4.2) hence $\tilde{\mathcal{W}}_{\mathbf{X}_0^\bullet}$ and $\tilde{\mathcal{W}}_{\mathbf{X}_1^\bullet}$ are homotopy equivalent. $\square$

3.3. Triangulated A_∞ -pre-categories

We defined triangulated A_∞ -pre-categories in §1.4.3. In this section we give more details on that definition. In §3.3.1 we define the triangulated envelope of an A_∞ -pre-category. In §3.3.2 we recall shifts and cones for twisted complexes. In §3.3.3 we prove an expected property of a triangulated A_∞ -pre-category with enough homotopy equivalences.

3.3.1. Triangulated envelope. The triangulated envelope of an A_∞ -pre-category is defined similarly to the case of A_∞ -categories, we follow the construction of Efimov [17, §4] with slight modifications taking in account that transverse sequences form transverse sets in our A_∞ -pre-categories.

Let $\mathcal{W}$ be a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category where transverse sequences form transverse sets. The *triangulated envelope* $\text{Tw } \mathcal{W}$ of $\mathcal{W}$ is the $\mathbb{Z}$ -graded R -linear A_∞ -pre-category defined as follows.

Objects. An object of $\text{Tw } \mathcal{W}$ is a twisted complex, i.e. a pair (X, δ) where

- $X = \{X_i[m_i]\}_{i \in I}$ is a collection of formal shifts of objects in $\mathcal{W}$ such that I is finite, $\{X_i\}_{i \in I}$ is a transverse set in $\mathcal{W}$, and $m_i \in \mathbb{Z}$ for $i \in I$;

- δ is an element of degree 1 in the graded R -module

$$\bigoplus_{i_0, i_1 \in I: i_0 \neq i_1} \mathcal{W}(X_{i_0}, X_{i_1})[m_{i_1} - m_{i_0}]$$

and there exists a partial order $<$ on I such that

$$\delta \in \mathcal{A}_X^< = \bigoplus_{i_0, i_1 \in I: i_0 < i_1} \mathcal{W}(X_{i_0}, X_{i_1})[m_{i_1} - m_{i_0}]$$

is a bounding cochain in the nilpotent (uncurved) A_∞ -algebra $\mathcal{A}_X^<$ where the A_∞ -operations in $\mathcal{A}_X^<$ are given by that in $\mathcal{W}$ with a Koszul sign rule,

$$(3.1) \quad \mu^n(a_n[i_n], \dots, a_1[i_1]) = (-1)^{\sum_{1 \leq \alpha < \beta \leq n} i_\alpha(|a_\beta| - 1)} \tilde{\mu}^n(a_n, \dots, a_1)[i_1 + \dots + i_n]$$

for homogeneous morphisms $a_1, \dots, a_n$ in $\mathcal{W}$ and $i_1, \dots, i_n \in \mathbb{Z}$.

We have

$$\delta = \sum_{i_0, i_1 \in I} \delta_{i_0 i_1} [m_{i_1} - m_{i_0}] \text{ where } \delta_{i_0 i_1} \in \mathcal{W}^{m_{i_1} - m_{i_0} + 1}(X_{i_0}, X_{i_1}),$$

the morphisms $\delta_{i_0 i_1}$ are called *components* of δ . Note that if $\delta_{i_0 i_1} \neq 0$, then $i_0 < i_1$. The partial order on I is not part of the data, but we can always consider the weakest partial order such that $i_0 < i_1$ whenever $\delta_{i_0 i_1} \neq 0$.

Transverse sequences. A sequence of twisted complexes $((X^0, \delta^0), \dots, (X^n, \delta^n))$ is transverse if and only if $X^0 \sqcup \dots \sqcup X^n$ is a transverse set in $\mathcal{W}$.

Morphisms. For a transverse pair of twisted complexes $((X^0, \delta^0), (X^1, \delta^1))$ with $X^j = \{X_i^j\}_{i \in I^j}$ for $j = 0, 1$ the morphism space is the graded R -module

$$(3.2) \quad \text{Tw } \mathcal{W}((X^0, \delta^0), (X^1, \delta^1)) = \bigoplus_{i^0 \in I^0, i^1 \in I^1} \mathcal{W}(X_{i^0}^0, X_{i^1}^1)[m_{i^1}^1 - m_{i^0}^0].$$

For a morphism $a \in \text{Tw } \mathcal{W}((X^0, \delta^0), (X^1, \delta^1))$ of degree d we have

$$a = \sum_{i^0 \in I^0, i^1 \in I^1} a_{i^0 i^1} [m_{i^1}^1 - m_{i^0}^0] \text{ where } a_{i^0 i^1} \in \mathcal{W}^{m_{i^1}^1 - m_{i^0}^0 + d}(X_{i^0}^0, X_{i^1}^1),$$

the morphisms $a_{i^0 i^1}$ are called *components* of a .

Structure maps. For a transverse sequence $((X^0, \delta^0), \dots, (X^n, \delta^n))$ with $X^j = \{X_i^j[m_i^j]\}_{i \in I^j}$ for $j = 0, \dots, n$, the A_∞ -operation $\tilde{\mu}^n$ is given by μ_δ from (1.21) in the A_∞ -algebra $\mathcal{A}_X^<$ where $\delta = \delta^0 + \dots + \delta^n$ is the bounding cochain in $\mathcal{A}_X^<$ with $X = X^0 \sqcup \dots \sqcup X^n$ and a partial order $<$ on $I^0 \sqcup \dots \sqcup I^n$ that includes partial orders on each I^j and satisfies $i^j < i^{j'}$ for any $j < j'$ and $i^j \in I^j, i^{j'} \in I^{j'}$.

It is instructive to think of $X[m]$ for $X \in \mathcal{W}$ and $m \in \mathbb{Z}$ as a formal tensor product $R[m] \otimes X$. The Koszul sign in (3.1) comes from this point of view and the fact that μ^n acts from the right.

Note that $\mathcal{W}$ is naturally a full subcategory of $\text{Tw } \mathcal{W}$ by identifying X and the twisted complex $(X[0], 0)$ for each $X \in \text{Ob } \mathcal{W}$. Observe that transverse sequences in $\text{Tw } \mathcal{W}$ form transverse sets.

3.3.2. Shifts and cones for twisted complexes. The *zero object* $\mathbf{0}$ in $\text{Tw } \mathcal{W}$ is the twisted complex $(\emptyset, 0)$.

The *shift functor* Σ^s for $s \in \mathbb{Z}$ is defined as follows. For a twisted complex $(X = \{X_i[m_i]\}_{i \in I}, \delta)$ the twisted complex $\Sigma^s(X, \delta) = (\Sigma^s X, \Sigma^s \delta)$ is defined by $\Sigma^s X = \{X_i[m_i + s]\}_{i \in I}$ and $(\Sigma^s \delta)_{i_0 i_1} = (-1)^{s(m_{i_1} - m_{i_0})} \delta_{i_0 i_1} = (-1)^{s(|\delta_{i_0 i_1}| - 1)} \delta_{i_0 i_1}$.

For a transverse pair of twisted complexes $(\mathbf{X}^0, \mathbf{X}^1)$ where $\mathbf{X}^j = (\{X_i^j[m_i^j]\}_{i \in I^j}, \delta^j)$, $j = 0, 1$ and $s^0, s^1 \in \mathbb{Z}$ define a map of degree $s^0 - s^1$

$$\begin{aligned} \Sigma^{s^0, s^1} : \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) &\rightarrow \text{Tw } \mathcal{W}(\Sigma^{s^0} \mathbf{X}^0, \Sigma^{s^1} \mathbf{X}^1), \\ \left(\Sigma^{s^0, s^1} a \right)_{i^0 i^1} &= (-1)^{s^0(|a_{i^0 i^1}| - 1)} a_{i^0 i^1}. \end{aligned}$$

Note that $\Sigma^{s^2} \circ \Sigma^{s^0} = \Sigma^{s^0 + s^2}$ on objects and $\Sigma^{s^2, s^3} \circ \Sigma^{s^0, s^1} = \Sigma^{s^0 + s^2, s^1 + s^3}$ on morphisms. The shift respects the A_∞ -operations in the natural sense, that is,

$$(3.3) \quad \Sigma^{s^0, s^n} \circ \tilde{\mu}^n(a^n, \dots, a^1) = \tilde{\mu}^n(\Sigma^{s^{n-1}, s^n}(a^n), \dots, \Sigma^{s^0, s^1}(a^1)).$$

for any sequence of morphisms $\mathbf{X}^0 \xrightarrow{a^1} \mathbf{X}^1 \rightarrow \dots \xrightarrow{a^n} \mathbf{X}^n$.

For $s \in \mathbb{Z}$ set $\Sigma^s = \Sigma^{s, s}$ on morphisms. An immediate consequence of (3.3) is that $s \mapsto \Sigma^s$ induces a group homomorphism from $\mathbb{Z}$ to automorphisms of the triangulated envelope $\text{Tw } \mathcal{W}$

Let $(\mathbf{X}^0, \mathbf{X}^1)$ be a transverse pair of twisted complexes with $\mathbf{X}^j = (X^j = \{X_i^j[m_i^j]\}_{i \in I^j}, \delta^j)$ for $j = 0, 1$. Their *direct sum* $\mathbf{X}^0 \oplus \mathbf{X}^1$ is the twisted complex $(X^0 \sqcup X^1, \delta^0 + \delta^1)$. Note that $\mathbf{X}^0 \oplus \mathbf{X}^1 = \mathbf{X}^1 \oplus \mathbf{X}^0$.

Now let $a \in \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1)$ be a degree one closed morphism of twisted complexes. The *cone* of a is the twisted complex $\mathbf{C}(a) = (X^0 \sqcup X^1, \delta_{\mathbf{C}(a)})$ where

$$\delta_{\mathbf{C}(a)} \in \mathcal{A}_{\mathbf{C}(a)}^< = \mathcal{A}_{X^0}^< \oplus \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \oplus \mathcal{A}_{X^1}^<$$

is given by $\delta_{\mathbf{C}(a)} = \delta^0 + a + \delta^1$. Note that $\delta_{\mathbf{C}(a)}$ is indeed a bounding cochain in $\mathcal{A}_{\mathbf{C}(a)}^<$ since $\mu_{\delta_{\mathbf{C}(a)}}^0(1) = \tilde{\mu}^1(a) = 0$.

We extend the definition of a cone to closed morphisms of degree other than one (in practice we will only use it for morphisms of degree zero) by setting $\mathbf{C}(a) = \mathbf{C}\left(\Sigma^{1-d} \mathbf{X}^0 \xrightarrow{\Sigma^{1-d, 0} a} \mathbf{X}^1\right)$ for a closed morphism $a \in \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1)$ of degree d .

Observe that any twisted complex $\mathbf{X} = (X = \{X_i[m_i]\}_{i \in I}, \delta)$ can be represented as a cone. Namely, for any partition $I = I^0 \sqcup I^1$ such that $i^1 \not\prec i^0$ for

all $i^0 \in I^0$ and $i^1 \in I^1$ we have twisted complexes

$$\mathbf{X}^j = \left(X^j = \{X_i[m_i]\}_{i \in I^j}, \delta^j = \sum_{i_0, i_1 \in I^j} \delta_{i_0 i_1} [m_{i_1} - m_{i_0}] \right)$$

and $\mathbf{X} = \mathbf{C}(\mathbf{X}^0 \xrightarrow{a} \mathbf{X}^1)$ where

$$a = \sum_{i^0 \in I^0, i^1 \in I^1} a_{i^0 i^1} [m_{i^1} - m_{i^0}] = \delta - \delta^0 - \delta^1 \in \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1).$$

In particular, any twisted complex is an iterated cone of shifts of objects in $\mathcal{W}$.

3.3.3. Triangulated A_∞ -pre-category. Recall from §1.4.3 that an A_∞ -pre-category $\mathcal{W}$ is triangulated if there is a strict totalisation functor $\mathrm{Tot}: \mathrm{Tw} \mathcal{W} \rightarrow \mathcal{W}$. It is not hard to show that $\mathrm{Tw} \mathcal{W}$ is triangulated for any A_∞ -pre-category $\mathcal{W}$.

If $\mathcal{W}$ has enough homotopy equivalences, then one could also define $\mathcal{W}$ to be triangulated if any twisted complex over $\mathcal{W}$ is homotopy equivalent to an object of $\mathcal{W}$. The next statement shows that this alternative definition is implied by ours.

Lemma 3.9. *Let $\mathcal{W}$ be a triangulated A_∞ -pre-category with enough homotopy equivalences. Then*

- (1) *Two twisted complexes $\mathbf{X}^0, \mathbf{X}^1 \in \mathrm{Ob} \mathrm{Tw} \mathcal{W}$ are homotopy equivalent in $\mathrm{Tw} \mathcal{W}$ if and only if $X^0 = \mathrm{Tot}^0(\mathbf{X}^0)$ is homotopy equivalent to $X^1 = \mathrm{Tot}^0(\mathbf{X}^1)$ in $\mathcal{W}$.*
- (2) *Any twisted complex over $\mathcal{W}$ is homotopy equivalent to an object of $\mathcal{W}$.*

Lemma 3.9 implies, in particular, that we do not need to distinguish between homotopy equivalences in $\mathcal{W}$ and $\mathrm{Tw} \mathcal{W}$, that $\mathrm{Tw} \mathcal{W}$ has enough homotopy equivalences, and that a homotopy equivalence of twisted complexes can be tested only on objects of $\mathcal{W}$.

If $\mathcal{W}$ has enough homotopy equivalences, but not triangulated, then we do not know how to prove that homotopy inverse morphisms in $\mathcal{W}$ remain homotopy inverse in $\mathrm{Tw} \mathcal{W}$ or that $\mathrm{Tw} \mathcal{W}$ has enough homotopy equivalences. The issue is that we did not impose any conditions on $\tilde{\mu}^n(\dots, q, \dots)$, $n \geq 3$ for homotopy equivalences q . This can probably be fixed in a way similar to the rather involved homotopy unit framework of [23, §3.3], but we do not need it since the Fukaya categories that we construct geometrically are triangulated.

PROOF OF LEMMA 3.9. To prove (1) it suffices to show that if $e^1 \in \mathcal{W}(X^0, X^1)$ and $e^0 \in \mathcal{W}(X^1, X^0)$ are homotopy inverse morphisms in $\mathcal{W}$, then $\mathbf{e}^1 = \mathrm{Tot}_{\mathbf{X}^0, \mathbf{X}^1}^{-1}(e^1) \in \mathrm{Tw} \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1)$ and

$\mathbf{e}^0 = \text{Tot}_{\mathbf{X}^1, \mathbf{X}^0}^{-1}(e^0) \in \text{Tw } \mathcal{W}(\mathbf{X}^1, \mathbf{X}^0)$ are homotopy inverse morphisms in $\text{Tw } \mathcal{W}$. To see that this is indeed the case let $\{\mathbf{T}, \mathbf{X}^0, X^1\}$ be a transverse set in $\text{Tw } \mathcal{W}$. Denote $T = \text{Tot}^0(\mathbf{T})$. If H is a homotopy of $\mathcal{W}(T, X^0)$ certifying that $e^0 \circ (e^1 \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(T, X^0)}$, then $\text{Tot}_{\mathbf{T}, \mathbf{X}^0}^{-1} H \text{Tot}_{\mathbf{T}, \mathbf{X}^0}$ certifies that $\mathbf{e}^0 \circ (\mathbf{e}^1 \circ -)$ is homotopic to $\text{id}_{\text{Tw } \mathcal{W}(\mathbf{T}, \mathbf{X}^0)}$. The argument for the other three homotopies is analogous.

We prove (2). Consider a twisted complex $\mathbf{X}^0 \in \text{Tw } \mathcal{W}$. Since $\mathcal{W}$ has enough homotopy equivalences, there exists $X^1 \in \text{Ob } \mathcal{W}$ such that $e^1 \in \mathcal{W}(\text{Tot}^0(\mathbf{X}^0), X^1)$ and $e^0 \in \mathcal{W}(X^1, \text{Tot}^0(\mathbf{X}^0))$ are homotopy inverse morphisms in $\mathcal{W}$ and $\{\mathbf{X}^0, X^1\}$ is a transverse set in $\text{Tw } \mathcal{W}$. Applying part (1) of this lemma to $\mathbf{X}^0$ and X^1 concludes the proof. $\square$

CHAPTER 4

Proof of Theorem 1

4.1. Obstruction algebra

Let $\mathcal{L} = (L = (I, \gamma, P, \text{gr}), E)$ be a Lagrangian pre-brane in a weighted surface $(\mathcal{S} = (S; \lambda, \Lambda, D; \eta), w: D \rightarrow R)$. Recall that in §1.2.3 we defined the obstruction algebra $\mathcal{A}_{\mathcal{L}}$ of $\mathcal{L}$. In this section we check that $\mathcal{A}_{\mathcal{L}}$ is a structured A_{∞} -algebra. We prove this in §4.1.2 after reducing to the case of sufficiently self-wrapped Lagrangians from §2.3.3 in §4.1.1. In §4.1.3 we observe that Lagrangian isotopy induces isomorphism of obstruction algebras.

4.1.1. We show that the obstruction algebra does not change when passing from an arbitrary Lagrangian pre-brane to a sufficiently self-wrapped one as in Proposition 2.20.

Lemma 4.1. *Let $L^w = (I, \gamma^w, P^w, \text{gr}^w)$ be a sufficiently self-wrapped Lagrangian from Proposition 2.20 applied to L and a neighbourhood U not containing any points from $D \cup X_L$. Then $\psi: Z_L \rightarrow X_{L^w} = Z_{L^w}$ induces an isomorphism of obstruction algebras $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{L}^w}$ where $\mathcal{L}^w = (L^w, E)$.*

PROOF. ψ from Proposition 2.20 establishes a bijection between summands in (1.15) and ψ_* gives a bijection between summands in (1.17). Also note that the set $\{t \in I: \gamma^w(t) \neq \gamma(t)\}$ is contained in a neighbourhood of ∂I that retracts to ∂I , so we can assume that $E_{z_{\pm}} = E_{\psi(z)_{\pm}}$ for any $z \in Z_L$ and the transition maps of E along the boundaries of $\mathbf{u}$ and $\psi_*\mathbf{u}$ are the same for any $\mathbf{u}$ contributing to the structure maps. It therefore only remains to check the orientation classes from Construction 1.15 are compatible. By induction it suffices to check they are for a single Reidemeister move of type $II_{>0}$.

Let $C_0, \dots, C_n$ be connected components of I and $z^0, \dots, z^n \in Z_L$ be such that

$$z_+^0 \in C_0, z_-^0 \in C_n \text{ and } z_+^i \in C_{i-1}, z_-^i \in C_i \text{ for } i = 1, \dots, n.$$

Consider $\mathbf{u} \in \Delta^0(L; z^0; z^n, \dots, z^1)$ and $\mathbf{u}' = \psi_*(\mathbf{u}) \in \Delta^0(L'; \psi(z^0); \psi(z^n), \dots, \psi(z^1))$ where L' is the Lagrangian obtained from L by a Reidemeister move $II_{>0}$, ψ is the map from (2.9), and ψ_* is the map from (2.10). For $i = 1, \dots, n$ (resp. $i = 0$) let $r_i, r'_i \in |C_i|_R$ (resp. $r_0, r'_0 \in |C_n|_R$) be the orientation classes from Construction 1.15 applied for $\mathbf{u}$ and $\mathbf{u}'$. We need

to check that

$$(4.1) \quad r_1^{\otimes |z^1|} \otimes \dots \otimes r_n^{\otimes |z^n|} \otimes r_0^{\otimes (-|z^0|)} = (r'_1)^{\otimes |z^1|} \otimes \dots \otimes (r'_n)^{\otimes |z^n|} \otimes (r'_0)^{\otimes (-|z^0|)}$$

in

$$(|C_1|_R)^{\otimes |z^1|} \otimes \dots \otimes (|C_n|_R)^{\otimes (|z^n| - |z^0|)}.$$

In all cases from Construction 2.18 except the case b) we have $r_i = r'_i$ for all $i = 0, \dots, n$. In the case b) we have $r'_i = -r_i$ for all $i = 0, 1, 2$ and (4.1) still holds since $|z^0| = |z^2| + |z^1|$. $\square$

4.1.2. We check that the obstruction algebra is a structured A_∞ -algebra.

PROPOSITION 4.2. *The obstruction algebra $\mathcal{A}_\mathcal{L}$ is a $\mathbb{Z}$ -graded structured A_∞ -algebra over R .*

The argument is based on the decomposition of polygons with one concave corner by Chekanov [15, Theorem 3.3]. There are versions of this argument that take into account signs or local systems, see e.g. Abouzaid [1, §3] for the case of immersed curves in closed surfaces that do not bound any teardrops and Etnyre–Ng [20, §3] for an overview of various flavours of Chekanov–Eliashberg DGAs. However, we could not find a convenient reference for our setting, so we provide a proof.

PROOF. The set $Z = Z_L$ of double points of L and Reeb chords between the ends of L of positive action is finite by Lemma 2.10. The sums defining the structure maps μ^n are well-defined by Lemma 2.16(3). The degree of any μ^n , $n \geq 0$ is equal to $2 - n$ and the action property from Definition 1.13 is satisfied by Lemma 2.16(2). It remains to check the A_∞ -relations (1.14). By Lemma 4.1 we can assume L is sufficiently self-wrapped.

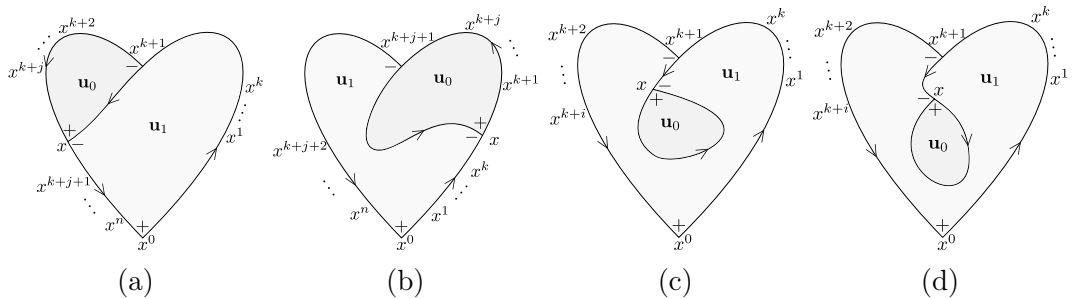


FIGURE 13. Extension of the incoming side of a negative concave corner.

Let $C_0, \dots, C_n$ be connected components of I and $x^0, \dots, x^n \in X_L = Z_L$ be such that

$$x_+^0 \in C_0, x_-^0 \in C_n \text{ and } x_+^i \in C_{i-1}, x_-^i \in C_i \text{ for } i = 1, \dots, n.$$

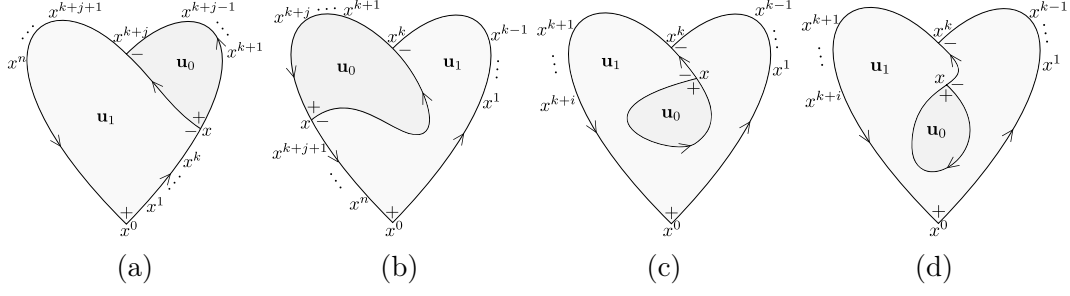


FIGURE 14. Extension of the outgoing side of a negative concave corner.

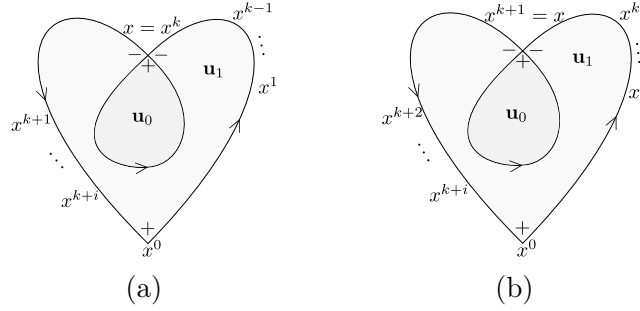


FIGURE 15. Decompositions of a polygon with one negative concave corner arising when extensions of the incoming and outgoing sides coincide.

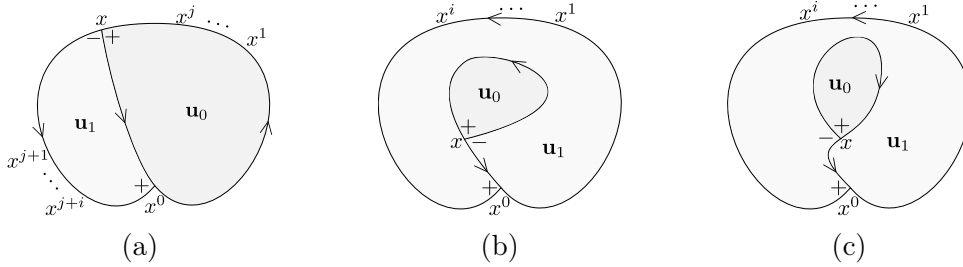


FIGURE 16. Extension of the outgoing side of a positive concave corner.

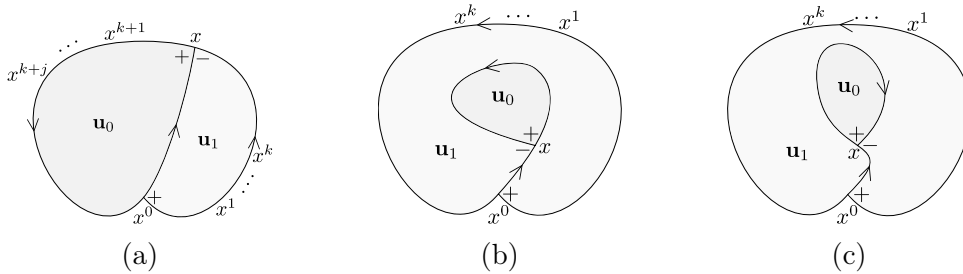


FIGURE 17. Extension of the incoming side of a positive concave corner.

We show that the entry

$$\mathcal{A}_{\mathcal{L}}(x^n) \otimes \dots \otimes \mathcal{A}_{\mathcal{L}}(x^1) \rightarrow \mathcal{A}_{\mathcal{L}}(x^0)$$

of the map in the left hand side of the A_∞ -relation (1.14) vanishes. Each summand of this entry is given by

$$(4.2) \quad (-1)^{|x^k|+\dots+|x^1|-k} \mu^{\mathbf{u}_1}(-^n \otimes \dots \otimes -^{k+j+1} \otimes \mu^{\mathbf{u}_0}(-^{k+j} \otimes \dots \otimes -^{k+1}) \otimes -^k \otimes \dots \otimes -^1)$$

where

- $i, j, k \in \mathbb{Z}_{\geq 0}$ and $i + j + k = n$;
- x is a double point of L ;
- $\mathbf{u}_0 = (\Pi_0, u_0) \in \Delta^0(L; x; x^{k+j}, \dots, x^{k+1})$;
- $\mathbf{u}_1 = (\Pi_1, u_1) \in \Delta^0(L; x^0; x^n, \dots, x^{k+j+1}, x, x^k, \dots, x^1)$.

Denote by T the set of the tuples $\mathbf{t} = ((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1)$ as above.

By [15, §7] there is an involution ι on T and a ι -invariant double cover $\theta: T \rightarrow \Delta^1(L; x^0; x^n, \dots, x^1)$ such that

- if $\theta((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1) = \mathbf{u} = (\Pi, u)$, then the pullback of $\mathbf{u}_0$ and $\mathbf{u}_1$ to Π via u is diffeomorphic to one of the pictures in Figs. 13 to 17 where Π has vertices at each point marked by x^i for $i = 0, \dots, n$, the corners which are not depicted are all negative;
- the involution ι swaps the pictures in Fig. 13 with that in Fig. 14, the two pictures in Fig. 15, and the pictures in Fig. 16 with that in Fig. 17.

Fix $\mathbf{u} = (\Pi, u) \in \Delta^1(L; x^0; x^n, \dots, x^1)$ and $\varphi^i \in \text{Hom}_R(E_{x_+^i}, E_{x_-^i})$ for $i = 1, \dots, n$. For $i = 1, \dots, n$ (resp. $i = 0$) let $r_i \in |C_i|_R$ (resp. $r_0 \in |C_n|_R$) be the orientation induced by the orientation of $\partial\Pi$. Let $\varphi^0 \in \text{Hom}_R(E_{x_+^0}, E_{x_-^0})$ be the composition

$$E_{x_+^0} \rightarrow E_{x_+^1} \xrightarrow{\varphi^1} E_{x_-^1} \rightarrow \dots \rightarrow E_{x_+^n} \xrightarrow{\varphi^n} E_{x_-^n} \rightarrow E_{x_-^0}$$

where the unnamed maps are the transition maps of E along $\partial\Pi$. Then for

$$\mathbf{t} = ((i, j, k), x, \mathbf{u}_0, \mathbf{u}_1) \in T \text{ with } \theta(\mathbf{t}) = \mathbf{u}$$

a straightforward computation using that $w(\mathbf{u}) = w(\mathbf{u}_0)w(\mathbf{u}_1)$, $|x| = |x^1| + \dots + |x^j| + 2 - j$, and $|x^0| = |x^1| + \dots + |x^n| + 3 - n$, shows that the map from (4.2) applied to $\varphi^n \otimes r_n^{\otimes |x^n|} \otimes \dots \otimes \varphi^1 \otimes r_1^{\otimes |x^1|}$ is equal to

- $(-1)^{|x^l|+\dots+|x^1|-l} w(\mathbf{u}) \varphi^0 \otimes r_0^{\otimes |x^0|}$ for $\mathbf{t}$ from Figs. 13 and 15(b);
- $(-1)^{|x^l|+\dots+|x^1|-l-1} w(\mathbf{u}) \varphi^0 \otimes r_0^{\otimes |x^0|}$ for $\mathbf{t}$ from Figs. 14 and 15(a);
- $w(\mathbf{u}) \cdot \varphi^0 \otimes r_0^{\otimes |x^0|}$ for $\mathbf{t}$ from Fig. 16;
- $-w(\mathbf{u}) \cdot \varphi^0 \otimes r_0^{\otimes |x^0|}$ for $\mathbf{t}$ from Fig. 17,

where for the first two cases l is the number of vertices of Π on the open interval of $\partial\Pi$ from x^0 to the concave corner. Now it is clear that the summands from (4.2) corresponding to $\mathbf{t}$ and $\iota(\mathbf{t})$ cancel out for each $\mathbf{t} \in T$. $\square$

4.1.3. We finish this section with the following simple observation.

Lemma 4.3. *Let $L_t = (I, \gamma_t, P_t, \text{gr}_t)$, $t \in [0, 1]$ be a Lagrangian isotopy between Lagrangians L_0 and L_1 and E a local system of R -modules on I . Then the isotopy induces an isomorphism of obstruction algebras $\mathcal{A}_{(L_0, E)} \rightarrow \mathcal{A}_{(L_1, E)}$.*

The isomorphism in Lemma 4.3 does not necessarily preserve the action. If the double points or Reeb chords $z_0^0, z_0^1 \in Z_{L_0}$ correspond to $z_1^0, z_1^1 \in Z_{L_1}$, then it is possible to have $A(z_0^0) < A(z_0^1)$, but $A(z_1^0) > A(z_1^1)$.

A similar statement for Legendrian isotopy is more involved and is proven in §5.3.

PROOF. Denote the Lagrangian pre-branes (L_0, E) and (L_1, E) by $\mathcal{L}_0$ and $\mathcal{L}_1$ respectively. For any $z^0 \in Z = Z_{L_0}$ we get the corresponding double points or Reeb chords $z^t \in L_t, t \in [0, 1]$ and hence the paths $z_+^t \in C_{z_+^0}, t \in [0, 1]$ from z_+^0 to z_+^1 and $z_-^t \in C_{z_-^0}, t \in [0, 1]$ from z_-^0 to z_-^1 . These paths induce isomorphisms of the respective fibres of the local system $E_{z_\pm^0} \rightarrow E_{z_\pm^1}$. These isomorphisms induce an isomorphism of graded R -modules $\mathcal{F}_1^1: \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ by definition (1.15).

Extending $\mathcal{F}_1^1$ to $\mathcal{F}_1 = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1): \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ by setting $\mathcal{F}_1^n = 0$ for $n \neq 1$ we obtain an isomorphism of A_∞ -algebras. Indeed, the isotopy gives a bijection between the polygons contributing to $\mu_{\mathcal{L}_0}$ and $\mu_{\mathcal{L}_1}$ in (1.17) respectively. That the contributions $\mu_{\mathcal{L}_0}^{\mathbf{u}_0}$ and $\mu_{\mathcal{L}_1}^{\mathbf{u}_1}$ from (1.18) of the corresponding polygons $\mathbf{u}_0$ and $\mathbf{u}_1$ satisfy

$$\mathcal{F}_1^1(\mu_{\mathcal{L}_0}^{\mathbf{u}_0}(-, \dots, -)) = \mu_{\mathcal{L}_1}^{\mathbf{u}_1}(\mathcal{F}_1^1(-), \dots, \mathcal{F}_1^1(-))$$

follows from the definition in (1.19) and the fact that E is a functor. The inverse to $\mathcal{F}_1$ is given by $\mathcal{F}_0: \mathcal{A}_{\mathcal{L}_1} \rightarrow \mathcal{A}_{\mathcal{L}_0}$ constructed analogously from the Lagrangian isotopy $L_{1-t}, t \in [0, 1]$ from L_1 to L_0 . $\square$

4.2. Fukaya A_∞ -pre-category

Let $(\mathcal{S}, w)$ be a weighted surface. In this section we demonstrate the following.

PROPOSITION 4.4. *The relative wrapped Fukaya category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ defined in §1.3.3 is a $\mathbb{Z}$ -graded R -linear A_∞ -pre-category such that transverse sequences form transverse sets.*

Any graded exact symplectomorphism $\varphi = (\varphi, \tilde{\varphi}) \in \text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}, \mathcal{S}')$ induces an isomorphism $\mathcal{I}_\varphi: \mathcal{W}_w(\mathcal{S}) \rightarrow \mathcal{W}_{w \circ \varphi^{-1}}(\mathcal{S}')$. In particular, $\mathcal{W}_w(\mathcal{S})$ does not depend on the Liouville structure and $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}, w)$ acts on $\mathcal{W}_w(\mathcal{S})$ by isomorphisms.

The group $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}, w)$ in Proposition 4.4 is the subgroup of $\text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S})$ consisting of symplectomorphisms φ such that $w \circ \varphi = w$.

After establishing the necessary lemmas in §4.2.1 and §4.2.2, we prove Proposition 4.4 in §4.2.3.

4.2.1. Shifting transverse Lagrangians. Let

$$L_i = (I_i, \gamma_i, P_i), \quad i = 0, \dots, n$$

be transverse Lagrangians in $\mathcal{S}$, see §1.3.1. Recall that for $i \neq j$ we have the set $Z(L_i, L_j)$ of intersection points between L_i and L_j and Reeb chords from an end of L_i to an end of L_j and that for a shift vector $h \in \mathbb{R}^{n+1}$ we defined the action A_h on $Z(L_i, L_j)$. Note that

$$A_h(z) - A_0(z) = h_i - h_j \text{ for any } z \in Z(L_i, L_j).$$

We therefore have a characterisation of admissible shift vectors (i.e. such that the composite Lagrangian $L_h(L_0, \dots, L_n)$ is actually a Lagrangian),

$$\mathcal{H}(L_0, \dots, L_n) = \left\{ h \in \mathbb{R}^{n+1} : h_j - h_i \neq A_0(z) \text{ for all } \begin{array}{l} i, j \in \{0, \dots, n\}, i \neq j \\ \text{and } z \in Z(L_i, L_j) \end{array} \right\}.$$

That is, $\mathcal{H}(L_0, \dots, L_n)$ is the complement of a hyperplane arrangement in $\mathbb{R}^{n+1}$. The number of hyperplanes in the arrangement can be infinite, but the arrangement is discrete since $\{z \in Z(L_i, L_j) : A_0(z) > B\}$ is finite for all $i, j \in \{0, \dots, n\}, i \neq j$ and $B \in \mathbb{R}$ by Lemma 2.10.

Let L_h be a composite Lagrangian of $L_0, \dots, L_n$. Fix $i, j \in \{0, \dots, n\}, i \neq j$. Then A_h and A_{L_h} are two action functions defined on $Z(L_i, L_j)$. Note that

$$A_{L_h}(z) = \begin{cases} |A_h(z)|, & \text{if } z \in X(L_i, L_j); \\ A_h(z), & \text{if } z \in Y(L_i, L_j). \end{cases}$$

We finish this discussion with the following easy but useful lemma.

Lemma 4.5. *Given finite sets $Z_{ij} \subset Z(L_i, L_j)$, $0 \leq i < j \leq n$ there exists an admissible shift vector $h \in \mathbb{R}^{n+1}$ such that for any $0 \leq i < j \leq n$ we have*

$$(4.3) \quad A_h(z) > 0 \text{ for any } z \in Z_{ij} \text{ and } A_h(z) < 0 \text{ for any } z \in Z(L_j, L_i)$$

PROOF. Recall that $A_h(z) = A_0(z) + h_i - h_j$ for $z \in Z(L_i, L_j)$ and that $\{z \in Z(L_i, L_j) : A_0(z) > 0\}$ is finite for all $i, j \in \{0, \dots, n\}, i \neq j$. Therefore, for each $0 \leq i < j \leq n$ there exists

$$M_{ij} = \min(\{0\} \cup \{A_0(z) \mid z \in Z_{ij}\} \cup \{-A_0(z) \mid z \in Z(L_j, L_i)\}) \in \mathbb{R}.$$

Observe that if $h_j - h_i < M_{ij}$ for each $0 \leq i < j \leq n$, then (4.3) holds. Since the left-hand sides of these inequalities are given by linear combinations with non-negative coefficients of n linearly independent functionals $h \mapsto h_i -$

h_{i-1} , $i = 1, \dots, n$, these inequalities define a non-empty open subset of $\mathbb{R}^{n+1}$. The lemma follows since $\mathcal{H}(L_0, \dots, L_n)$ is dense in $\mathbb{R}^{n+1}$. $\square$

4.2.2. Composite Lagrangian branes. In §1.3.3 we defined the A_∞ -operations $\tilde{\mu}$ in $\mathcal{W}$ by computing them in certain composite Lagrangian branes. We check that this computation does not depend on the choice of a composite Lagrangian brane.

Lemma 4.6. *Let $\mathbb{L}_i = (I_i, \gamma_i, P_i, \text{gr}_i, E_i, \delta_i)$, $i = 0, \dots, n$ be Lagrangian branes, $h, h' \in \mathbb{R}^{n+1}$ two admissible shift vectors, and $\mathbb{L} = \mathbb{L}_h = (I, \gamma, P, \text{gr}, E, \delta)$, $\mathbb{L}' = \mathbb{L}_{h'} = (I, \gamma, P', \text{gr}, E, \delta)$ the corresponding composite Lagrangian branes. Then*

- (1) *For any $z \in Z(\mathbb{L}_i, \mathbb{L}_j)$ if the actions $A_h(z)$ and $A_{h'}(z)$ are positive, then $A_L(z) = A_h(z)$, $A_{L'}(z) = A_{h'}(z)$, and $\mathcal{A}_L(z) = \mathcal{A}_{L'}(z)$.*
- (2) *Let $z^0 \in Z(\mathbb{L}_0, \mathbb{L}_n)$ and $z^i \in Z(\mathbb{L}_{i-1}, \mathbb{L}_i)$, $i = 1, \dots, n$ be such that $A_h(z^i) > 0$ and $A_{h'}(z^i) > 0$ for all $i = 1, \dots, n$. Suppose that*

$$\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} : \mathcal{A}_{\mathbb{L}}(z^n) \otimes \dots \otimes \mathcal{A}_{\mathbb{L}}(z^1) \rightarrow \mathcal{A}_{\mathbb{L}}(z^0)$$

is non-zero. Then $A_h(z^0) > 0$, $A_{h'}(z^0) > 0$ and $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}'}^{z^0; z^n, \dots, z^1}$.

PROOF. To show (1) first note that $A_h(z) \neq 0$ and $A_{h'}(z) \neq 0$ since h and h' are admissible. Positivity of these actions implies that the definitions of z_+ and z_- are the same for $\mathbb{L}$, $\mathbb{L}'$ (these are given in §1.1.3) and for $(L_0, \dots, L_n)$ (this is given in §1.3.1). Namely, we have $z_+ \in I_i$ and $z_- \in I_j$. This implies $A_L(z) = A_h(z)$, $A_{L'}(z) = A_{h'}(z)$ by comparing definitions in (1.4), (1.6), and (1.23), and $\mathcal{A}_L(z) = \mathcal{A}_{L'}(z)$ by (1.16).

We prove (2). Observe that

$$A_h(z^i) = A_L(z^i) > 0 \text{ and } A_{h'}(z^i) = A_{L'}(z^i) > 0 \text{ for } i = 1, \dots, n.$$

by (1) of this lemma and

$$A_{L_i}(z) = A_h(z) = A_{h'}(z) = A_L(z) \text{ for any } z \in Z_{L_i}, \ i = 0, \dots, n$$

by definition. Non-vanishing of $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1}$ implies that there exists

$$\mathbf{u} \in \Delta^0(L; z^0; z_{j_n}^n, \dots, z_1^n, z^n, z_{j_{n-1}}^{n-1}, \dots, z_1^1, z^1, z_{j_0}^0, \dots, z_1^0)$$

for some $j_0, \dots, j_n \geq 0$ and $z_j^i \in Z_{L_i}$.

Note that $z_+^0 \in I_0$ and $z_-^0 \in I_n$ by any definition. For L it follows from the existence of $\mathbf{u}$, and for L' this follows by considering $\mathbf{u}$ as an element in $\Delta^0(L'; z^0; \dots)$ and applying Lemma 2.16(2). Therefore, we have $A_h(z^0) = A_L(z^0) > 0$ and $A_{h'}(z^0) = A_{L'}(z^0) > 0$. Since any such $\mathbf{u}$ can be viewed as an element of $\Delta^0(L'; z^0; \dots)$ and vice versa, and as the definition of $\mu_{\mathbb{L}}$ and

$\mu_{\mathbb{L}'}$ does not depend on the values of the actions, but only on their signs, we conclude $\mu_{\mathbb{L}}^{z^0; z^n, \dots, z^1} = \mu_{\mathbb{L}'}^{z^0; z^n, \dots, z^1}$. $\square$

4.2.3. Proof of Proposition 4.4. That subsets of transverse sets of Lagrangians are transverse is clear from the definition in §1.3.1 hence transverse sequences form transverse sets in $\mathcal{W}$. $\mathcal{W}$ is R -linear by construction.

That the choice of an admissible shift vector required in the definition of A_∞ -operations in $\mathcal{W}$ from §1.3.3 can be made follows from Lemma 4.5. By Lemma 4.6 the operation $\tilde{\mu}^{z^0; z^n, \dots, z^1}$ does not depend on this choice, and has the correct degree since the A_∞ -algebra $\mathcal{A}_{\mathbb{L}}$ is $\mathbb{Z}$ -graded.

To prove that the A_∞ -relations hold in $\mathcal{W}$ consider a transverse sequence $(\mathbb{L}_0, \dots, \mathbb{L}_n)$ of Lagrangian branes and a finite set of morphisms $\{a^k \in \mathcal{W}(\mathbb{L}_{i_k}, \mathbb{L}_{j_k})\}_{k \in K}$ such that $i_k < j_k$ for all $k \in K$. By Lemma 4.5 there exists a composite Lagrangian brane $\mathbb{L} = \mathbb{L}_h(\mathbb{L}_0, \dots, \mathbb{L}_n)$ such that $A_h(z) > 0$ whenever the $\mathcal{W}(z)$ -component of one of the morphisms a^k is non-zero. We call such composite Lagrangian brane $\mathbb{L}$ *adapted* to $\{a^k\}_{k \in K}$. Note that if $\mathbb{L}$ is adapted to $\{a^k\}_{k \in K}$, then for any $l \geq 1$, $k_1, \dots, k_l \in K$ we have

$$(4.4) \quad \mu_{\mathbb{L}}^l(a^{k_l}, \dots, a^{k_1}) = \begin{cases} \tilde{\mu}_{\mathcal{W}}^l(a^{k_l}, \dots, a^{k_1}), & \text{if } \tilde{\mu}_{\mathcal{W}}^l(a^{k_l}, \dots, a^{k_1}) \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases}$$

That is, the A_∞ -relation on a given sequence $\mathbb{L}_0 \xrightarrow{a^1} \dots \xrightarrow{a^n} \mathbb{L}_n$ of morphisms in $\mathcal{W}$ boils down to an A_∞ -relation in a composite Lagrangian brane $\mathbb{L}$ and holds since $\mathcal{A}_{\mathbb{L}}$ is an uncurved A_∞ -algebra.

Now consider a graded exact symplectomorphism $\varphi = (\varphi, \tilde{\varphi}) \in \text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}, \mathcal{S}')$. Recall that it induces a bijection on Lagrangian pre-branes in $\mathcal{S}$ and $\mathcal{S}'$ by

$$\varphi_*(I, \gamma, P, \text{gr}, E) = (I, \varphi \circ \gamma, P - g \circ \gamma, \varphi_*(\text{gr}) \cdot \tilde{\varphi}, E)$$

where $g \in C^\infty(S, \mathbb{R})$ is such that $\varphi^* \lambda_1 - \lambda_0 = dg$ and $g(q_0) = 0$ for some fixed basepoint $q_0 \in S$. By inspecting the definitions of obstruction algebra in §1.2.3 and of the Fukaya category in §1.3.3 we observe that this correspondence induces an isomorphism of obstruction algebras $\varphi_*: \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\varphi_* \mathcal{L}}$ for any Lagrangian pre-brane $\mathcal{L}$ in $\mathcal{S}$ and an isomorphism $\mathcal{I}_\varphi: \mathcal{W}_{\text{w}}(\mathcal{S}) \rightarrow \mathcal{W}_{\text{w} \circ \varphi^{-1}}(\mathcal{S}')$.

If $\mathcal{S}'$ only differs from $\mathcal{S}$ by the Liouville structure, then by Corollary 2.8 there exists a symplectomorphism $\varphi = (\varphi, \tilde{\varphi}) \in \text{Symp}_{\text{ex}}^{\text{gr}}(\mathcal{S}, \mathcal{S}')$ such that $\varphi|_{\Lambda \cup D} = \text{id}$ hence we obtain an isomorphism $\varphi_*: \mathcal{W}_{\text{w}}(\mathcal{S}) \rightarrow \mathcal{W}_{\text{w}}(\mathcal{S}')$. $\square$

4.3. Nearby Lagrangians

In this section we study transverse perturbations of a Lagrangian. This is preparatory work needed to establish that the relative wrapped Fukaya category of a weighted surface has enough homotopy equivalences.

In §4.3.1 we describe the self-intersection points and Reeb chords of a composite Lagrangian of a Lagrangian and its transverse perturbation. In §4.3.2 we classify the small polygons with boundary on a Lagrangian and its transverse perturbation. In §4.3.3 we explain how some polygons with boundary on a Lagrangian can be perturbed to have the boundary on the transverse perturbation.

Throughout this section we work in a fixed graded Liouville marked surface $\mathcal{S} = (S; \Lambda, D; \lambda; \eta)$.

4.3.1. Transverse double of a Lagrangian. Let $L_0 = (I, \gamma_0, P_0, \text{gr}_0)$ be a graded Lagrangian.

Definition 4.7. A graded Lagrangian $L_1 = (I, \gamma_1, P_1, \text{gr}_1)$ is called a *transverse perturbation* of L_0 if L_1 is Lagrangian isotopic to L_0 through an isotopy L_t , $t \in [0, 1]$ such that L_t is transverse (in the sense of §1.3.1) to L_0 for all $t \in (0, 1]$.

Note that by the Legendrian neighbourhood theorem [16, Proposition 6.20] if $L_1 = (I, \gamma_1, P_1, \text{gr}_1)$ is sufficiently C^∞ -close to L_0 in $\text{Lag}_I^{\text{gr}}(\mathcal{S})$, then there is a contactomorphism $\varphi: (J^1 I = T^*I \times \mathbb{R}_z, \lambda_{\text{can}} + dz) \rightarrow (S^+, \lambda^+)$ such that $(\gamma_0, P_0) = \varphi \circ (t \mapsto (t, 0, 0))$ and $(\gamma_1, P_1) = \varphi \circ (t \mapsto (t, df(t), f(t)))$ for some $f \in C^\infty(I, \mathbb{R})$. If moreover L_1 is transverse to L_0 , then $s \mapsto sf$, $s \in [0, 1]$ gives an isotopy from Definition 4.7. We therefore obtain the following statement.

Lemma 4.8. *There is a neighbourhood $\mathcal{U}$ of L_0 in $\text{Lag}_I^{\text{gr}}(\mathcal{S})$ such that any graded Lagrangian L_1 from $\mathcal{U}$ that is transverse to L_0 is a transverse perturbation of L_0 .*

Let L_1 be a transverse perturbation of L_0 . By Lemma 4.5 we can pick an admissible shift vector $h \in \mathbb{R}^2$ such that $L_h = L_h(L_0, L_1)$ satisfies properties (1)–(5) below. In this case we call L_h a *transverse double* of L_0 .

- (1) For each double point x^0 of L_0 , there is a double point $x^1 = x^1(x^0)$ of L_1 , intersection points $x^{01} = x^{01}(x^0), \bar{x}^{01} = \bar{x}^{01}(x^0) \in X(L_0, L_1)$ and a neighbourhood U_{x^0} such that $\{U_{x^0}\}_{x^0 \in X_{L_0}}$ are pairwise disjoint and L_h in U_x is diffeomorphic to one of the pictures in Fig. 18.
- (2) For each connected component C of I that is a circle, there is an integer $k_C \geq 1$ such that the intersection points in $Z(L_0|_C, L_1|_C) = X(L_0|_C, L_1|_C)$ that are not of the form x^{01} or $\bar{x}^{01}$ from

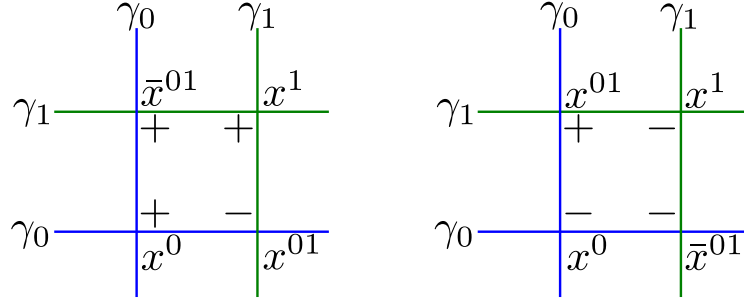


FIGURE 18. Neighbourhood of a perturbed self-intersection

(2) split into two sets, $Q_C = \{q^i\}_{i \in \mathbb{Z}/k_C\mathbb{Z}}$ and $P_C = \{p^i\}_{i \in \mathbb{Z}/k_C\mathbb{Z}}$. These points satisfy $A_h(q^i) > 0$, $A_h(p^i) > 0$, $|q^i| = 0$, and $|p^i| = 1$ for each $i \in \mathbb{Z}/k_C\mathbb{Z}$ and

$$|\Delta^0(L_h; p^i; q^j)| = \begin{cases} 1, & \text{if } i = j \text{ or } i + 1 = j \text{ in } \mathbb{Z}/k_C\mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

That is, there is an immersion $i_C: T_C \subset \mathbb{R}^2 \rightarrow S \setminus (\Lambda \cup D)$, where T_C is the shaded area in Fig. 19(a), that extends the immersions given by bigons from $\Delta^0(L_h; p^i; q^j)$ (that the numbering goes counter-clockwise is a condition on the indexing of Q_C and P_C).

- (3) For each $t \in \partial I$ there exists a unique Reeb chord $y_t \in Z(L_0, L_1) \cup Z(L_1, L_0)$ that is isotoped to the constant path at $\gamma_1(t)$ by the isotopy from L_0 to L_1 . $A_h(y_t) > 0$ if $y_t \in Z(L_0, L_1)$ and $A_h(y) < 0$ for any $y \in Z(L_1, L_0)$.
- (4) For each connected component C of I that is a segment, there is an integer $k_C \geq 1$ such that the intersection points in $Z(L_0|_C, L_1|_C)$ that are not of the form x^{01} or $\bar{x}^{01}$ from (2) and the Reeb chords $y_t \in Z(L_0|_C, L_1|_C)$ for $t \in \partial I$ split into two sets, $Q_C = \{q^i\}_{i=1}^{k_C}$ and $P_C = \{p^i\}_{i=1}^{k_C-1}$. These points and chords satisfy $A_h(q^i) > 0$, $A_h(p^i) > 0$, $|q^i| = 0$, and $|p^i| = 1$ for each i and

$$|\Delta^0(L_h; p^i; q^j)| = \begin{cases} 1, & \text{if } i = j \text{ or } i + 1 = j; \\ 0, & \text{otherwise.} \end{cases}$$

That is, there is an immersion $i_C: T_C \subset \mathbb{R}^2 \rightarrow S \setminus (\Lambda \cup D)$, where T_C is the closed shaded area in Fig. 19(b), that extends the immersions given by bigons from $\Delta^0(L_h; p^i; q^j)$.

- (5) The set of self-intersection points of L_h satisfies

$$X_{L_h} \subset \bigsqcup_{x^0 \in X_{L_0}} \{x^0, x^1, x^{01}, \bar{x}^{01}\} \sqcup \bigsqcup_{C \in \pi_0 I} Q_C \sqcup P_C.$$

and the difference between the right hand side and X_{L_h} is $\{y_t \mid t \in \partial I\} \cap Z(L_0, L_1)$.

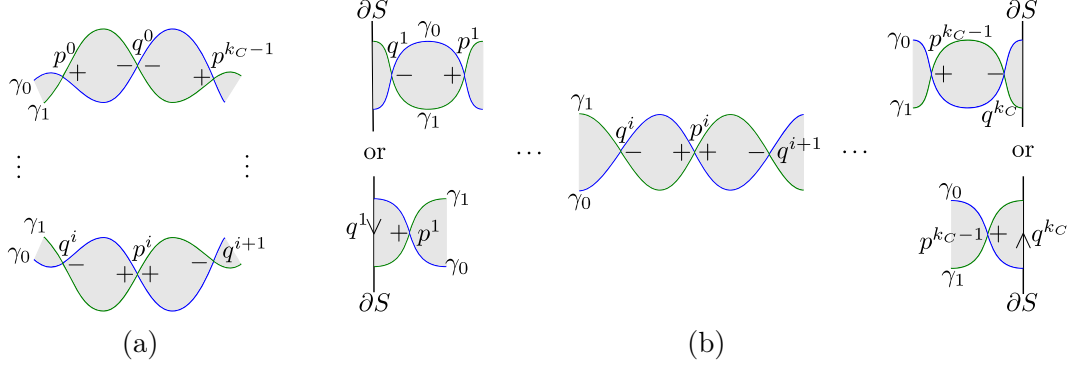


FIGURE 19. Pullbacks of perturbed components of the Lagrangian along bigons

Note that even if L_1 is very C^∞ -close to L_0 , we might not be able to choose the admissible vector h close to $0 \in \mathbb{R}^2$. Namely, for $x^0 \in X_{L_0}$ the local model from (2) around this self-intersection implies $A_0(\bar{x}^{01}) + h_0 - h_1 = A_h(\bar{x}^{01}) > 0$, but $A_0(\bar{x}^{01}) = P_0(\bar{x}_+^{01}) - P_1(\bar{x}_-^{01})$ is close to $P_0(x_-^0) - P_0(x_+^0) = -A_{L_0}(x^0)$ which is a negative number that depends only on L_0 , not on the perturbation.

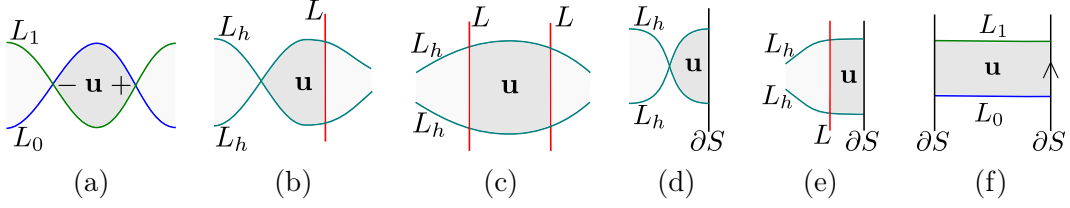
4.3.2. Lagrangians containing a transverse double. Let $L_0 = (I, \gamma_0, P_0, \text{gr}_0)$ and $L = (J, \gamma, P, \text{gr})$ be two graded Lagrangians. We say that L contains a transverse double of L_0 nicely if L is the composite Lagrangian of L_0 , L_1 , and L_2 with a shift vector $h = (h_0, h_1, h_2) \in \mathbb{R}^3$ such that

- (1) $L_1 = (I, \gamma_1, P_1, \text{gr}_1)$ is a transverse perturbation of L_0 and $L_{(h_0, h_1)}(L_0, L_1)$ is a transverse double of L_0 .
- (2) For a connected component C of I the closure (in $\mathbb{R}^2$) of the pullback of γ by the immersion i_C to the interior of the shaded region T_C in Fig. 19 is a collection of disjoint segments with one end on γ_0 and another end on γ_1 .

Note that the local model for self-intersection from (2) in §4.3.1 implies that a transverse double of L_0 contains itself nicely.

We give a criterion for a polygon with convex corners bounded by L to be contained in the shaded region T_C from Fig. 19.

Lemma 4.9. *Let $L = (J, \gamma, P, \text{gr})$ be a Lagrangian containing a transverse double $L_h = L_h(L_0, L_1)$ of $L_0 = (I, \gamma_0, P_0, \text{gr}_0)$ nicely, C a connected component of I , $\mathbf{u} = (\Pi, u)$ a polygon with convex corners bounded by L , and $v, v' \in \Pi$ two vertices of Π such that $v = v'$ or $[v, v']$ is a side of Π . Suppose that $u(v) \in \text{Im } \gamma_0$, $u(v') \in \text{Im } \gamma_1$, and there is an embedding $\varphi: U \rightarrow T_C$ where U is a neighbourhood of $[v, v']$ in Π such that $i_C \circ \varphi = u|_U$. Then φ can be extended to an embedding $\varphi: \Pi \rightarrow T_C$ such that $i_C \circ \varphi = u$. That is, $\mathbf{u}$ is one of the polygons in Fig. 20.*

FIGURE 20. Polygons contained in T_C

Note that convex polygons bounded by L that have a negative corner at Q_C or a positive corner at P_C satisfy the assumption of Lemma 4.9. This lemma is similar to [1, Lemma 4.4], our proof is somewhat different.

PROOF. Note that i_C induces an orientation on C : it is the counter-clockwise orientation in Fig. 19(a) if C is a circle and the orientation from left to right in Fig. 19(b) if C is a segment. By abuse of notation we will not always distinguish between the objects in S and their pullbacks to $\mathbb{R}^2$ via i_C .

Let the regions $R_1, \dots, R_r \subset S$ be the closures of connected components of $S \setminus \text{Im } \gamma$. Since L contains a transverse double of L_0 nicely, the pullback of γ to $T = T_C$ consists of the boundary of T and a collection of disjoint segments with one end on γ_0 and another end on γ_1 . These segments divide T into regions each of which is a copy of some R_i , $i = 1, \dots, r$ and is diffeomorphic to one of the darker shaded regions in Fig. 20 (with an additional condition that there are no segments of γ inside the region). Let $T_1, \dots, T_l$ be these regions, listed in the direction of the orientation of C . When C is a circle we think of the indices $1, \dots, l$ as elements of $\mathbb{Z}/l\mathbb{Z}$, the indexing as above is not unique in this case, we pick any.

Pulling γ to Π along u we see that Π is glued from copies of the regions R_i , $i = 1, \dots, r$. To prove the lemma it suffices to show that these regions are $T_j, T_{j+1}, \dots, T_{j+m}$ or $T_j, T_{j-1}, \dots, T_{j-m}$ for some $m \geq 0$ and Π is glued from them the same way as T .

Let R_{i_0} be the region that contains a neighbourhood of $[v, v']$ in Π . That $[v, v']$ is contained in one region is clear when $v = v'$ and follows from the existence of φ from the assumption when $[v, v']$ is a side of Π . The existence of φ also implies that R_{i_0} has a copy $T_j \subset T$. Set $\sigma = 1$ if T_j follows $[u(v), u(v')]$ in the direction of C , set $\sigma = -1$ otherwise.

Let v^1 (resp. v^{-1}) be the vertex of Π following (resp. preceding) the (possibly degenerate) segment $[v, v']$ in the counter-clockwise order on $\partial\Pi$. Let w^1 (resp. w^{-1}) be the vertex of R_{i_0} that follows (resp. precedes) the segment $[v, v']$ in the counter-clockwise order on ∂R_{i_0} . If $w^1 \neq v^1$, then define R_{i_1} to be the region that contains the corner at w^1 not contained in R_{i_0} . Observe that this region is the same as the region $T_{j+\sigma}$. Therefore, w^{-1} is a vertex of R_{i_1}

and the regions R_{i_0} and R_{i_1} share the side $[w^{-1}, w^1]$. Let w^2 (resp. w^{-2}) be the vertex of R_{i_1} that follows (resp. precedes) w^1 (resp. w^{-1}) in the counter-clockwise order on ∂R_{i_1} . If $w^2 \neq v^1$, then define R_{i_2} to be the region that contains the corner at w^2 not contained in R_{i_1} . Observe that this region is the same as the region $T_{j+2\sigma}$. Therefore, w^{-2} is a vertex of R_{i_2} and the regions R_{i_1} and R_{i_2} share the side $[w^{-2}, w^2]$. Continuing this process while we can we obtain a sequence of regions $R_{i_0}, \dots, R_{i_m} \subset \Pi$ and the corresponding sequence of regions $T_j, \dots, T_{j+m\sigma} \subset T$ for some $m \geq 0$. Since we cannot continue, $w^{m+1} = v^1$.

If $w^{-m-1} = w^{m+1}$, then $R_{i_0}, \dots, R_{i_m}$ cover all Π and we are done. In this case $\mathbf{u}$ is a polygon from Figs. 20(a), 20(b), and 20(d) with $u(w^{m+1}) \in P_C \cup Q_C$. Suppose $w^{-m-1} \neq w^{m+1}$. Then $[w^{-m-1}, w^{m+1}]$ is a side of R_{i_m} since $[u(w^{-m-1}), u(w^{m+1})]$ is a side of $T_{j+m\sigma}$. Moreover, as $v^1 = w^{m+1}$, we have $[w^{-m-1}, w^{m+1}] \subset \partial \Pi$. Therefore, going from v^0 to w^{m+1} to w^{-m-1} to v is a closed curve on $\partial \Pi$ hence $R_{i_0}, \dots, R_{i_m}$ cover all Π and we are done. If $[u(w^{-m-1}), u(w^{m+1})]$ is a sub-segment of γ , then $\mathbf{u}$ is a polygon from Figs. 20(b), 20(c), and 20(e) with $[u(w^{-m-1}), u(w^{m+1})]$ being a red side. If $[u(w^{-m-1}), u(w^{m+1})]$ is a sub-segment of ∂S , then $\mathbf{u}$ is a polygon from Figs. 20(d) to 20(f) with $[u(w^{-m-1}), u(w^{m+1})]$ being a side lying on the boundary.

□

4.3.3. Transport of data to a transverse perturbation. Let $L = L_h(L_0, L_1, L_2)$ be a Lagrangian containing the transverse double $L_{(h_0, h_1)}(L_0, L_1)$ of L_0 nicely. Then, pushing the pullback of L_0 towards the pullback of L_1 in T_C , we can choose the Lagrangian isotopy $L_t, t \in [0, 1]$ from Definition 4.7 such that L_t is transverse to L_2 for all $t \in [0, 1]$, see Figs. 19 and 20.

We describe partially defined functions $\theta_{\pm}, \bar{\theta}_{\pm}: Z_L \rightarrow Z_L$ that transport double points and Reeb chords along the isotopy L_t . Fix $z \in Z_L$.

- If z is a Reeb chord that starts on L_0 or L_1 , then $\theta_+(z)$ is the Reeb chord obtained from z by isotoping its starting point by L_t provided that $A(\theta_+(z)) > 0$.
- If z is a Reeb chord that ends on L_0 or L_1 , then $\theta_-(z)$ is the Reeb chord obtained from z by isotoping its ending point by L_t provided that $A(\theta_+(z)) > 0$.
- All other $\theta_{\pm}(z), \bar{\theta}_{\pm}(z)$ are not defined if z is a Reeb chord.

- If $z \in \{x^0, x^1, x^{01}, \bar{x}^{01}\}$ for some $x^0 \in Z_{L_0}$ as in Fig. 18, then

$$\begin{aligned} \bar{\theta}_+(x^0) &= \bar{x}^{01}, \quad \theta_-(x^0) = x^{01}; & \theta_+(x^{01}) &= x^1, \quad \theta_-(x^{01}) = x^0; \\ \bar{\theta}_+(x^{01}) &= x^1, \quad \bar{\theta}_-(x^{01}) = x^0; & \theta_+(x^1) &= x^{01}, \quad \bar{\theta}_-(x^1) = \bar{x}^{01}. \end{aligned}$$

All other values of $\theta_\pm(z)$ and $\bar{\theta}_\pm(z)$ are not defined.

- If $z \in Z_{L_2}$ or $z \in Q_C \cup P_C$ for a connected component C of the domain of L_0 , then $\theta_\pm(z)$ and $\bar{\theta}_\pm(z)$ are not defined.
- If z is any other double point, then by (2) in §4.3.2 z is an end of a segment $[z^0, z^1]$ where z^i has one of the strands lying in L_i and another strand lying in L_2 for $i = 0, 1$. We have

$$\begin{aligned} \theta_+(z^i) &= z^{1-i} \text{ if the overstrand at } z^i \text{ lies in } L_i \text{ and the overstrand at } z^{1-i} \text{ lies in } L_{1-i}; \\ \theta_-(z^i) &= z^{1-i} \text{ if the understrand at } z^i \text{ lies in } L_i \text{ and the understrand at } z^{1-i} \text{ lies in } L_{1-i}; \\ \bar{\theta}_+(z^i) &= z^{1-i} \text{ if the overstrand at } z^i \text{ lies in } L_i \text{ and the understrand at } z^{1-i} \text{ lies in } L_{1-i}; \\ \bar{\theta}_-(z^i) &= z^{1-i} \text{ if the understrand at } z^i \text{ lies in } L_i \text{ and the overstrand at } z^{1-i} \text{ lies in } L_{1-i}. \end{aligned}$$

All other values of $\theta_\pm(z)$ and $\bar{\theta}_\pm(z)$ are not defined.

Note that we have $|\theta_\pm(z)| = |z|$ and $|\bar{\theta}_\pm(z)| = 1 - |z|$ whenever the left hand side is defined. Observe that if a Reeb chord z starts on L_1 , then $\theta_+(z)$ is always defined and if a Reeb chord z ends on L_0 , then $\theta_-(z)$ is always defined. Consider a double point $z \in X_{L_0}$. If $\theta_\pm(z)$ is defined, then L_t isotopes positive (negative) corners of z to positive (resp. negative) corners of $\theta_\pm(z)$. If $\bar{\theta}_\pm(z)$ is defined, then L_t isotopes positive (negative) corners of z to negative (resp. positive) corners of $\theta_\pm(z)$. Finally, note that θ_+ and θ_- are involutions of their domains.

We now analyse how θ affects the polygons.

Lemma 4.10. *Let $L = L_h(L_0, L_1, L_2)$ be a Lagrangian containing the transverse double $L_{(h_0, h_1)}(L_0, L_1)$ of L_0 nicely, and $z^0, \dots, z^n \in Z_L$ double points and Reeb chords of L .*

- (1) *Suppose that $\theta_-(z^i)$ and $\theta_+(z^{i+1})$ are defined for some $i = 1, \dots, n-1$. Then the isotopy L_t induces a bijection*

$$\begin{array}{c} \Delta^0(L; z^0, z^n, \dots, z^1) \\ \theta_* \downarrow \\ \Delta^0(L; z^0, z^n, \dots, z^{i+2}, \theta_+(z^{i+1}), \theta_-(z^i), z^{i-1}, \dots, z^1). \end{array}$$

(2) Suppose that $\theta_+(z^0)$ and $\theta_+(z^1)$ are defined. Then the isotopy L_t induces a bijection

$$\Delta^0(L; z^0; z^n, \dots, z^1) \xrightarrow{\theta_*} \Delta^0(L; \theta_+(z^0); z^n, \dots, z^2, \theta_+(z^1)).$$

(3) Suppose that $\theta_-(z^0)$ and $\theta_-(z^n)$ are defined. Then the isotopy L_t induces a bijection

$$\Delta^0(L; z^0; z^n, \dots, z^1) \xrightarrow{\theta_*} \Delta^0(L; \theta_-(z^0); \theta_-(z^n), z^{n-1}, \dots, z^1).$$

PROOF. If $n = 2$, z^0, z^1, z^2 are Reeb chords and $z^0 = z^2 z^1$ (i.e. we are in case (2c) of Definition 1.5), then it is straightforward to see that both the domain and codomain of θ_* are singletons.

In the rest of the cases in Definition 1.5, $\theta_*(\mathbf{u})$ is obtained from $\mathbf{u}$ by moving one of its sides by L_t . Since $\theta_+(\theta_+(z)) = z$ (resp. $\theta_-(\theta_-(z)) = z$) whenever $\theta_+(z)$ (resp. $\theta_-(z)$) is defined, the map $\theta_* \circ \theta_*$ is well defined. It is the identity of $\Delta^0(L; z^0; z^n, \dots, z^1)$ since the first application of θ_* moves a side of a polygon by L_t and the second application of θ_* moves this side back. $\square$

4.4. Enough homotopy equivalences

Let $(\mathcal{S}, w)$ be a weighted surface. In this section we prove the following unitality result.

PROPOSITION 4.11. *Then the relative wrapped Fukaya category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ has enough homotopy equivalences (see Definition 1.29).*

In §4.4.1 we define morphisms between a Lagrangian brane and its transverse perturbation and deduce Proposition 4.11 provided that the morphisms we defined are homotopy equivalences. In §4.4.2 we prove that these morphisms are closed of degree zero. In §4.4.3 we prove that these morphisms for two transverse perturbations compose as expected. In §4.4.4 we prove that these morphisms behave like strict units. In §4.4.5 we give a criterion for two closed morphisms of degree zero to be homotopy inverse. In §4.4.6 we finish the proof that these morphisms are homotopy equivalences.

4.4.1. Let $\mathbb{L}_0 = (L_0, E, \delta_0)$ be a Lagrangian brane in $(\mathcal{S}, w)$. Let the Lagrangian L_1 be a transverse perturbation of L_0 , as in §4.3.1. Then L_1 and L_0 are Lagrangian isotopic, so the obstruction algebras $\mathcal{A}_{L_0, E}$ and $\mathcal{A}_{L_1, E}$ can be identified by Lemma 4.3 hence we have a Lagrangian brane $\mathbb{L}_1 = (L_1, E, \delta_1)$ where the bounding cochain δ_1 corresponds to δ_0 under the identification. We call the Lagrangian brane $\mathbb{L}_1$ a *transverse perturbation* of $\mathbb{L}_0$. Note that if $\mathbb{L}_1$ is a transverse perturbation of $\mathbb{L}_0$, then $\mathbb{L}_0$ is a transverse perturbation of $\mathbb{L}_1$.

Recall from §4.3.1 that for each connected component C of the domain of L_0 we have finite sets $Q_C, P_C \subset Z(L_0|_C, L_1|_C)$ with $|z| = 0$ if $z \in Q_C$ and

$|z| = 1$ if $z \in P_C$. In particular, (1.25) reads $\mathcal{W}(q) = \text{Hom}(E_{q_+}, E_{q_-})$ for $q \in Q_C$. For any $z \in Q_C \cup P_C$ the points $z_+, z_- \in C$ are close hence we get a morphism $\varphi_z: E_{z_+} \rightarrow E_{z_-}$ induced by the short path in C from z_+ to z_- . Define

$$(4.5) \quad e_{\mathbb{L}_0, \mathbb{L}_1} = \sum_{C \in \pi_0 I} \sum_{q \in Q_C} \varphi_q \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1).$$

Lemma 4.12. *Let $\mathbb{L}_1$ be a transverse perturbation of a Lagrangian brane $\mathbb{L}_0$. Then $e_{\mathbb{L}_0, \mathbb{L}_1} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$ and $e_{\mathbb{L}_1, \mathbb{L}_0} \in \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0)$ are homotopy inverse morphisms in $\mathcal{W}$ (see Definition 1.28).*

We prove Lemma 4.12 later in this section, but first we deduce Proposition 4.11 from it.

PROOF OF PROPOSITION 4.11. We check the relative wrapped Fukaya category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ satisfies Definition 1.29. Let $\mathbb{L}_0 = (L = (I, \gamma_0, P_0, \text{gr}_0), E_0, \delta_0)$ be a Lagrangian brane and $\{\{\mathbb{L}_j^k\}_{k=0}^{l_j}\}_{j=1}^m$ be a set of transverse sets of Lagrangian branes. For each $j = 1, \dots, m$ let $\mathbb{L}_j$ be a composite Lagrangian brane of $\mathbb{L}_j^0, \dots, \mathbb{L}_j^{l_j}$ supported on the immersed curve γ_j .

By Lemma 4.12 it suffices to find a Lagrangian $L' = (I, \gamma', P', \text{gr}')$ sufficiently C^∞ -close to L_0 such that L_j and L' are transverse for each $j = 0, \dots, m$. That is, we want $\gamma' \sqcup \gamma_j$, $j = 0, \dots, m$ to have at worst transverse double points and to restrict to an embedding on the boundary. This is a generic condition hence the desired L' can be chosen arbitrarily C^∞ -close to L_0 . $\square$

4.4.2. We start the proof of Lemma 4.12 by checking that $e_{\mathbb{L}_0, \mathbb{L}_1}$ is a closed morphism of degree 0. We do it in a slightly more general context allowing for gauge equivalence.

Two bounding cochains δ_0, δ_1 in a structured A_∞ -algebra $\mathcal{A}$ are called *gauge equivalent* if there exists $a \in \mathcal{A}^0$ such that

$$(4.6) \quad \mu_{\delta_1, \delta_0}^1(a) = \sum_{k_0, k_1 \in \mathbb{Z}_{\geq 0}} \mu^{k_1+1+k_0}(\underbrace{\delta_1, \dots, \delta_1}_{k_1}, a, \underbrace{\delta_0, \dots, \delta_0}_{k_0}) = \delta_1 - \delta_0.$$

Gauge equivalence is an equivalence relation on the set $\widehat{\text{MC}}(\mathcal{A})$ of bounding cochains in $\mathcal{A}$. Given $a_1, a_2 \in \mathcal{A}^0$ such that $\mu_{\delta_1, \delta_0}^1(a_1) = \delta_1 - \delta_0$ and $\mu_{\delta_2, \delta_1}^1(a_2) = \delta_2 - \delta_1$ we obtain

$$\mu_{\delta_2, \delta_0}^1(a_1 + a_2 - \mu_{\delta_2, \delta_1, \delta_0}^2(a_2, a_1)) = \delta_2 - \delta_0;$$

we do need the explicit formula for $a' \in \mathcal{A}^0$ such that $\mu_{\delta_0, \delta_1}^1(a') = \delta_0 - \delta_1$ in terms of $a \in \mathcal{A}^0$ such that $\mu_{\delta_1, \delta_0}^1(a') = \delta_1 - \delta_0$, the existence of such a' follows from the equivalence of any two models of $[0, 1] \times \mathcal{A}$, see [23, §4.3].

Consider a Lagrangian brane $\mathbb{L}_0 = (\mathcal{L}_0 = (L_0, E), \delta_0)$ and let δ'_0 be a bounding cochain in the obstruction algebra $\mathcal{A}_{\mathcal{L}_0}$ such that δ_0 and δ'_0 are gauge

equivalent, i.e. there exists an element $a_0 \in \mathcal{A}_{\mathcal{L}_0}$ of degree 0 solving (4.6) with δ'_0 in place of δ_1 . We obtain another Lagrangian brane $\mathbb{L}'_0 = (\mathcal{L}_0, \delta'_0)$.

Let L_1 be a transverse perturbation of L_0 , $\mathbb{L}_1 = (\mathcal{L}_1 = (L_1, E), \delta_1)$ and $\mathbb{L}'_1 = (\mathcal{L}_1, \delta'_1)$ the corresponding transverse perturbations of $\mathbb{L}_0$ and $\mathbb{L}'_0$ respectively. We also get $a_1 \in \mathcal{A}_{\mathcal{L}_1}$ corresponding to $a_0 \in \mathcal{A}_{\mathcal{L}_0}$. Note that $\mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$ is identical to $\mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$ as graded R -modules. To get a closed morphism from $\mathbb{L}_0$ to $\mathbb{L}'_1$ we add a correction term a_{01} to $e_{\mathbb{L}_0, \mathbb{L}_1} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$.

The morphism $a_{01} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$ is defined as follows. Recall from §4.3.3 that for any $z^0 \in Z_{L_0}$ we have $\theta_-(z^0) \in Z(L_0, L_1)$ obtained by the Lagrangian isotopy from L_0 to L_1 and $|\theta_-(z^0)| = |z^0|$. The isotopy from L_0 to L_1 gives an isomorphism of R -modules $\mathcal{A}_{\mathbb{L}_0}(z^0) \cong \mathcal{W}(\theta_-(z^0))$. Combining these isomorphisms for all $z^0 \in Z_{L_0}$ we obtain an embedding of graded R -modules $\mathcal{A}_{\mathbb{L}_0} \hookrightarrow \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$. Define a_{01} to be the image of a_0 under this embedding.

Set $e_{\mathbb{L}_0, \mathbb{L}'_1} = -a_{01} + e_{\mathbb{L}_0, \mathbb{L}_1} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$.

Lemma 4.13. *Let Lagrangian branes $\mathbb{L}_0$ and $\mathbb{L}'_1$ be as above. Then $e' = e_{\mathbb{L}_0, \mathbb{L}'_1} \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$ is a closed morphism of degree zero, i.e. $\tilde{\mu}^1(e') = 0$ and $|e'| = 0$.*

Before the proof of Lemma 4.13 we make the following observation. Let $\mathbb{L}$ be a Lagrangian brane supported on the Lagrangian L . Suppose L contains a transverse double of $L_h(L_0, L_1)$ of a Lagrangian $L_0 = (I, \gamma_0, P_0, \text{gr}_0)$ nicely, as in §4.3.2. Recall the partially defined maps $\theta_{\pm}: Z_L \rightarrow Z_L$ from §4.3.3.

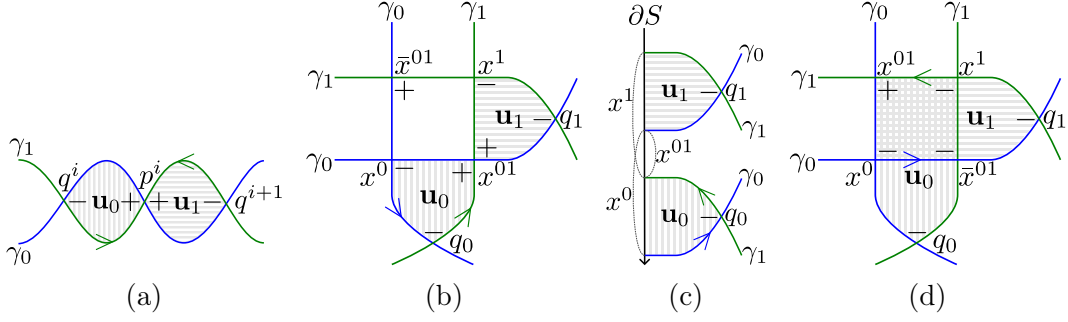
Lemma 4.9 immediately yields the following.

Lemma 4.14. *Let $\mathbb{L}$ be a Lagrangian brane as above and $q \in Q_C$ for some connected component C of I . Assume that $z^0, \dots, z^n \in Z(L)$ are such that $q = z^i$ for some $i \in \{1, \dots, n\}$ and $\tilde{\mu}_{\mathbb{L}}^{z^0; z^n, \dots, z^1}$ does not vanish.*

Then one of the following possibilities holds.

- (1) $n = 1$, $z^1 = q$, and $\tilde{\mu}_{\mathbb{L}}^{z^0; q} = \mu^{\mathbf{u}}$ for $\mathbf{u}$ as in Fig. 20(a).
- (2) $n = 2$, $z^1 = q$, $z^0 = \theta_+(z^2)$, and $\tilde{\mu}_{\mathbb{L}}^{z^0; z^2, q} = \mu^{\mathbf{u}}$ for $\mathbf{u}$ as in Figs. 20(b) and 20(d) to 20(f).
- (3) $n = 2$, $z^2 = q$, $z^0 = \theta_-(z^1)$, and $\tilde{\mu}_{\mathbb{L}}^{z^0; q, z^1} = \mu^{\mathbf{u}}$ for $\mathbf{u}$ as in Figs. 20(b) and 20(d) to 20(f).

PROOF OF LEMMA 4.13. We have $|e'| = 0$ by construction since $|q| = 0$ for each $q \in Q_C$, $C \in \pi_0 I$ and $|a_{01}| = |a_0| = 0$. Let $L_h = L_h(L_0, L_1)$ be a transverse double of L_0 , as in §4.3.1. By Lemma 4.5 we can ensure that $\mathbb{L}_h = \mathbb{L}_h(L_0, L_1)$ is adapted to e' , a_{01} , and $e = e_{\mathbb{L}_0, \mathbb{L}_1}$. Perform the move $II_{>0}$ on all $q \in Q_C$, $C \in \pi_0 I$ which are Reeb chords. By Lemma 4.1 the obstruction algebra $\mathcal{A}_{\mathcal{L}_h}$ didn't change, so we can assume that all $q \in Q_C$, $C \in \pi_0 I$ are double points.

FIGURE 21. Polygons contributing to $\tilde{\mu}^1(e)$

Recall that the isotopy from L_0 to L_1 induces embeddings of graded R -modules $\mathcal{A}_{\mathcal{L}_0} \hookrightarrow \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$ and $\mathcal{A}_{\mathcal{L}_1} \hookrightarrow \mathcal{W}(\mathbb{L}_0, \mathbb{L}'_1)$. Denote by δ_{01} (resp. δ'_{01}) the image of δ_0 (resp. δ'_0) under the first embedding, which is equal to the image of δ_1 (resp. δ'_1) under the second embedding. Lemma 4.10 and (4.6) imply that

$$\tilde{\mu}^1(a_{01}) = \sum_{k_0, k_1 \in \mathbb{Z}_{\geq 0}} \mu_{\mathcal{L}_h}^{k_1+1+k_0}(\underbrace{\delta'_1, \dots, \delta'_1}_{k_1}, a, \underbrace{\delta_0, \dots, \delta_0}_{k_0}) = \delta'_{01} - \delta_{01}.$$

It therefore remains to prove that $\tilde{\mu}^1(e) = \delta'_{01} - \delta_{01}$. Lemma 4.14 applied to $\mathbb{L}_h$ implies that the polygons contributing to $\tilde{\mu}^1(e)$ come in pairs $(\mathbf{u}_0, \mathbf{u}_1)$ as depicted in Fig. 21. We prove that $\tilde{\mu}^{\mathbf{u}_0}(e) + \tilde{\mu}^{\mathbf{u}_1}(e) = 0$ for Fig. 21(a) and $\tilde{\mu}^{\mathbf{u}_0}(e) + \tilde{\mu}^{\mathbf{u}_1}(e) = (\delta'_{01})_{x^{01}} - (\delta_{01})_{x^{01}}$ for Figs. 21(b) to 21(d). Note that all involved polygons have weight 1.

Fig. 21(a). We have $\mathbf{u}_0 \in \Delta^0(L_h; p^i; q^i)$ and $\mathbf{u}_1 \in \Delta^0(L_h; p^i; q^{i+1})$ where $p^i \in P_C$ and $q^i, q^{i+1} \in Q_C$ for some $C \in \pi_0 I$ (the indexing of the points in $Q_C \sqcup P_C$ is as in §4.3.1). For $j = 0, 1$ we have

$$\tilde{\mu}^{\mathbf{u}_j}(e) = \mu^{\mathbf{u}_j}(\varphi_{q^{i+j}}) = (E_{p_+^i} \rightarrow E_{q_+^{i+j}} \xrightarrow{\varphi_{q^{i+j}}} E_{q_-^{i+j}} \rightarrow E_{p_-^i}) \otimes r_j$$

where the unnamed maps between the fibres of E are induced by following the boundary of $\mathbf{u}_j$ and r_j is the orientation of C shown by the green arrow on the boundary of $\mathbf{u}_j$ in Fig. 21(a). Since the paths in C inducing the morphisms $E_{p_+^i} \rightarrow E_{p_-^i}$ are homotopic to the short path from p_+^i to p_-^i and $r_0 + r_1 = 0$, we conclude

$$\tilde{\mu}^{\mathbf{u}_0}(e) + \tilde{\mu}^{\mathbf{u}_1}(e) = \varphi_{p_i} \otimes r_0 + \varphi_{p_i} \otimes r_1 = 0.$$

Figs. 21(b) and 21(c). We have $\mathbf{u}_0 \in \Delta^0(L_h; x^{01}; q_0, x^0)$ and $\mathbf{u}_1 \in \Delta^0(L_h; x^{01}; x^1, q_1)$ for some $x^0 \in X_{L_0}$, $q_0 \in Q_{C_0}$, $q_1 \in Q_{C_1}$, and $C_0, C_1 \in \pi_0 I$. Note that $|x^0| = |x^1|$ and the polygons contribute to $\tilde{\mu}(e)$ only if this degree is equal to 1. Trivialising the local system on the four strands involved we can assume that $E_{x_-^0} = E_{(q_0)_+} = E_{q_0_-} = E_{x_+^{01}} = E_{x_-^1} = E_0$ and $E_{x_-^{01}} = E_{(q_1)_+} = E_{(q_1)_-} = E_{x_+^1} = E_{x_+^0} = E_1$ for some R -modules E_0, E_1 ; $\varphi_{q_0} = \text{id}_{E_0}$, $\varphi_{q_1} = \text{id}_{E_1}$, and $(\delta_0)_{x^0} = (\delta_{01})_{x^{01}}, (\delta'_1)_{x^1} = (\delta'_{01})_{x^{01}} \in$

$\text{Hom}_R(E_1, E_0) \otimes |C_0|_R$. With all the identifications done we obtain

$$\tilde{\mu}^{\mathbf{u}_0}(e) + \tilde{\mu}^{\mathbf{u}_1}(e) = \mu^{\mathbf{u}_0}(\text{id}_{E_0}, (\delta_0)_{x^0}) + \mu^{\mathbf{u}_1}((\delta'_1)_{x^1}, \text{id}_{E_1}) = -(\delta_{01})_{x^{01}} + (\delta'_{01})_{x^{01}}$$

where the negative sign appears since the blue and green arrows on the boundary of $\mathbf{u}_0$ in Figs. 21(b) and 21(c) induce opposite orientations of C_0 .

Fig. 21(d). We have $\mathbf{u}_0 \in \Delta^0(L_h; x^{10}; x^1, q_0)$ and $\mathbf{u}_1 \in \Delta^0(L_h; x^{10}; q_1, x^0)$ for some $x^0 \in X_{L_0}$, $q_0 \in Q_{C_0}$, $q_1 \in Q_{C_1}$, and $C_0, C_1 \in \pi_0 I$. Note that $|x^0| = |x^1|$ and the polygons contribute to $\tilde{\mu}(e)$ only if this degree is equal to 1. Trivialising the local system on the four strands involved we can assume that $E_{x^0_+} = E_{(q_0)_+} = E_{q_0_-} = E_{x^1_+} = E_{x^0_+} = E_0$ and $E_{x^0_-} = E_{(q_1)_+} = E_{(q_1)_-} = E_{x^{01}} = E_{x^1_-} = E_1$ for some R -modules E_0, E_1 ; $\varphi_{q_0} = \text{id}_{E_0}$, $\varphi_{q_1} = \text{id}_{E_1}$, and $(\delta_0)_{x^0} = (\delta_{01})_{x^{01}}, (\delta'_1)_{x^1} = (\delta'_{01})_{x^{01}} \in \text{Hom}_R(E_0, E_1) \otimes |C_1|_R$. With all the identifications done we obtain

$$\tilde{\mu}^{\mathbf{u}_0}(e) + \tilde{\mu}^{\mathbf{u}_1}(e) = \mu^{\mathbf{u}_0}((\delta'_1)_{x^1}, \text{id}_{E_0}) + \mu^{\mathbf{u}_1}(\text{id}_{E_1}, (\delta_0)_{x^0}) = (\delta'_{01})_{x^{01}} - (\delta_{01})_{x^{01}}$$

where the negative sign appears since the blue and green arrows on the boundary of $\mathbf{u}_1$ in Fig. 21(d) induce opposite orientations of C_1 . $\square$

4.4.3. Let $\mathbb{L}_1 = (L_1, E, \delta_1)$ be a transverse perturbation of a Lagrangian brane $\mathbb{L}_0 = (L_0, E, \delta_0)$. For each component C of the domain of $\mathbb{L}_0$ let i_C be the immersion as in Fig. 19 for a transverse double $L_{(h_0, h_1)}(L_0, L_1)$ of L_0 . Then in any neighbourhood of L_0 in $\text{Lag}(\mathcal{S})$ we can find a Lagrangian L_2 and $h_2 \in \mathbb{R}$ such that the composite Lagrangian $L = L_{(h_0, h_1, h_2)}(L_0, L_1, L_2)$ contains the transverse doubles $L_{(h_0, h_2)}(L_0, L_2)$ of L_0 and $L_{(h_1, h_2)}(L_1, L_2)$ of L_1 nicely, as in Fig. 22. Let $\mathbb{L}_2$ be the corresponding transverse perturbation of $\mathbb{L}_0$.

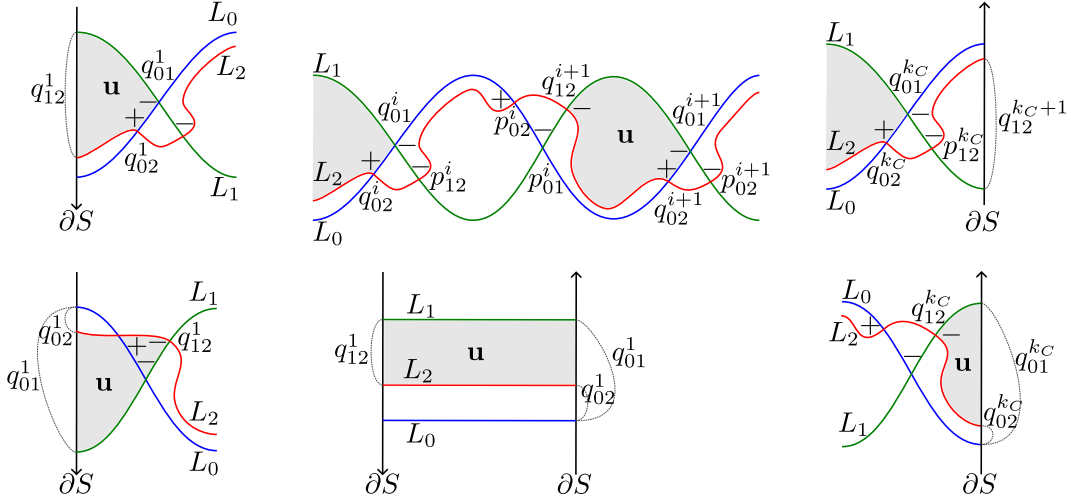


FIGURE 22. Pullback of L_2 by i_C

Lemma 4.15. *Let $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$ be as above. Then*

$$e_{\mathbb{L}_1, \mathbb{L}_2} \circ e_{\mathbb{L}_0, \mathbb{L}_1} = e_{\mathbb{L}_0, \mathbb{L}_2}$$

PROOF. Denote $e_{01} = e_{\mathbb{L}_0, \mathbb{L}_1}$, $e_{12} = e_{\mathbb{L}_1, \mathbb{L}_2}$, and $e_{02} = e_{\mathbb{L}_0, \mathbb{L}_2}$. Since $L_{(h_0, h_1)}(L_0, L_1)$, $L_{(h_1, h_2)}(L_1, L_2)$, and $L_{(h_0, h_2)}(L_0, L_2)$ are transverse doubles, the Lagrangian brane $\mathbb{L} = \mathbb{L}_{(h_0, h_1, h_2)}(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2)$ is adapted to $\{e_{01}, e_{12}, e_{02}\}$.

Let C be a connected component of the common domain of $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$. Since L contains the transverse double $L_{(h_1, h_2)}(L_1, L_2)$ nicely, Lemma 4.10 implies that the polygons contributing to $\tilde{\mu}_{\mathbb{L}}(e_{12}, e_{01})$ are the shaded polygons $\mathbf{u}$ in Fig. 22.

Let $\mathbf{u} \in \Delta^0(L; q_{02}^i; q_{12}^i, q_{01}^i)$ be one of these contributing polygons. Recall that the component of e_{ab} at q_{ab}^i is given by $\varphi_{ab} \in \text{Hom}_R(E_{(q_{ab}^i)_+}, E_{(q_{ab}^i)_-})$ induced by a short path in C from $(q_{ab}^i)_+$ to $(q_{ab}^i)_-$. Observe that the concatenation of the short paths from $(q_{01}^i)_+$ to $(q_{01}^i)_-$ and from $(q_{12}^i)_+$ to $(q_{12}^i)_-$ with the paths in C induced by the boundary of $\mathbf{u}$ is homotopic to the short path from $(q_{02}^i)_+$ to $(q_{02}^i)_-$. Therefore, $\mu^{\mathbf{u}}(\varphi_{12}, \varphi_{01}) = \varphi_{02}$. As this is true for any contributing polygon $\mathbf{u}$, we conclude $\tilde{\mu}_{\mathbb{L}}(e_{12}, e_{01}) = e_{02}$. $\square$

4.4.4. Let $\mathbb{L}_0$ and $\mathbb{L}_1$ be transverse Lagrangian branes supported on Lagrangians L_0 and L_1 respectively. Then in any neighbourhood of L_0 in $\text{Lag}(\mathcal{S})$ we can find a Lagrangian L'_0 such that the Lagrangian $L = L_{(h_0, h_1, h'_0)}(L_0, L_1, L'_0)$ contains the transverse double $L_{(h_0, h'_0)}(L_0, L'_0)$ of L_0 nicely, as in §4.3.2. Let $\mathbb{L}'_0$ be the small perturbation of $\mathbb{L}_0$ supported on L'_0 and $\mathbb{L} = \mathbb{L}_{h_0, h_1, h'_0}(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}'_0)$ the corresponding composite Lagrangian brane.

Note that $\mathbb{L}$ is adapted to $e_{\mathbb{L}_0, \mathbb{L}'_0}$. The following is an immediate corollary of Lemma 4.14.

Lemma 4.16. *Let $\mathbb{L}$ be the Lagrangian brane as above. Then*

$$\mu_{\mathbb{L}}^n(\dots, e_{\mathbb{L}_0, \mathbb{L}'_0}, \dots) = 0 \text{ if } n \geq 3.$$

Recall that $\mu_{\mathbb{L}}^1(e_{\mathbb{L}_0, \mathbb{L}'_0}) = 0$ by Lemma 4.13. It remains to describe $\mu_{\mathbb{L}}^2$ when one of the arguments is $e_{\mathbb{L}_0, \mathbb{L}'_0}$.

Given $z \in Z(\mathbb{L}_1, \mathbb{L}_0)$ (resp. $z \in Z(\mathbb{L}'_0, \mathbb{L}_1)$) we can pick h_1 above in such a way that $A_{(h_0, h_1)}(z) > 0$ (resp. $A_{(h'_0, h_1)}(z) > 0$). Then $\theta_-(z) \in Z(\mathbb{L}_1, \mathbb{L}'_0)$ (resp. $\theta_-(z) \in Z(\mathbb{L}_0, \mathbb{L}_1)$) as in §4.3.3 is defined and the Lagrangian isotopy between L_0 and L'_0 induces an isomorphism $\theta_{\mathbb{L}_0, \mathbb{L}'_0}: \mathcal{W}(z) \rightarrow \mathcal{W}(\theta_{\pm}(z))$. Combining these isomorphisms for all $z \in Z(\mathbb{L}_1, \mathbb{L}_0)$ (resp. $z \in Z(\mathbb{L}'_0, \mathbb{L}_1)$) we obtain an isomorphism of graded R -modules

$$\theta_{\mathbb{L}_0, \mathbb{L}'_0}: \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{L}_1, \mathbb{L}'_0) \text{ (resp. } \mathcal{W}(\mathbb{L}'_0, \mathbb{L}_1) \rightarrow \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)).$$

$\theta_{\mathbb{L}_0, \mathbb{L}'_0}$ is a chain map by Lemma 4.10.

Lemma 4.17. *Let $\mathbb{L}_0$, $\mathbb{L}_1$, and $\mathbb{L}'_0$ be Lagrangian branes as above. Then*

$$(1) \ e_{\mathbb{L}_0, \mathbb{L}'_0} \circ - = \theta_{\mathbb{L}_0, \mathbb{L}'_0}: \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{L}_1, \mathbb{L}'_0);$$

$$(2) \quad - \circ e_{\mathbb{L}_0, \mathbb{L}'_0} = \theta_{\mathbb{L}_0, \mathbb{L}'_0} : \mathcal{W}(\mathbb{L}'_0, \mathbb{L}_1) \rightarrow \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1).$$

PROOF. We prove (1), (2) is similar. It suffices to prove that $e_{\mathbb{L}_0, \mathbb{L}'_0} \circ - = \theta_{\mathbb{L}_0, \mathbb{L}'_0} : \mathcal{W}(z) \rightarrow \mathcal{W}(\mathbb{L}_1, \mathbb{L}'_0)$ for any $z \in Z(\mathbb{L}_1, \mathbb{L}_0)$.

Fix $z \in Z(\mathbb{L}_1, \mathbb{L}_0)$, and pick h_1 such that $A_{(h_0, h_1)}(z) > 0$. Lemma 4.14 implies

$$(e_{\mathbb{L}_0, \mathbb{L}'_0} \circ -)|_{\mathcal{W}(z)} = (-1)^{|z|} \mu^{\mathbf{u}}(\varphi_q, -)$$

where $\mathbf{u} \in \Delta^0(L; \theta_-(z); q, z)$ for some $q \in Q_C$. Performing a move $II_{>0}$ if necessary we can assume that q is a double point. Then $\mathbf{u}$ is a polygon like in Figs. 20(b) and 20(d) and $\mu^{\mathbf{u}}(\varphi_q, -) = (-1)^{|z|} \theta_{\mathbb{L}_0, \mathbb{L}'_0}$ since the sides of $\mathbf{u}$ lying on L_0 and L'_0 induce opposite orientations of C . $\square$

4.4.5. As the last ingredient in the proof of Proposition 4.11 we give the following criterion for two morphisms in $\mathcal{W}$ to be homotopy inverse.

Lemma 4.18. *Let $q^1 \in \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1)$ and $q^0 \in \mathcal{W}(\mathbb{L}_1, \mathbb{L}_0)$ be two closed morphisms of degree 0 in $\mathcal{W}$. Suppose that in any neighbourhood of L_0 in $\text{Lag}(\mathcal{S})$ there exists a Lagrangian L'_0 such that $L_{(h_0, h_1, h'_0)}(L_0, L_1, L'_0)$ contains the transverse double $L_{(h_0, h'_0)}(L_0, L'_0)$ of L_0 nicely and $\theta_{\mathbb{L}_0, \mathbb{L}'_0}(q^0) \circ q^1$ is cohomologous to $e_{\mathbb{L}_0, \mathbb{L}'_0}$.*

Then for any transverse set $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_2\}$ in $\mathcal{W}$

- (1) *the map $q^0 \circ (q^1 \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_0)}$;*
- (2) *the map $(- \circ q^0) \circ q^1$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{L}_0, \mathbb{T})}$.*

PROOF. Let $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_2\}$ be a transverse set. We only prove (1), the proof of (2) is similar.

The assumption of the lemma and the fact that $\mathbb{T}$ and $\mathbb{L}_0$ are transverse imply that we can find a Lagrangian L'_0 such that the Lagrangian $L_{(h_T, h_0, h_1, h'_0)}(T, L_0, L_1, L'_0)$ contains the transverse double $L_{(h_0, h'_0)}(L_0, L'_0)$ of L_0 nicely and $\theta_{\mathbb{L}_0, \mathbb{L}'_0}(q^0) \circ q^1$ is cohomologous to $e_{\mathbb{L}_0, \mathbb{L}'_0}$. The A_2 - and A_3 -relations then imply that

$$\theta_{\mathbb{L}_0, \mathbb{L}'_0}(q^0) \circ (q^1 \circ -) \text{ and } e_{\mathbb{L}_0, \mathbb{L}'_0} \circ - \text{ are homotopic as maps } \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}'_0).$$

Lemma 4.10 implies that

$$\theta_{\mathbb{L}_0, \mathbb{L}'_0}(q^0) \circ (q^1 \circ -) = \theta_{\mathbb{L}_0, \mathbb{L}'_0}(q^0 \circ (q^1 \circ -)) : \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}'_0)$$

and Lemma 4.17 yields

$$e_{\mathbb{L}_0, \mathbb{L}'_0} \circ - = \theta_{\mathbb{L}_0, \mathbb{L}'_0} : \mathcal{W}(\mathbb{T}, \mathbb{L}_0) \rightarrow \mathcal{W}(\mathbb{T}, \mathbb{L}'_0).$$

As $\theta_{\mathbb{L}_0, \mathbb{L}'_0}$ is an isomorphism, $q^0 \circ (q^1 \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_0)}$ as required. $\square$

4.4.6. We assemble a proof of Lemma 4.12 from Lemmas 4.13, 4.15, and 4.18.

PROOF OF LEMMA 4.12. We want to prove that $e^1 = e_{\mathbb{L}_1, \mathbb{L}_0}$ and $e^0 = e_{\mathbb{L}_0, \mathbb{L}_1}$ are homotopy inverse morphisms. By Lemma 4.13 they are closed morphisms of degree 0. It is enough to prove that for any transverse set $\{\mathbb{T}, \mathbb{L}_0, \mathbb{L}_1\}$ in $\mathcal{W}$ the map $e^0 \circ (e^1 \circ -)$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{T}, \mathbb{L}_0)}$ and the map $(- \circ e^0) \circ e^1$ is homotopic to $\text{id}_{\mathcal{W}(\mathbb{L}_0, \mathbb{T})}$ since we can swap e^0 and e^1 . By Lemma 4.18 it suffices to find in any neighbourhood of L_0 in $\text{Lag}(\mathcal{S})$ a Lagrangian L'_0 such that $L_{(h_0, h_1, h'_0)}(L_0, L_1, L'_0)$ contains the transverse double $L_{(h_0, h'_0)}(L_0, L'_0)$ of L_0 nicely and $\theta_{\mathbb{L}_0, \mathbb{L}'_0}(e^0) \circ e^1$ is cohomologous to $e_{\mathbb{L}_0, \mathbb{L}'_0}$. By Lemma 4.15 we can indeed find such Lagrangian L'_0 and get

$$\theta_{\mathbb{L}_0, \mathbb{L}'_0}(e^0) \circ e^1 = e_{\mathbb{L}_1, \mathbb{L}'_0} \circ e_{\mathbb{L}_0, \mathbb{L}_1} = e_{\mathbb{L}_0, \mathbb{L}'_0}.$$

□

4.5. Triangulated structure

In this section we prove the following statement.

PROPOSITION 4.19. *Let $(\mathcal{S}, w)$ be a weighted surface. Then the relative wrapped Fukaya category $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$ is triangulated (see §1.4.3).*

We prove Proposition 4.19 by definition in §1.4.3, i.e. we construct a strict totalisation functor $\text{Tot}: \text{Tw } \mathcal{W} \rightarrow \mathcal{W}$ from the triangulated envelope defined in §3.3.1.

In §4.5.1 we define Tot^0 , in §4.5.2 we define Tot^1 , and in §4.5.3 we prove that Tot is a strict totalisation functor.

4.5.1. We define a Lagrangian brane $\text{Tot}^0(\mathbf{X})$ for any twisted complex $\mathbf{X}$ over $\mathcal{W} = \mathcal{W}_w(\mathcal{S})$.

First, let $\mathbb{L} = (\mathcal{L} = (I, \gamma, P, \text{gr}, E), \delta)$ be a Lagrangian brane and m an integer. Define the Lagrangian pre-brane $\mathcal{L}[m] = (I, \gamma, P, \text{gr}[m], E)$ where the path $\text{gr}[m]$ from $\gamma^*\eta$ to $\dot{\gamma}$ is obtained by concatenation of gr and a *clockwise* rotation by πm . Observe that $\mathcal{A}_{\mathcal{L}} = \mathcal{A}_{\mathcal{L}[m]}$ since the only way the grading enters into the definition of obstruction algebra is by determining the degree of double points and Reeb chords, and by definition in §1.2.3 the degree $|z|$ does not change when $\text{gr}(z_+)$ and $\text{gr}(z_-)$ are both changed by a rotation by the same multiple of π . We set

$$\text{Tot}^0((\{\mathbb{L}[m]\}, 0)) = (\mathcal{L}[m], \delta) =: \Sigma^m(\mathbb{L}).$$

Now let $\mathbb{L}_0, \mathbb{L}_1$ be two transverse Lagrangian branes and m_0, m_1 two integers. We define a linear map of degree $m_0 - m_1$

$$\Sigma^{m_0, m_1} : \mathcal{W}(\mathbb{L}_0, \mathbb{L}_1) \rightarrow \mathcal{W}(\Sigma^{m_0} \mathbb{L}_0, \Sigma^{m_1} \mathbb{L}_1)$$

entry-wise. Let $z \in Z(\mathbb{L}_0, \mathbb{L}_1)$ be an intersection point or a Reeb chord of degree $|z| = d$. Then the degree of z viewed as an element of $Z(\Sigma^{m_0} \mathbb{L}_0, \Sigma^{m_1} \mathbb{L}_1)$ is $d + m_0 - m_1$. We set

$$\begin{array}{c} \text{Hom}_R((E_0)_{z_+}, (E_1)_{z_-}) \otimes (|C_{z_-}|_R)^{\otimes d} \\ \Sigma_z^{m_0, m_1} \downarrow \\ \text{Hom}_R((E_0)_{z_+}, (E_1)_{z_-}) \otimes (|C_{z_-}|_R)^{\otimes d + m_0 - m_1} \end{array}$$

to be $\Sigma_z^{m_0, m_1}(\varphi \otimes r^{\otimes d}) = (-1)^{m_1 - m_0} \varphi \otimes r^{\otimes d + m_0 - m_1}$ where r is either of the two orientations of C_{z_-} . From the definition in (1.18) and the fact that $\tilde{\mu}$ on given arguments can be computed in a composite Lagrangian brane adapted to the arguments it follows that

$$\Sigma^{m_0, m_n} \circ \tilde{\mu}(-, \dots, -) = \tilde{\mu}(\Sigma^{m_{n-1}, m_n}(-), \dots, \Sigma^{m_0, m_1}(-)).$$

We set $\text{Tot}^0(\mathbf{0})$ to be the Lagrangian brane with empty domain. Now let $\mathbf{X} = (\{\mathbb{L}_i[m_i]\}_{i \in I}, \delta^{\mathbf{X}})$ be a twisted complex over $\mathcal{W}$. By definition of a twisted complex we can replace the index set I with $\{1, \dots, n\}$ in such a way that $\delta_{i_0 i_1}^{\mathbf{X}} = 0$ unless $i_0 < i_1$. By Lemma 4.5 we can find a composite Lagrangian brane $(\mathcal{L}, \delta^{\mathcal{L}})$ of $\Sigma^{m_1} \mathbb{L}_1, \dots, \Sigma^{m_n} \mathbb{L}_n$ which is adapted to $\{\Sigma^{i_0 i_1} \delta_{i_0 i_1}^{\mathbf{X}}\}_{i_0 < i_1}$.

Lemma 4.20.

$$\delta = \delta^{\mathcal{L}} + \sum_{i_0 < i_1} (-1)^{m_{i_0}(m_{i_1} - m_{i_0})} \Sigma^{i_0 i_1} \delta_{i_0 i_1}^{\mathbf{X}}$$

is a bounding cochain in the obstruction algebra $\mathcal{A}_{\mathcal{L}}$.

PROOF. Since $\delta^{\mathbf{X}}$ is a bounding cochain for any $1 \leq s < f \leq n$ and $z^0 \in Z(\mathbb{L}_s, \mathbb{L}_f)$ we have

$$(4.7) \quad \sum_{s=i_0 < \dots < i_r=f} \sum_{\substack{z^j \in Z(\mathbb{L}_{i_{j-1}}, \mathbb{L}_{i_j}) \\ j=1, \dots, r}} (-1)^{\dagger} \tilde{\mu}^{z^0; z^r, \dots, z^1} ((\delta_{i_{r-1} i_r}^{\mathbf{X}})_{z^r}, \dots, (\delta_{i_0 i_1}^{\mathbf{X}})_{z^1}) = 0$$

where

$$\dagger = \sum_{1 \leq \alpha < \beta \leq r} (m_{i_\alpha} - m_{i_{\alpha-1}})(|z^\beta| - 1) \equiv \frac{1}{2} m_s(m_f - m_s) + \sum_{\alpha=1}^r m_{i_{\alpha-1}}(m_{i_\alpha} - m_{i_{\alpha-1}}).$$

Applying Σ^{m_s, m_f} to (4.7) and writing $\tilde{\mu}$ as in (1.20) we obtain the equation certifying that δ is a bounding cochain. $\square$

We define $\text{Tot}^0(\mathbf{X}) = (\mathcal{L}, \delta)$. In the process of defining $\text{Tot}^0(\mathbf{X})$ we made some choices. We can make sure $\text{Tot}^0(\mathbb{L}) = \mathbb{L}$, other choices do not matter and can be made independently for different $\mathbf{X}$. Observe that Tot^0 maps transverse sequences of twisted complexes to transverse sequences of Lagrangian branes.

4.5.2. Let $\mathbf{X}^j = (\{\mathbb{L}_i^j[m_i^j]\}_{i \in I^j}, \delta^j)$, $j = 0, 1$ be two transverse twisted complexes over $\mathcal{W}$. Define Lagrangian branes $\mathbb{L}^j = \text{Tot}^0(\mathbf{X}^j)$, $j = 0, 1$. We define

$$\text{Tot}^1: \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) \rightarrow \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1).$$

Observe that

$$\begin{aligned} \text{Tw } \mathcal{W}(\mathbf{X}^0, \mathbf{X}^1) = \\ \bigoplus_{i^0 \in I^0, i^1 \in I^1} \bigoplus_{z \in Z(\mathbb{L}_{i^0}^0, \mathbb{L}_{i^1}^1)} \text{Hom}_R((E_0)_{z+}, (E_1)_{z-}) \otimes (|C_{z-}|_R)^{\otimes |z|} [-|z|][m_{i^1}^1 - m_{i^0}^0] \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}(\mathbb{L}^0, \mathbb{L}^1) = \\ \bigoplus_{i^0 \in I^0, i^1 \in I^1} \bigoplus_{z \in Z(\mathbb{L}_{i^0}^0, \mathbb{L}_{i^1}^1)} \text{Hom}_R((E_0)_{z+}, (E_1)_{z-}) \otimes \\ (|C_{z-}|_R)^{\otimes |z| + m_{i^0}^0 - m_{i^1}^1} [-|z| + m_{i^1}^1 - m_{i^0}^0]. \end{aligned}$$

For $z \in Z(\mathbb{L}_{i^0}^0, \mathbb{L}_{i^1}^1)$ we set

$$\begin{aligned} \text{Tot}^1(\varphi \otimes r^{\otimes |z|}[m_{i^1}^1 - m_{i^0}^0]) &= (-1)^{m_{i^0}^0(|z|-1)\Sigma m_{i^0}^0, m_{i^1}^1} (\varphi \otimes r^{\otimes |z|}) \\ &= (-1)^{m_{i^0}^0|z| + m_{i^1}^1} \varphi \otimes r^{\otimes |z| + m_{i^0}^0 - m_{i^1}^1}. \end{aligned}$$

Note that Tot^1 is an isomorphism of graded modules and $\text{Tot}_{\mathbb{L}^0, \mathbb{L}^1}^1 = \text{id}_{\mathcal{W}(\mathbb{L}^0, \mathbb{L}^1)}$ for any transverse pair of Lagrangian branes $(\mathbb{L}^0, \mathbb{L}^1)$.

4.5.3. It remains to check that Tot constructed above satisfies the properties in §3.3.3.

PROOF OF PROPOSITION 4.19. The only property of Tot in §3.3.3 that is not immediate is (3). Since A_∞ -operations for twisted complexes and Lagrangian branes are computed in some composite Lagrangian brane, to verify (3) it suffices for a transverse set $\{\mathbb{L}^0, \dots, \mathbb{L}^n\}$ of Lagrangian branes, integers $m^0, \dots, m^n$, and intersection points or Reeb chords $z^0 \in Z(\mathbb{L}^0, \mathbb{L}^n)$, $z^j \in Z(\mathbb{L}^{j-1}, \mathbb{L}^j)$ for $j = 1, \dots, n$ to check that

$$\begin{aligned} (-1)^{m^0(|z^0|-1)\Sigma m^0, m^n} \circ \left((-1)^{\Sigma_{1 \leq \alpha < \beta \leq n} (m^\alpha - m^{\alpha-1})(|z^\beta|-1)} \tilde{\mu}^{z^0; z^n, \dots, z^1} \right) = \\ (-1)^{m^0} \tilde{\mu}^{z^0; z^n, \dots, z^1} \circ \left((-1)^{m^{n-1}(|z^n|-1)\Sigma m^{n-1}, m^n} \otimes \dots \otimes (-1)^{m^0(|z^1|-1)\Sigma m^0, m^1} \right) \end{aligned}$$

as maps

$$\mathcal{W}(z^1) \otimes \dots \otimes \mathcal{W}(z^n) \rightarrow \mathcal{W}\left(\Sigma^{m^0} \mathbb{L}^0 \xrightarrow{z^0} \Sigma^{m^n} \mathbb{L}^n\right).$$

Since $\Sigma^{-,-}$ commutes with $\tilde{\mu}$ and $\sum_{1 \leq \beta \leq n} (|z^\beta| - 1) \equiv \frac{1}{2} |z^0|$ we conclude that this is indeed the case. $\square$

4.6. Fukaya A_∞ -category

In this section we apply the constructions from §3.2 to relative wrapped Fukaya categories of surfaces. The result is the following.

PROPOSITION 4.21. *Let $\mathcal{W}$ be the relative wrapped Fukaya A_∞ -pre-category of a weighted surface $(\mathcal{S}, w)$. Then there exists a strictly unital A_∞ -category $\tilde{\mathcal{W}}$ with $\text{Ob } \tilde{\mathcal{W}} = \text{Ob } \mathcal{W}$ such that*

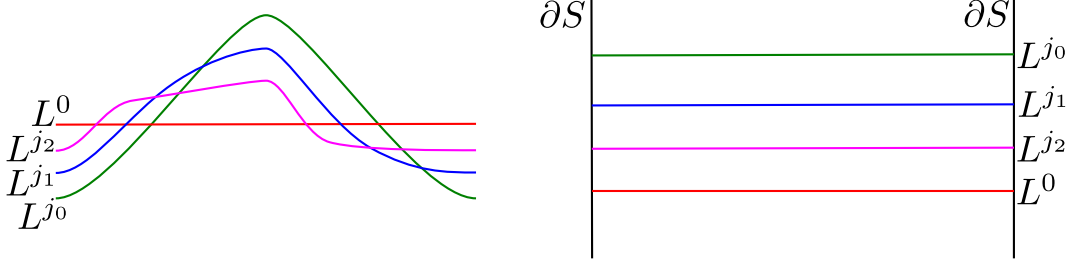
- (1) *two Lagrangian branes $\mathbb{L}_0, \mathbb{L}_1$ are homotopy equivalent in $\mathcal{W}$ if and only if they are homotopy equivalent in $\tilde{\mathcal{W}}$; and*
- (2) *the homotopy equivalence class of $\tilde{\mathcal{W}}$ is independent of any choices involved in its construction.*

Proposition 4.21 will follow from Lemmas 3.6 and 3.8. In §4.6.1 we construct the perturbations of a given Lagrangian that give rise to a system of strict units on an arbitrary transverse set of Lagrangian branes in $(\mathcal{S}, w)$, which we explain in §4.6.2. In §4.6.3 we show that Lagrangian isotopic Lagrangian branes are homotopy equivalent as objects of $\mathcal{W}$. We finish the proof of Proposition 4.21 in §4.6.4

4.6.1. Let $L^0 = (I, \gamma^0, P^0, \text{gr}^0)$ be a graded Lagrangian. Then there exists a transverse set $\{L^j\}_{j=0}^\infty$ such that

- (1) if $j_0 \neq j_1$, then L^{j_1} is a transverse perturbation of L^{j_0} ;
- (2) L^j tends to L_0 in $\text{Lag}_I^{\text{gr}}(\mathcal{S})$ as $j \rightarrow \infty$;
- (3) for pairwise distinct $j_0, j_1, j_2 \geq 0$ there exists a composite Lagrangian $L = L_{h_0, h_1, h_2}(L^{j_0}, L^{j_1}, L^{j_2})$ that contains the transverse double $L_{h_0, h_1}(L^{j_0}, L^{j_1})$ of L^{j_0} nicely;
- (4) for any $j_0 < \dots < j_{l-2} < \min\{j_{l-1}, j_l\} < \max\{j_{l-1}, j_l\} < j_{l+1} < \dots < j_n$ there exists a composite Lagrangian $L = L_{h_0, \dots, h_n}(L^{j_0}, \dots, L^{j_n})$ that contains the transverse double $L_{h_{l-1}, h_l}(L^{j_{l-1}}, L^{j_l})$ nicely.

Such a transverse set can be obtained, for example, by perturbing each connected component C of I that is a circle as in Fig. 23, left (the parts of L^j that are not depicted run parallel to L^0) in a small enough neighbourhood of a point $\gamma(t)$ for $t \in C$ and by perturbing each connected component C of I that is a segment as in Fig. 23, right (the picture is of the pullback to $\mathbb{R}^2$ via the immersion i_C for L_0 and L_1 as in §4.3.1).

FIGURE 23. $L^0, L^{j_0}, L^{j_1}, L^{j_2}$ for $0 < j_0 < j_1 < j_2$

4.6.2. Let $\{L_k^0\}_{k \in K}$ be a transverse set of Lagrangians in $\mathcal{S}$ and $\{\mathbb{L}_i^0\}_{i \in I}$ a collection of Lagrangian branes in $(\mathcal{S}, w)$ such that each $\mathbb{L}_i^0$ is supported on some $L_{\kappa(i)}^0$. For each $k \in K$ pick a transverse set $\{L_k^j\}_{j=0}^\infty$ as in §4.6.1 and for each $i \in I$ and $j \geq 0$ let $\mathbb{L}_i^j$ be the Lagrangian perturbation of $\mathbb{L}_i^0$ supported on $L_{\kappa(i)}^j$ as in §4.4.1. Note that $\mathbb{L}_i^{j_1}$ is a transverse perturbation of $\mathbb{L}_i^{j_0}$ for any $i \in I$ and $j_0 \neq j_1$ hence we have a homotopy equivalence $e_i^{j_0, j_1} \in \mathcal{W}(\mathbb{L}_i^{j_0}, \mathbb{L}_i^{j_1})$ as in (4.5).

Lemma 4.22. $\mathbf{L}^\bullet = \left\{ \left(\{\mathbb{L}_i^j\}_{j=0}^\infty, \{e_i^{j_0, j_1}\}_{j_0 \neq j_1} \right) \right\}_{i \in I}$ is a system of strict units (see §3.2.1) on the collection $\mathbf{L}^0 = \{\mathbb{L}_i^0\}_{i \in I}$ of objects in $\mathcal{W}$.

By Lemma 3.6 we obtain a strictly unital A_∞ -category $\mathcal{W}_{\mathbf{L}^\bullet}$.

PROOF. We check properties (1)–(4) from §3.2.1. Property (1) follows from (3) in §4.6.1 and Lemma 4.17.

To prove Property (2) fix $i_0, i_1 \in I$ and let $k_0 = \kappa(i_0)$ and $k_1 = \kappa(i_1)$. By Lemma 4.17 it suffices to show that there exists $N > 0$ such that for any $j_0, j_1, j'_1 \in \{0\} \cup \mathbb{Z}_{\geq N}$ satisfying $j_0 < j_1 < j'_1$ there exists a composite Lagrangian $L = L_{h_0, h_1, h'_1}(L_{k_0}^{j_0}, L_{k_1}^{j_1}, L_{k_1}^{j'_1})$ that contains the transverse double $L_{h_1, h'_1}(L_{k_1}^{j_1}, L_{k_1}^{j'_1})$ nicely. If $k_0 = k_1$, then this follows from (3) in §4.6.1. If $k_0 \neq k_1$, then this is true since $L_{k_0}^0$ and $L_{k_1}^0$ are transverse, $L_{k_0}^j \xrightarrow{j \rightarrow \infty} L_{k_0}^0$, and $L_{k_1}^j \xrightarrow{j \rightarrow \infty} L_{k_1}^0$. Property (3) is proved analogously.

Property (4) follows from Lemma 4.16, (4) in §4.6.1, and the fact that $L_k^j \xrightarrow{j \rightarrow \infty} L_k^0$ for any $k \in K$. $\square$

4.6.3. The construction of homotopy equivalences in §4.4 implies that Lagrangian isotopic Lagrangian branes are homotopy equivalent.

Namely, let $\mathbb{L}_0 = (L_0, E, \delta_0)$ be a Lagrangian brane and $L_t, t \in [0, 1]$ a Lagrangian isotopy. Then the obstruction algebras $\mathcal{A}_{(L_0, E)}$ and $\mathcal{A}_{(L_t, E)}$ are isomorphic by Lemma 4.3 for any $t \in [0, 1]$. Let δ_t be the bounding cochain in $\mathcal{A}_{(L_t, E)}$ corresponding to δ_0 under this isomorphism. We say that $\mathbb{L}_t = (L_t, E, \delta_t), t \in [0, 1]$ is a *Lagrangian isotopy* of Lagrangian branes. We say that two Lagrangian branes $\mathbb{L}_0$ and $\mathbb{L}_1$ with domains I_0 and I_1 respectively

are *Lagrangian isotopic* if there exists a diffeomorphism $\varphi: I_0 \rightarrow I_1$ and a Lagrangian isotopy from $\mathbb{L}_0$ to $\varphi^*\mathbb{L}_1$.

Lemma 4.23. *Let $\mathbb{L}_0, \mathbb{L}_1$ be two Lagrangian isotopic Lagrangian branes. Then $\mathbb{L}_0$ and $\mathbb{L}_1$ are homotopy equivalent as objects of the relative wrapped Fukaya category $\mathcal{W}$.*

PROOF. Observe that the diffeomorphism of the domain induces an isomorphism of all algebraic structures associated with a Lagrangian brane, hence we can assume that $\mathbb{L}_0$ and $\mathbb{L}_1$ have the same domain. By Lemma 4.12 for any Lagrangian brane $\mathbb{L} = (L, E, \delta)$ there exists a contractible neighbourhood U_L of L in $\text{Lag}(\mathcal{S})$ such that for any Lagrangian $L' \in U_L$ which is transverse to L the transverse perturbation $\mathbb{L}' = (L', E, \delta')$ of $\mathbb{L}$ is homotopy equivalent to $\mathbb{L}$. For a Lagrangian isotopy $\mathbb{L}_t = (L_t, E, \delta_t), t \in [0, 1]$ there exists $\varepsilon > 0$ such that $L_{t_0}, L_{t_1} \in U_{L_{t(t_0, t_1)}}$ whenever $|t_0 - t_1| < \varepsilon$. Fix some $t_0, t_1 \in [0, 1]$ such that $|t_0 - t_1| < \varepsilon$. For $i = 0, 1$ pick $L'_{t_i} \in U_{L_{t_i}} \cap U_{L_{t(t_0, t_1)}}$ such that L'_{t_i} is transverse to L_{t_i} and $L_{t(t_0, t_1)}$ and observe that the corresponding Lagrangian branes satisfy

$$\mathbb{L}_{t_0} \cong \mathbb{L}'_{t_0} \cong \mathbb{L}_{t(t_0, t_1)} \cong \mathbb{L}'_{t_1} \cong \mathbb{L}_{t_1}$$

where $\cong$ is the homotopy equivalence in $\mathcal{W}$. This implies that $\mathbb{L}_0 \cong \mathbb{L}_1$. $\square$

4.6.4. Proof of Proposition 4.21. By Lemmas 3.6 and 3.8 to prove Proposition 4.21 it is enough to find a saturated collection $\mathcal{SU}$ of systems of strict units in $\mathcal{W}$, see Definition 3.7.

Lemma 2.12 and the fact that there are countably many classes of immersed curves up to a reparametrisation of the domain and the action of $\text{Diff}(\mathcal{S})$ imply that the number of connected components of the space $\text{Lag}_I(\mathcal{S})$ of all Lagrangians with given domain I is countable. The number of diffeomorphisms classes of the domains (which are disjoint unions of finitely many circles and segments) and of gradings on a given Lagrangian is also countable. Therefore, we can pick a transverse set of graded Lagrangians $\{L_k^0\}_{k=0}^\infty$ in $\mathcal{S}$ such that for any Lagrangian L in $\mathcal{S}$ there exists a Lagrangian L_k^0 such that the domains of L and L_k^0 are diffeomorphic and under this diffeomorphism L and L_k^0 are Lagrangian isotopic.

Let $\mathcal{SU}$ be the collection of all systems of strict units obtained from a transverse set of Lagrangians $\{L_k^0\}_{k=0}^\infty$ as above and all brane structures on L_k^0 , as in §4.6.2. It remains to prove that $\mathcal{SU}$ is saturated.

We prove $\mathcal{SU}$ satisfies (1) from Definition 3.7. Let $\mathbf{L}^\bullet \in \mathcal{SU}$ be a system of strict units on a collection $\mathbb{L}^0 = \{\mathbb{L}_i^0\}_{i \in I}$ of Lagrangian branes and $\mathbb{L}$ a Lagrangian brane. Then by construction and Lemma 4.23 there exists a Lagrangian brane $\mathbb{L}_i^0, i \in I$ homotopy equivalent to $\mathbb{L}$.

We prove $\mathcal{SU}$ satisfies (2) from Definition 3.7. Recall the space $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ of Hamiltonian functions from §2.1.2 and note that the standard Frechét structure makes it into a Baire space. The time-1 map of the Hamiltonian flow gives the map $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R}) \rightarrow \text{Ham}(\mathcal{S})$, $H \mapsto \Phi^H$ and $\text{Ham}(\mathcal{S})$ acts on $\mathcal{W}$ as described in §4.2.3. Therefore, it suffices for any two transverse sets $\{L_k^0\}_{k=0}^\infty$ and $\{L_k^1\}_{k=0}^\infty$ to find a Hamiltonian $H \in C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$ such that $\{\Phi_*^H L_k^0\}_{k=0}^\infty \sqcup \{L_k^1\}_{k=0}^\infty$ is transverse. Such Hamiltonian H exists since for each $0 \leq k_0^0 < \dots < k_a^0$ and $0 \leq k_0^1 < \dots < k_b^1$ the condition that $\left\{ \Phi_*^H L_{k_0^0}^0, \dots, \Phi_*^H L_{k_a^0}^0, L_{k_0^1}^1, \dots, L_{k_b^1}^1 \right\}$ is transverse defines an open dense subset of $C_{[0,1]}^\infty(\mathcal{S}, \mathbb{R})$.

CHAPTER 5

Proof of Theorem 2

5.1. Geometric preparations

Recall the Legendrian Reidemeister moves from §2.2.4. In this section we describe how the sets of polygons from Definition 1.5 change under moves II , III_a , and III_b . This is a restatement of the results of Chekanov [15, §8], the proofs there are for Legendrians in $\mathbb{R}^3$, but the proofs in our setting are the same, so we omit them. By Lemma 2.14 the only other move we should consider is the move $II_{>0}$, we did it in §2.3.2.

Until the end of this section L_0 and L_1 are graded Lagrangians in a graded Liouville surface $\mathcal{S}$.

5.1.1. Move II . Let L_1 be obtained from L_0 by the Legendrian Reidemeister move II , as in Fig. 10(a). We have $Z_{L_1} = Z_{L_0} \cup \{a^1, b^1\}$ and $|b^1| = |a^1| + 1$.

Let $z^0, \dots, z^n \in Z_{L_0}$ and $0 \leq i_1 \leq i'_1 \leq \dots \leq i_k \leq i'_k \leq n$. Given polygons $\mathbf{u} \in \Delta^0(L_1; z^0; z^n, \dots, z^{i'_k+1}, a^1, z^{i_k}, \dots, z^{i'_1+1}, a^1, z^{i_1}, \dots, z^1)$ and $\mathbf{u}_j \in \Delta^0(L_1; b^1; z^{i'_j}, \dots, z^{i_j+1})$ for $j = 1, \dots, k$ we define $\psi_*(\mathbf{u}, \mathbf{u}_k, \dots, \mathbf{u}_1) \in \Delta^0(L_0; z^0; z^n, \dots, z^1)$ by gluing the polygon $\mathbf{u}_j$ to the j -th corner of $\mathbf{u}$ at a^1 for each $j = 1, \dots, k$, i.e. by removing the region C from $\mathbf{u}$, removing the region A from $\mathbf{u}_j$, and adding the region E to their union in Fig. 10(a).

Lemma 5.1 (Chekanov [15, §8.4]).

- (1) Let $\mathbf{u}$ be a polygon from some $\Delta^0(L_1; b^1; \dots, a^1, \dots)$. Then $\mathbf{u}$ is the polygon in $\Delta^0(L_1; b^1; a^1)$ corresponding to the bigon H in Fig. 10(a).
- (2) All $\Delta^0(L_1; a^1; \dots, b^1, \dots)$ are empty.
- (3) For any $z^0, \dots, z^n \in Z_{L_0}$ the map

$$\bigsqcup_{\substack{k \geq 0 \\ 0 \leq i_1 \leq i'_1 \leq \dots \leq i_k \leq i'_k \leq n}} \Delta^0(L_1; z^0; z^n, \dots, z^{i'_1+1}, a^1, z^{i_1}, \dots, z^1) \\ \times \prod_{j=1}^k \Delta^0(L_1; b^1; z^{i'_j}, \dots, z^{i_j+1}) \xrightarrow{\psi_*} \Delta^0(L_0; z^0; z^n, \dots, z^1)$$

defined above is a bijection.

5.1.2. Moves III_a and III_b . Let L_1 be obtained from L_0 by a Legendrian Reidemeister move III_a or III_b , as in Fig. 10(d) or Fig. 10(e). Note that then L_0 is obtained from L_1 by a Legendrian Reidemeister move III_a or III_b respectively.

We have the bijection $\psi: Z_{L_0} \rightarrow Z_{L_1}$ defined by $\psi(a^0) = a^1$, $\psi(b^0) = b^1$ and $\psi(z) = z$ for $z \in Z_{L_0} \setminus \{a^0, b^0\}$. Note that ψ preserves the degree. Also $|c| = |a^0| + |b^0|$ for move III_a , and $|c| = |a^0| + |b^0| - 1$ for move III_b by (2.8).

If $\mathbf{u}$ is the polygon from $\Delta^0(L_0; c; b^0, a^0)$ given by the triangle H in Fig. 10(d), then define $\psi_*(\mathbf{u}) \in \Delta^0(L_1; c; b^1, a^1)$ to be the polygon given by the triangle K .

Let $\mathbf{u} = (\Pi, u)$ be a polygon with convex corners bounded by L_0 . Suppose that no side of Π is mapped bijectively by u to one of the sides of the triangle H in Fig. 10(d) or Fig. 10(e). Then the isotopy from L_0 to L_1 gives an isotopy from $\mathbf{u}$ to a polygon $\psi_*(\mathbf{u}) = (\Pi, \psi_*(u))$ bounded by L_1 . For example, if $u(v) = a^0$ and a contractible neighbourhood V of a vertex v of Π is mapped bijectively by u to the union of regions A, B, C , then V is mapped bijectively by $\psi_*(u)$ to the union of regions A, C, G, H .

We are ready to state the effect of move III_a on polygons bounded by L_0 .

Lemma 5.2 (Chekanov [15, §8.2]). *Suppose that L_1 is obtained from L_0 by the Legendrian Reidemeister move III_a .*

- (1) *For $i = 0, 1$ all sets $\Delta^0(L_i; a^i; \dots, c, \dots)$ and $\Delta^0(L_i; b^i; \dots, c, \dots)$ are empty.*
- (2) *For any $z^0, \dots, z^n \in Z_{L_0}$ the map*

$$\psi_*: \Delta^0(L_0; z^0, z^n, \dots, z^1) \rightarrow \Delta^0(L_0; \psi(z^0), \psi(z^n), \dots, \psi(z^1))$$

is well defined and a bijection.

It remains to see what happens under the move III_b to polygons bounded by L_0 that have a side coinciding with a side of triangle H in Fig. 10(e). Consider $\mathbf{u} = (\Pi, u) \in \Delta^0(L_0; z^0; z^n, \dots, z^1)$ for some $z^0, \dots, z^n \in Z_{L_0}$.

Suppose $z^0 = c$. If $z^1 = b^0$ (resp. $z^n = a^0$) and the side of Π between w^0 and w^1 (resp. w^n) is mapped by u to the segment $[c, b^0]$ (resp. $[c, a^0]$) in Fig. 10(e), then set $\chi_0(\mathbf{u}) \in \Delta^0(L_0; a^0; z^n, \dots, z^2)$ (resp. $\Delta^0(L_0; b^0; z^{n-1}, \dots, z^1)$) to be the polygon obtained from $\mathbf{u}$ by adding the triangle H .

Now suppose $z^0 \neq c$. For each $i = 1, \dots, n-1$ such that $z^i = b^0$, $z^{i+1} = a^0$ and the side of Π between w^i and w^{i+1} is mapped by u to the segment $[b^0, a^0]$ add the triangle H to the polygon $\mathbf{u}$. This way we obtain $\chi_0(\mathbf{u}) \in \Delta^0(L_0; z^0; \dots)$, where $\dots$ is obtained from $z^n, \dots, z^1$ by replacing each consecutive piece a^0, b^0 such that $[b^0, a^0]$ is a side of $\mathbf{u}$ with c .

Due to the symmetry between L_0 and L_1 we also have the analogously defined χ_1 for polygons bounded by L_1 .

Lemma 5.3 (Chekanov [15, §8.3]). *Suppose that L_1 is obtained from L_0 by the Legendrian Reidemeister move III_b .*

(1) *For any $z^0, \dots, z^n \in Z_{L_0}$ the partially defined map*

$$\psi_*: \Delta^0(L_0; z^0, z^n, \dots, z^1) \rightarrow \Delta^0(L_0; \psi(z^0), \psi(z^n), \dots, \psi(z^1))$$

is a bijection between its domain and image.

(2) *ψ_* is well-defined on the image of χ_0 .*

(3) *For any $z^2, \dots, z^n \in Z_{L_0}$ the partially defined map*

$$\begin{array}{c} \Delta^0(L_0; c; z^n, \dots, z^2, b^0) \sqcup \Delta^0(L_1; c; \psi(z^n), \dots, \psi(z^2), b^1) \\ \downarrow \psi_* \circ \chi_0 \sqcup \chi_1 \\ \Delta^0(L_1; a^1; \psi(z^n), \dots, \psi(z^2)) \end{array}$$

is a bijection between its domain and codomain.

(4) *For any $z^1, \dots, z^{n-1} \in Z_{L_0}$ the partially defined map*

$$\begin{array}{c} \Delta^0(L_0; c; a^0, z^{n-1}, \dots, z^1) \sqcup \Delta^0(L_1; c; a^1, \psi(z^{n-1}), \dots, \psi(z^1)) \\ \downarrow \psi_* \circ \chi_0 \sqcup \chi_1 \\ \Delta^0(L_1; b^1; \psi(z^{n-1}), \dots, \psi(z^1)) \end{array}$$

is a bijection between its domain and codomain.

(5) *Let $z^0 \in Z_{L_0} \setminus \{a^0, b^0, c\}$, $z^1, \dots, z^n \in Z_{L_0}$, and $0 < j_1 < \dots < j_r < n-1$ be such that $z^{j_i+1} = b^0$ and $z^{j_i+2} = a^0$ for each $i = 1, \dots, r$. Then the complement of $\psi_* \circ \chi_0(\Delta^0(L_0; z^0; z^n, \dots, z^1))$ in $\Delta^0(L_1; z^0; \psi(z^n), \dots, \psi(z^{j_r+3}), c, \psi(z^{j_r}), \dots, \psi(z^{j_1+3}), c, \psi(z^{j_1}), \dots, \psi(z^1))$ consists of the polygons $\mathbf{u}$ such that for each $0 < j < n-1$ satisfying $z^{j+1} = b^0$ and $z^{j+2} = a^0$*

- *if $j = j_i$ for some $i = 1, \dots, r$, then the corresponding corner of $\mathbf{u}$ at c lies in the region H from Fig. 10(e);*
- *otherwise, the side of $\mathbf{u}$ between $\psi(z^{j+1}) = b^1$ and $\psi(z^{j+2}) = a^1$ is not the segment $[b^1, a^1]$ from Fig. 10(e).*

5.2. Algebraic preparations

Recall structured A_∞ -algebras from §1.2.2, A_∞ -morphisms from §1.5.2, and A_∞ -homotopies from §1.5.3. In §5.2.1 we explain how to transfer the elements of an A_∞ -algebra along A_∞ -morphisms. In §5.2.2 we explicitly construct homotopy equivalences between an A_∞ -algebra and its stabilisation that respect a given bounding cochain.

5.2.1. Let $\mathcal{F}: (\mathcal{A}, \mu) \rightarrow (\mathcal{A}', \mu')$ be an A_∞ -morphism between structured A_∞ -algebras and δ a bounding cochain in $\mathcal{A}$. Recall that we have an uncurved A_∞ -algebra structure on $\mathcal{A}$ given by

$$\mu_\delta(a_n, \dots, a_1) = \sum_{l_0, \dots, l_n \geq 0} \mu(\delta^{l_n}, a_n, \dots, \delta^{l_1}, a_1, \delta^{l_0})$$

where the superscript for μ is suppressed and δ^l is a shorthand for $\underbrace{\delta, \dots, \delta}_l$.

Define the structured morphism $\mathcal{F}_\delta: \mathcal{A} \rightarrow \mathcal{A}'$ by

$$\mathcal{F}_\delta(a_n, \dots, a_1) = \sum_{l_0, \dots, l_n \geq 0} \mathcal{F}(\delta^{l_n}, a_n, \dots, \delta^{l_1}, a_1, \delta^{l_0})$$

and recall that $\delta' = \mathcal{F}_\delta(\delta) = \sum_{n \geq 0} \mathcal{F}(\delta^n)$ is a bounding cochain in $\mathcal{A}'$.

The A_∞ -morphism-relation (1.29) for $\mathcal{F}$ implies that $\tilde{\mathcal{F}}_\delta: (\mathcal{A}, \mu_\delta) \rightarrow (\mathcal{A}', \mu'_{\delta'})$ given by $\tilde{\mathcal{F}}_\delta^0 = 0$ and $\tilde{\mathcal{F}}_\delta^n = \mathcal{F}_\delta^n$ for $n \geq 1$ is an A_∞ -morphism. We will only need the A_1 and A_2 -relations. Namely,

$$(5.1) \quad \mu'_{\delta'}(\mathcal{F}_\delta(a)) = \mathcal{F}_\delta(\mu_\delta(a))$$

for any $a \in \mathcal{A}$ and

$$(5.2) \quad \begin{aligned} & \mu'_{\delta'}(\mathcal{F}_\delta(a_2), \mathcal{F}_\delta(a_1)) + \mu'_{\delta'}(\mathcal{F}_\delta(a_2, a_1)) = \\ & \mathcal{F}_\delta(\mu_\delta(a_2, a_1)) + \mathcal{F}_\delta(a_2, \mu_\delta(a_1)) + (-1)^{|a_1|-1} \mathcal{F}_\delta(\mu_\delta(a_2), a_1) \end{aligned}$$

for any homogeneous $a_1, a_2 \in \mathcal{A}$.

Now let $\mathcal{F}^0, \mathcal{F}^1: (\mathcal{A}, \mu) \rightarrow (\mathcal{A}', \mu')$ be two A_∞ -morphisms and $\mathcal{H}: \mathcal{A} \rightarrow \mathcal{A}'$ an A_∞ -homotopy from $\mathcal{F}^0$ to $\mathcal{F}^1$ and δ a bounding cochain in $\mathcal{A}$. Define $\mathcal{H}_\delta: \mathcal{A} \rightarrow \mathcal{A}'$ by

$$\mathcal{H}_\delta(a_n, \dots, a_1) = \sum_{l_0, \dots, l_n \geq 0} \mathcal{H}(\delta^{l_n}, a_n, \dots, \delta^{l_1}, a_1, \delta^{l_0}).$$

The A_∞ -homotopy relation (1.31) yields

$$(5.3) \quad \delta^1 - \delta^0 = \mu'_{\delta^1, \delta^0}(\mathcal{H}_\delta())$$

and

$$(5.4) \quad \begin{aligned} & \mathcal{F}_\delta^1(a) - \mathcal{F}_\delta^0(a) = \mathcal{H}_\delta(\mu_\delta(a)) + \mu'_{\delta^1, \delta^0}(\mathcal{H}_\delta(a)) \\ & + \mu'_{\delta^1, \delta^1, \delta^0}(\mathcal{F}_\delta^1(a), \mathcal{H}_\delta()) + (-1)^{|a|-1} \mu'_{\delta^1, \delta^0, \delta^0}(\mathcal{H}_\delta(), \mathcal{F}_\delta^0(a)) \end{aligned}$$

where $\delta^i = \mathcal{F}_\delta^i(\delta)$, $i = 0, 1$ are the bounding cochains in $\mathcal{A}'$ obtained from δ via $\mathcal{F}^0$ and $\mathcal{F}^1$ and μ' with subscript is defined in (1.20).

5.2.2. Stabilisation of A_∞ -algebra. Let $(\mathcal{A}, \mu)$ be a structured A_∞ -algebra with $\mathcal{A} = \bigoplus_{z \in Z} \mathcal{A}(z)[-|z|]$. For $d \in \mathbb{Z}$ a *stabilisation (in degree d)* of $\mathcal{A}$ is a structured A_∞ -algebra $(\hat{\mathcal{A}}, \hat{\mu})$ of the form $\hat{\mathcal{A}} = \bigoplus_{z \in \hat{Z}} \hat{\mathcal{A}}(z)[-|z|]$ where $\hat{Z} = Z \sqcup \{a, b\}$, $|a| = |b| - 1 = d$, and $\hat{\mathcal{A}}(z) = \mathcal{A}(z)$ for $z \in Z$ with the only

non-zero entry of $\hat{\mu} - \mu$ given by an isomorphism

$$(\hat{\mu} - \mu)^{b;a} = \hat{\mu}^{b;a}: \mathcal{A}'(a) \rightarrow \mathcal{A}'(b).$$

The action function required in Definition 1.13 can be extended from Z to $\hat{Z}$ by any $A(b) > A(a) > 0$.

Let $(\hat{\mathcal{A}}, \hat{\mu})$ be a stabilisation in degree d of a structured A_∞ -algebra $(\mathcal{A}, \mu)$. Given $c_1 \in \hat{\mathcal{A}}(b)$ of degree one (so $c_1 \neq 0$ implies $d = 0$) set

$$c_0 = (\hat{\mu}^{b;a})^{-1}(c_1) \in \hat{\mathcal{A}}(a).$$

Recall the A_∞ -morphisms from §1.5.2. Define the structured morphisms $\mathcal{P}: \hat{\mathcal{A}} \rightarrow \mathcal{A}$ and $\mathcal{I}_{c_1}: \mathcal{A} \rightarrow \hat{\mathcal{A}}$ with the only non-zero entries given by $\mathcal{P}^{z;z} = \text{id}: \hat{\mathcal{A}}(z) \rightarrow \mathcal{A}(z)$ and $\mathcal{I}_{c_1}^{z;z} = \text{id}: \mathcal{A}(z) \rightarrow \hat{\mathcal{A}}(z)$ for $z \in Z$ and $\mathcal{I}_{c_1}^{b;()} = c_1$. A straightforward calculation shows that $\mathcal{P}$ and $\mathcal{I}_{c_1}$ are A_∞ -morphisms, $\mathcal{P} \circ \mathcal{I}_{c_1} = \text{Id}_{\mathcal{A}}$, and the only non-zero entries of $\mathcal{I}_{c_1} \circ \mathcal{P}$ are

$$(\mathcal{I}_{c_1} \circ \mathcal{P})^{z;z} = \text{id}_{\mathcal{A}(z)} \text{ for } z \in Z \text{ and } (\mathcal{I}_{c_1} \circ \mathcal{P})^{b;()} = c_1.$$

Recall the A_∞ -homotopies from §1.5.3. Define the structured morphism $\mathcal{H}_{c_1}: \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}[-1]$ with the only non-zero entries $\mathcal{H}_{c_1}^{a;()} = -c_0$ and $\mathcal{H}_{c_1}^{a;b} = (\hat{\mu}^{b;a})^{-1}$. We have the following.

Lemma 5.4. *$\mathcal{H}_{c_1}$ is an A_∞ -homotopy from $\mathcal{I}_{c_1} \circ \mathcal{P}$ to $\text{Id}_{\hat{\mathcal{A}}}$ and $-\mathcal{H}_{c_1}$ is an A_∞ -homotopy from $\text{Id}_{\hat{\mathcal{A}}}$ to $\mathcal{I}_{c_1} \circ \mathcal{P}$.*

Moreover, for any bounding cochain δ in $(\hat{\mathcal{A}}, \hat{\mu})$ if c_1 is the component of δ in $\hat{\mathcal{A}}(b)$, then $(\mathcal{H}_{c_1})_\delta() = 0$, which implies $(\mathcal{I}_{c_1} \circ \mathcal{P})_\delta() = \delta$ by (5.3)

PROOF. We check (1.31) for $\mathcal{H}_{c_0}$ and $\text{Id}_{\hat{\mathcal{A}}} - \mathcal{I}_{c_0} \circ \mathcal{P}$.

- The $(b;)$ -entry reads $-c_1 = \hat{\mu}^{b;a}(-c_0) = \hat{\mu}^{b;a} \circ \mathcal{H}_{c_1}^{a;()}();$
- the $(a; a)$ -entry reads $\text{id}_{\mathcal{A}(a)} = (\hat{\mu}^{b;a})^{-1} \circ \hat{\mu}^{b;a} = \mathcal{H}_{c_1}^{a;b} \circ \hat{\mu}^{b;a};$
- the $(b; b)$ -entry reads $\text{id}_{\mathcal{A}(b)} = \hat{\mu}^{b;a} \circ (\hat{\mu}^{b;a})^{-1} = \hat{\mu}^{b;a} \circ \mathcal{H}_{c_1}^{a;b};$ and
- in all other entries of (1.31) both sides vanish.

The same computation shows that $-\mathcal{H}_{c_0}$ is an A_∞ -homotopy from $\text{Id}_{\hat{\mathcal{A}}}$ to $\mathcal{I}_{c_0} \circ \mathcal{P}$.

If δ is a bounding cochain in $\hat{\mathcal{A}}$, then $(\mathcal{H}_{c_1})_\delta = \mathcal{H}_{c_1}^0() + \mathcal{H}_{c_1}^1(\delta) = -c_0 + (\hat{\mu}^{b;a})^{-1}(c_1) = 0$. $\square$

5.3. Effect of Legendrian isotopy on obstruction algebra

The aim of this section is to prove the following statement.

PROPOSITION 5.5. *Let $\mathcal{L}_i = (L_i, E)$, $i = 0, 1$ be two Lagrangian pre-branes in $(\mathcal{S}, w)$ such that L_1 is either Lagrangian isotopic to L_0 or differs from L_0 by a Legendrian Reidemeister move. Then*

- (1) *There are homotopy inverse A_∞ -morphisms $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1): \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ and $\mathcal{F}(\mathcal{L}_1, \mathcal{L}_0): \mathcal{A}_{\mathcal{L}_1} \rightarrow \mathcal{A}_{\mathcal{L}_0}$.*
- (2) *If δ is a bounding cochain in $\mathcal{A}_{\mathcal{L}_0}$, then these homotopy inverse A_∞ -morphisms can be chosen in such a way that an A_∞ -homotopy $\mathcal{H}$ from $\mathcal{F} = \mathcal{F}(\mathcal{L}_1, \mathcal{L}_0) \circ \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)$ to $\text{Id}_{\mathcal{A}_{\mathcal{L}_0}}$ satisfies $\mathcal{H}_\delta() = 0$.*
- (3) *The homotopy equivalences above can be chosen to respect the restriction to sub-pre-branes.*

We explain what we mean by respecting the restriction to sub-pre-branes. Let $\mathcal{L} = (L = (I, \gamma, P, \text{gr}), E)$ be a Lagrangian pre-brane and $J \subset I$ a union of connected components of I . Then $\mathcal{L}|_J = (L|_J, E|_J)$ is also a Lagrangian pre-brane. (1.15) implies that $\mathcal{A}_{\mathcal{L}|_J}$ is a graded submodule of $\mathcal{A}_{\mathcal{L}}$. Suppose that $\Delta^0(L; z^0; z^n, \dots, z^1)$ is non-empty for some $z^0 \in Z_L$ and $z^1, \dots, z^n \in Z_{\mathcal{L}|_J}$. Then $z_+^0 \in C_0$, $z_-^0 \in C_n$ and $z_+^i \in C_{i-1}$, $z_-^i \in C_i$ for $i = 1, \dots, n$ for some connected components $C_0, \dots, C_n$ of I . Therefore, $C_0, \dots, C_n \subset J$ hence $z^0 \in Z_{\mathcal{L}|_J}$. (1.17) now implies that $\mu_{\mathcal{L}}^{z^0; z^n, \dots, z^1} = \mu_{\mathcal{L}|_J}^{z^0; z^n, \dots, z^1}$. That is, $\mathcal{A}_{\mathcal{L}|_J}$ is an A_∞ -subalgebra of $\mathcal{A}_{\mathcal{L}}$. (3) in Proposition 5.5 means that

$$\mathcal{F}(\mathcal{L}_0|_J, \mathcal{L}_1|_J) = \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1) \circ (\mathcal{A}_{\mathcal{L}_0|_J} \hookrightarrow \mathcal{A}_{\mathcal{L}_0}): \mathcal{A}_{\mathcal{L}_0|_J} \rightarrow \mathcal{A}_{\mathcal{L}_1|_J}$$

for any union of connected components $J \subset I$.

In the proof of Proposition 5.5 the homotopy inverse morphisms are provided by the geometry of Lagrangians. Our proof is similar to the proof of [15, Theorem 3.4] stating that the Chekanov–Eliashberg DGAs of Legendrian isotopic Legendrian knots have the same stable type. Perhaps one could prove Proposition 5.5(1), at least when the local system is valued in projective modules, using an appropriate version of the Whitehead theorem for A_∞ -algebras [23, Theorem 4.2.45].

By Lemmas 2.14, 4.1, and 4.3 it is enough to only consider moves $II^{\pm 1}$, III_a and III_b from §2.2.4. Let L_1 be obtained from L_0 by a Legendrian Reidemeister move II , III_a or III_b . By Lemma 4.1 we can assume that L_0 (and hence L_1) is sufficiently self-wrapped. Trivialise E on the subsegments of I that feature in the move (i.e. two subsegments for move II and three subsegments for moves III_a and III_b) and therefore get the identifications $\text{Hom}(E_{z_+}, E_{z_-}) = \text{Hom}(E_{\psi(z)_+}, E_{\psi(z)_-})$ for any $z \in Z_{L_0}$. Set $(\mathcal{A}_i, \mu_i) = (\mathcal{A}_{\mathcal{L}_i}, \mu_i)$ for $i = 0, 1$. Since $C_{z_\pm}$ is the same connected component of I as $C_{\psi(z)_\pm}$ we also identify $\mathcal{A}_0(z) = \mathcal{A}_1(\psi(z))$ for any $z \in Z_{L_0}$, which in the case of moves III_a and III_b gives an identification $\mathcal{A}_0 = \mathcal{A}_1$ and in the case of move II an identification $\mathcal{A}_1 = \mathcal{A}_0 \oplus \mathcal{A}_1(a^1) \oplus \mathcal{A}_1(b^1)$ as graded R -modules.

In §§5.3.1, 5.3.2 and 5.3.3 we give explicit formulae for homotopy inverse A_∞ -morphisms between $\mathcal{A}_0$ and $\mathcal{A}_1$ separately for each of the moves III_a , III_b , and II (in the order of increasing complexity).

We prove parts (2) and (3) of Proposition 5.5 in §5.3.4.

5.3.1. Move III_a . Let L_1 be obtained from L_0 by a Legendrian Reidemeister move III_a , as in Fig. 10(d). Recall the notation and definitions from §5.1.2.

We define the structured morphisms $\mathcal{F}_1: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ by setting the only non-zero component to be $\mathcal{F}_1^1 = \text{id}: \mathcal{A}_0 \rightarrow \mathcal{A}_1$. Because of the symmetry between $\mathcal{L}_0$ and $\mathcal{L}_1$ we analogously get $\mathcal{F}_0: \mathcal{A}_1 \rightarrow \mathcal{A}_0$.

Lemma 5.6. $\mathcal{F}_0$ and $\mathcal{F}_1$ are inverse A_∞ -morphisms.

PROOF. $\mathcal{F}_0$ and $\mathcal{F}_1$ are clearly inverse to each other (see (1.28) for definition of composition) and $\mathcal{F}_0$ is obtained from $\mathcal{F}_1$ by swapping $\mathcal{L}_0$ and $\mathcal{L}_1$. Therefore, the only thing to check is that $\mathcal{F}_1$ is an A_∞ -morphism, i.e. that $\mu_0 = \mu_1$. By Lemma 5.2 it suffices to check $\mu_0^{\mathbf{u}} = \mu_1^{\psi_*(\mathbf{u})}$ for any $z^0, \dots, z^n \in Z_{L_0}$ and $\mathbf{u} \in \Delta^0(L_0; z^0, z^n, \dots, z^1)$. This follows from (1.18) since $w(\mathbf{u}) = w(\psi_*(\mathbf{u}))$, the transition maps of E are the same since E is trivialised on the strands of the move, and the orientation classes from Construction 1.15 are the same since a corner of $\psi_*(\mathbf{u})$ is obtained by an isotopy from the corresponding corner of $\mathbf{u}$ as in the proof of Lemma 4.3. $\square$

5.3.2. Move III_b . Let L_1 be obtained from L_0 by a Legendrian Reidemeister move III_b , as in Fig. 10(e). Recall the notation and definitions from §5.1.2. Let $I_- \subset C_-$, $I_\nearrow \subset C_\nearrow$, and $I_\setminus \subset C_\setminus$ be the subsegments of I parametrising the strands of the move and E_- , $E_\nearrow$ and $E_\setminus$ be the R -modules trivialising the local system E on I_- , $I_\nearrow$ and $I_\setminus$ respectively ($\gamma_0(I_-)$ separates regions A and F , $\gamma_0(I_\nearrow)$ separates regions F and M , and $\gamma_0(I_\setminus)$ separates regions J and M). We define the structured morphism $\mathcal{F}_1: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ by setting the only non-zero components to be $\mathcal{F}_1^1 = \text{id}: \mathcal{A}_0 \rightarrow \mathcal{A}_1$ and $\mathcal{F}_1^{c;a^0,b^0}: \mathcal{A}_0(a^0) \otimes \mathcal{A}_0(b^0) \rightarrow \mathcal{A}_1(c)$ given for $\varphi^{b^0} \in \text{Hom}(E_\setminus, E_-)$ and $\varphi^{a^0} \in \text{Hom}(E_-, E_\nearrow)$ by

$$(5.5) \quad \mathcal{F}_1^{c;a^0,b^0} \left(\varphi^{a^0} \otimes r_{a^0}^{\otimes |a^0|} \otimes \varphi^{b^0} \otimes r_{b^0}^{\otimes |b^0|} \right) = (\varphi^{a^0} \circ \varphi^{b^0}) \otimes r_{c^0}^{\otimes |c|}$$

where $r_{a^0} = r_{c^0} \in |C_\nearrow|_R$ is the orientation as the boundary of region C (there is no point c^0 , we just use this as a notation to indicate that we consider $c \in Z_{L^0}$) and $r_{b^0} \in |C_-|_R$ is the orientation as the boundary of region H , as indicated in Fig. 10(e).

Because of the symmetry between $\mathcal{L}_0$ and $\mathcal{L}_1$ we analogously get $\mathcal{F}_0: \mathcal{A}_1 \rightarrow \mathcal{A}_0$.

Lemma 5.7. $\mathcal{F}_0$ and $\mathcal{F}_1$ are inverse A_∞ -morphisms.

PROOF. Due to symmetry between $\mathcal{L}_0$ and $\mathcal{L}_1$ it suffices to check that $\mathcal{F}_1$ is an A_∞ -morphism and $\mathcal{F}_0 \circ \mathcal{F}_1 = \mathcal{Id}_{\mathcal{A}_0}$.

First we compute $\mathcal{F}_0 \circ \mathcal{F}_1$ by (1.28). Let $z^0, \dots, z^n \in Z_{L_0}$. If $z^0 \neq c$, then the $(z^0; z^n, \dots, z^1)$ -entry is

$$(\mathcal{F}_0 \circ \mathcal{F}_1)^{z^0; z^n, \dots, z^1} = \begin{cases} \text{id}, & \text{if } n = 1 \text{ and } z^1 = z^0; \\ 0, & \text{otherwise.} \end{cases}$$

If $z^0 = c$, then

$$(\mathcal{F}_0 \circ \mathcal{F}_1)^{c; z^n, \dots, z^1} = \begin{cases} \text{id}, & \text{if } n = 1 \text{ and } z^1 = c; \\ \mathcal{F}_1^{c; a^0, b^0} + \mathcal{F}_0^{c; a^1, b^1}, & \text{if } n = 2, z^1 = b^0, \text{ and } z^2 = a^0; \\ 0, & \text{otherwise.} \end{cases}$$

As seen from (5.5) and Fig. 10(e), $r_{a^1} = -r_{a^0}$, $r_{b^1} = -r_{b^0}$, and $r_{c^1} = -r_{c^0}$. Therefore, $\mathcal{F}_0^{c; a^1, b^1} = (-1)^{|a^0|+|b^0|-|c|} \mathcal{F}_1^{c; a^0, b^0} = -\mathcal{F}_1^{c; a^0, b^0}$. Thus, $\mathcal{F}_0 \circ \mathcal{F}_1 = \mathcal{Id}_{\mathcal{A}_0}$.

Now check $\mathcal{F}_1$ is an A_∞ -morphism. Fix $z^0, \dots, z^n \in Z_{L_0}$.

If $z^0 \neq c$, then the $(z^0; z^n, \dots, z^1)$ -entry of the A_∞ -morphism-relation (1.29) reads

(5.6)

$$\mu_0^{z^0; z^n, \dots, z^1} =$$

(5.7)

$$\sum \mu_1^{\psi(z^0); \psi(z^n), \dots, \psi(z^{j_r+3}), c, \psi(z^{j_r}), \dots, \psi(z^{j_1+3}), c, \psi(z^{j_1}), \dots, \psi(z^1)} \circ (\text{id} \otimes \mathcal{F}_1^{c; a^0, b^0} \dots \otimes \text{id}).$$

where the sum in (5.7) is taken over $0 < j_1 < \dots < j_r < n-1$ such that $z^{j_i+1} = b^0$ and $z^{j_i+2} = a^0$ for each $i = 1, \dots, r$. If z^0 is a^0 or b^0 , then the only term in (5.7) is $\mu_1^{\psi(z^0); \psi(z^n), \dots, \psi(z^1)}$ hence (5.6) is equal to (5.7) by Lemma 5.3(1). Suppose $z^0 \in Z_{L_0} \setminus \{a^0, b^0, c\}$.

Consider a polygon $\mathbf{u} \in \Delta^0(L_0; z^0; z^n, \dots, z^1)$. Then the polygon $\psi_* \circ \chi_0(\mathbf{u})$ contributes to (5.7) and since the orientation classes in the definition (5.5) of $\mathcal{F}_1^{c; a^0, b^0}$ are the same as the orientation classes in the definitions (1.18) of $\mu_0^{\mathbf{u}}$ and $\mu_1^{\psi_* \chi_0(\mathbf{u})}$, we get that

$$\mu_0^{\mathbf{u}} = \mu_1^{\psi_* \chi_0(\mathbf{u})} \circ (\text{id} \otimes \mathcal{F}_1^{c; a^0, b^0} \otimes \dots \otimes \text{id}).$$

Consider a polygon

$$\mathbf{u} = (\Pi, u) \in \Delta^0(L_1; z^0; \psi(z^n), \dots, \psi(z^{j_1+3}), c, \psi(z^{j_1}), \dots, \psi(z^1))$$

contributing to (5.7). Either $\mathbf{u} \in \psi_* \chi_0(\Delta^0(L_0; z^0; z^n, \dots, z^1))$ or, by Lemma 5.3(5), there exists the smallest $0 < j < n-1$ such that $z^{j+1} = b^0$, $z^{j+2} = a^0$ and one of the following holds:

- (1) $j = j_i$ for some $i = 1, \dots, r$ and the corresponding corner of $\mathbf{u}$ at c lies in the region K in Fig. 10(e); or
- (2) a neighbourhood of the side of $\mathbf{u}$ between z^{j+1} and z^{j+2} is contained in the region M .

If (1) (resp. (2)) holds, then we define the polygon $\mathbf{u}'$ also contributing to (5.7) by removing (resp. adding) triangle K to $\mathbf{u}$. As $\mathcal{F}_0^{c;a^1,b^1} = -\mathcal{F}_1^{c;a^0,b^0}$, the contributions of $\mathbf{u}$ and $\mathbf{u}'$ to (5.7) cancel. We have therefore shown that (5.6) equals (5.7).

If $z^0 = c$, then the $(z^0; z^n, \dots, z^1)$ -entry of the A_∞ -morphism-relation (1.29) reads

$$(5.8) \quad \begin{aligned} 0 &= \mu_0^{c;z^n,\dots,z^1} - \mu_1^{c;\psi(z^n),\dots,\psi(z^1)} \\ &+ (-1)^{|b^0|-1} \mathcal{F}_1^{c;a^0,b^0} \circ (\mu_0^{a^0;z^n,\dots,z^2} \otimes 1_{z^1=b^0}) + \mathcal{F}_1^{c;a^0,b^0} \circ (1_{z^n=a^0} \otimes \mu_0^{b^0;z^{n-1},\dots,z^1}). \end{aligned}$$

By Lemma 5.3 (1)–(4) the terms in (5.8) group into the following pairs

- (1) $\mu_0^{\mathbf{u}} - \mu_1^{\psi_*(\mathbf{u})}$ for $\mathbf{u} \in \Delta^0(L_0; c; z^n, \dots, z^1)$;
- (2) $\mu_0^{\mathbf{u}} + (-1)^{|b^0|-1} \mathcal{F}_1^{c;a^0,b^0} \circ (\mu_0^{\chi_0(\mathbf{u})} \otimes \text{id}_{\mathcal{A}_0(b_0)})$ for $\mathbf{u} \in \Delta^0(L_0; c; z^n, \dots, z^2, b^0)$ if $z^1 = b^0$;
- (3) $-\mu_1^{\mathbf{u}} + (-1)^{|b^0|-1} \mathcal{F}_1^{c;a^0,b^0} \circ (\mu_0^{\psi_*^{-1}\chi_1(\mathbf{u})} \otimes \text{id}_{\mathcal{A}_0(b_0)})$ for $\mathbf{u} \in \Delta^0(L_1; c; \psi(z^n), \dots, \psi(z^2), b^1)$ if $z^1 = b^0$;
- (4) $\mu_0^{\mathbf{u}} + \mathcal{F}_1^{c;a^0,b^0} \circ (\text{id}_{\mathcal{A}_0(a^0)} \otimes \mu_0^{\chi_0(\mathbf{u})})$ for $\mathbf{u} \in \Delta^0(L_0; c; a^0, z^{n-1}, \dots, z^1)$ if $z^n = a^0$;
- (5) $-\mu_1^{\mathbf{u}} + \mathcal{F}_1^{c;a^0,b^0} \circ (\text{id}_{\mathcal{A}_0(a^0)} \otimes \mu_0^{\psi_*^{-1}\chi_1(\mathbf{u})})$ for $\mathbf{u} \in \Delta^0(L_0; c; a^0, z^{n-1}, \dots, z^1)$ if $z^n = a^0$.

That the contribution in (1) vanishes is clear, and the vanishing for (2)–(5) follows from comparison of the orientation classes in (5.5) and in Construction 1.15, e.g. for (2) we have $r_{b_0} = -r_1$ where r_{b_0} is the orientation class for $\mathcal{F}_1^{c;a^0,b^0}$ and r_1 is the orientation class for $\mu_0^{\mathbf{u}}$. $\square$

5.3.3. Move II. Let L_1 be obtained from L_0 by a Legendrian Reidemeister move II, as in Fig. 10(a). Recall the notation and definitions from §5.1.1. Note that by Lemma 5.1 (1) $\mu_1^{b^1;a^1} : \mathcal{A}_1(a^1) \rightarrow \mathcal{A}_1(b^1)$ is an isomorphism. Let $|a^1| = d$.

Let $(\mathcal{A}, \mu) = (\mathcal{A}_0, \mu_0) \oplus (\mathcal{A}_1(a^1)[-d] \oplus \mathcal{A}_1(b^1)[-d-1], \mu_1^{b^1;a^1})$ be a stabilisation of $\mathcal{A}_0$, see §5.2.2. The nilpotent A_∞ algebras $(\mathcal{A}_0, \mu_0)$ and $(\mathcal{A}, \mu)$ are homotopy equivalent by Lemma 5.4. Note that $\mathcal{A} = \mathcal{A}_1$ as graded R -modules. We construct an invertible A_∞ -morphisms $\mathcal{F} : (\mathcal{A}, \mu) \rightarrow (\mathcal{A}, \mu_1)$.

Since the area of the bigon H in Fig. 10(a) is small we have $Z_{L_1} = Z = \{a_1^0, \dots, a_l^0, a^1, b^1, b_1^0, \dots, b_r^0\}$ where the actions satisfy

$$A(a_1^0) < \dots A(a_l^0) < A(a^1) < A(b^1) < A(b_1^0) < \dots < A(b_r^0).$$

For $k = 0, \dots, r$ define $Z_{[k]} = \{a_1^0, \dots, a_l^0, a^1, b^1, b_1^0, \dots, b_k^0\}$

Construction 5.8. We inductively construct structured morphisms $\mathcal{F}_0, \dots, \mathcal{F}_r: \mathcal{A} \rightarrow \mathcal{A}$ such that for any $k \in \{0, \dots, r\}$

- (1) if the component $(\mathcal{F}_k - \mathcal{I}d)^{z^0; z^n, \dots, z^1}$ does not vanish, then $A(z^0) > A(z^n) + \dots + A(z^1)$;
- (2) if the component $(\mathcal{F}_k - \mathcal{I}d_{\mathcal{A}})^{z^0; z^n, \dots, z^{i+1}, a^1, z^{i-1}, \dots, z^1}$ does not vanish, then $b^1 \in \{z^1, \dots, z^{i-1}\}$;
- (3) for any

$$(z^0; z^1, \dots, z^n) \in Z_{[k]} \times Z^n \cup Z \times (Z \setminus \{a^1, b^1\})^n$$

the $(z^0; z^n, \dots, z^1)$ -entry of the A_n -morphism-relation (1.29) for $\mathcal{F}_k$ is satisfied, that is,

$$(5.9) \quad \sum_{n \geq j \geq i \geq 0} \sum_{z \in Z} (-1)^{|z^i| + \dots + |z^1| - i} \mathcal{F}_k^{z^0; z^n, \dots, z^{j+1}, z, z^i, \dots, z^1} \circ (\text{id} \otimes \mu^{z; z^j, \dots, z^{i+1}} \otimes \text{id}) = \\ \sum_{\substack{s \geq 0 \\ n = i_s \geq \dots \geq i_0 = 0}} \sum_{y^1, \dots, y^s \in Z} \mu_1^{z^0; y^s, \dots, y^1} \circ (\mathcal{F}_k^{y^s; z^n, \dots, z^{i_{s-1}+1}} \otimes \dots \otimes \mathcal{F}_k^{y^1; z^1, \dots, z^1}).$$

After we have constructed such $\mathcal{F}_0, \dots, \mathcal{F}_r$, we are done since then $\mathcal{F} = \mathcal{F}_r: (\mathcal{A}, \mu) \rightarrow (\mathcal{A}, \mu_1)$ is an A_∞ -morphism by property (3) and it is invertible with structured inverse since $(\mathcal{F} - \mathcal{I}d)^{\circ m} = 0$ for some $m > 0$ by property (1).

Define the structured morphism $\mathcal{F}_0: \mathcal{A} \rightarrow \mathcal{A}$ by $\mathcal{F}_0 = \mathcal{I}d_{\mathcal{A}} + \mathcal{C}_0$ where the only non-vanishing components of the structured morphism $\mathcal{C}_0: \mathcal{A} \rightarrow \mathcal{A}$ are

$$(5.10) \quad \mathcal{C}_0^{a^1; z^n, \dots, z^1}: \mathcal{A}(z^n) \otimes \dots \otimes \mathcal{A}(z^1) \rightarrow \mathcal{A}(a^1).$$

given by $-(\mu_1^{b^1; a^1})^{-1} \circ \mu_1^{b^1; z^n, \dots, z^1}$ for any $z^1, \dots, z^n \in Z_{[0]} \setminus \{a^1, b^1\}$.

Note that $\mathcal{F}_0^n$ is of degree $1 - n$ since $|b^1| = |a^1| + 1$ and $\mathcal{F}_0$ has only finitely many non-zero entries since $(\mathcal{A}_1, \mu_1)$ is structured, i.e. $\mathcal{F}_0$ is indeed a structured morphism. Observe that $\mathcal{F}_0$ satisfies (1) and (2) of Construction 5.8. We prove that it satisfies (3) as well.

Lemma 5.9. *Let $z^0, \dots, z^n \in Z$. Suppose that (a) $z^0 \in Z_{[0]}$ or (b) $z^1, \dots, z^n \in Z \setminus \{a^1, b^1\}$.*

Then (5.9) with $k = 0$ is satisfied.

PROOF. Assume (a), i.e. $z^0 \in Z_{[0]}$. In this case, if there are non-vanishing summands in (5.9), then $A(z^0) > A(z^1), \dots, A(z^n)$ hence $z^1, \dots, z^n \in Z_{[0]}$.

If $z^0 = b^1$, then by Lemma 5.1(1) there are two cases when there are non-vanishing contributions in (5.9). One is when $n = 1$ and $z^1 = a^1$, in which case both sides are equal to $\mu_1^{b^1; a^1}$. Another is when $z^1, \dots, z^n \in Z_{[0]} \setminus \{a^1, b^1\}$.

Then (5.9) reads

$$0 = \mu_1^{b^1; z^n, \dots, z^1} + \mu_1^{b^1; a^1} \circ \mathcal{F}_0^{a^1; z^n, \dots, z^1}$$

and holds by definition of $\mathcal{F}_0$.

If $z^0 = a^1$, then the left or right hand side of (5.9) does not vanish only if $z^1, \dots, z^n \in Z_{[0]} \setminus \{a^1, b^1\}$, in which case, after an application of the isomorphism $-\mu_1^{b^1; a^1}$, it reads

$$\begin{aligned} & \sum_{n \geq j \geq i \geq 0} (-1)^{|z^i| + \dots + |z^1| - i} \sum_{z \in Z_{[0]} \setminus \{a^1, b^1\}} \mu_1^{b^1; z^n, \dots, z^{j+1}, z, z^i, \dots, z^1} \circ \left(\text{id} \otimes \mu^{z; z^j, \dots, z^{i+1}} \otimes \text{id} \right) \\ &= -\mu_1^{b^1; a^1} \circ \mu_1^{a^1; z^n, \dots, z^1}. \end{aligned}$$

This identity holds since it is the A_∞ -relation for μ_1 , as for the contributing summands in the left-hand side we have $A(b^1) > A(z)$ hence $\mu^{z; z^j, \dots, z^{i+1}} = \mu_1^{z; z^j, \dots, z^{i+1}}$ by Lemma 5.1(3).

If $z^0 \in Z_{[0]} \setminus \{a^1, b^1\}$, then (5.9) reads

$$\mu^{z^0; z^n, \dots, z^1} = \mu_1^{z^0; z^n, \dots, z^1}$$

and holds by Lemma 5.1(3).

Now assume (b), i.e. $z^1, \dots, z^n \in Z \setminus \{a^1, b^1\}$. The case $z^0 \in Z_{[0]}$ is covered by (a), so assume $z^0 \in Z \setminus Z_{[0]}$. Then (5.9) reads

$$\begin{aligned} \mu_0^{z^0; z^n, \dots, z^1} &= \sum_{k \geq 0} \sum_{0 \leq i_1 \leq i'_1 \leq \dots \leq i_k \leq i'_k \leq n} (-1)^k \mu_1^{z^0; z^n, \dots, z^{i'_k+1}, a^1, z^{i_k}, \dots, z^{i'_1+1}, a^1, z^{i_1}, \dots, z^1} \circ \\ &\left(\text{id} \otimes (\mu_1^{b^1; a^1})^{-1} \circ \mu_1^{b^1; z^{i'_k}, \dots, z^{i_k+1}} \otimes \text{id} \otimes \dots \otimes (\mu_1^{b^1; a^1})^{-1} \circ \mu_1^{b^1; z^{i'_1}, \dots, z^{i_1+1}} \otimes \text{id} \right) \end{aligned}$$

and holds by Lemma 5.1(3) since the orientation of the lower horizontal strand in Fig. 10(a) used to define $\mu_1^{b^1; a^1}$ is opposite to the orientations used to define the moduli spaces of polygons bounded by L_1 in Lemma 5.1(3). $\square$

Now, for $k = 1, \dots, r$, given a formal morphism $\mathcal{F}_{k-1}$ satisfying the requirements of Construction 5.8, we define a formal morphism $\mathcal{F}_k: \mathcal{A} \rightarrow \mathcal{A}$ by $\mathcal{F}_k = \mathcal{F}_{k-1} + \mathcal{C}_k$ where the only non-vanishing components of the formal morphism $\mathcal{C}_k: \mathcal{A} \rightarrow \mathcal{A}$ are defined by

$$\begin{aligned} (5.11) \quad & (-1)^{|z^{t-1}| + \dots + |z^1| - t + 1} \mathcal{C}_k^{b_k^0; z^n, \dots, z^{t+1}, b^1, z^{t-1}, \dots, z^1} \circ \left(\text{id} \otimes \mu_1^{b^1; a^1} \otimes \text{id} \right) = \\ & \sum_{\substack{s \geq 0 \\ n = i_s \geq \dots \geq i_0 = 0}} \sum_{y^1, \dots, y^s \in Z_{[k-1]}} \mu_1^{b_k^0; y^s, \dots, y^1} \circ \left(\mathcal{F}_{k-1}^{y^s; z^n, \dots, z^{i_{s-1}+1}} \otimes \dots \otimes \mathcal{F}_{k-1}^{y^1; z^{i_1}, \dots, z^1} \right). \end{aligned}$$

for any $z^1, \dots, z^{t-1} \in Z_{[k-1]} \setminus \{a^1, b^1\}$, $z^t = b^1$, and $z^{t+1}, \dots, z^n \in Z_{[k-1]}$.

Thus defined $\mathcal{F}_k$ satisfies (1) from Construction 5.8 since $A(b^1) - A(a^1) = \text{Area}(H)$ (recall that H is the bigon in Fig. 10(a)) is small compared to all other areas. $\mathcal{F}_k$ satisfies (2) by construction. We check (3).

Fix $(z^0; z^1, \dots, z^n) \in Z_{[k]} \times Z^n \cup Z \times (Z \setminus \{a^1, b^1\})^n$. If $z^0 \neq b_k^0$ or $\{z^1, \dots, z^n\} \cap \{a^1, b^1\} = \emptyset$, then the entries of $\mathcal{F}_k$ contributing to (5.9) are equal to the corresponding entries of $\mathcal{F}_{k-1}$ hence (5.9) for $\mathcal{F}_k$ is satisfied as it is satisfied for $\mathcal{F}_{k-1}$ by induction hypothesis. It remains to consider the case when $z^0 = b_k^0$, $z^1, \dots, z^{t-1} \in Z_{[k-1]} \setminus \{a^1, b^1\}$, and $z^t \in \{a^1, b^1\}$.

If $z^t = a^1$, then (5.9) is precisely (5.11) since all other $\mathcal{F}_k^{b_k^0; \dots}$ potentially contributing to the left hand side vanish as they are of the form $\mathcal{F}_k^{b_k^0; \dots, a^1, y^s, \dots, y^1}$ where $y^s, \dots, y^1 \in Z \setminus \{a^1, b^1\}$.

If $z^t = b^1$, then (5.9) precomposed with the isomorphism $\text{id} \otimes \mu^{b^1, a^1} \otimes \text{id}$ reads

$$(5.12) \quad 0 = \sum_{\substack{s \geq 0 \\ y^1, \dots, y^s \in Z_{[k-1]}}} \mu_1^{b_k^0; y^s, \dots, y^1} \circ \left(\sum_{\substack{n \geq j \geq i \geq 0 \\ x \in Z_{[k-1]}}} (-1)^{|x^i| + \dots + |x^1| - i} \mathcal{F}_{k-1}^{y^s, \dots, y^1; x^n, \dots, x^{j+1}, x, x^i, \dots, x^1} \circ \left(\text{id} \otimes \mu^{x; x^j, \dots, x^{i+1}} \otimes \text{id} \right) \right)$$

where we renamed $x^h = z^h$ for $h \neq t$ and $x^t = a^1$ and denoted

$$\mathcal{F}_{k-1}^{y^s, \dots, y^1; w^l, \dots, w^1} = \sum_{l=i_s \geq \dots \geq i_0=0} \mathcal{F}_{k-1}^{y^s; w^l, \dots, w^{i_{s-1}+1}} \otimes \dots \otimes \mathcal{F}_{k-1}^{y^1; w^{i_1}, \dots, w^1}.$$

Since $\mathcal{F}_{k-1}$ satisfies the A_∞ -morphism relation for all the points from Z involved in (5.12), the right hand side of (5.12) is equal to

$$\sum_{\substack{s \geq 0 \\ y^1, \dots, y^s \in Z_{[k-1]}}} \sum_{\substack{s \geq j \geq i \geq 0 \\ y \in Z_{[k-1]}}} \mu_1^{b_k^0; y^s, \dots, y^{j+1}, y, y^i, \dots, y^1} \circ (-1)^{|y^i| + \dots + |y^1| - i} \left(\text{id} \otimes \mu_1^{y; y^j, \dots, y^{i+1}} \otimes \text{id} \right) \circ \left(\mathcal{F}_{k-1}^{y^s; x^n, \dots, x^{i_{s-1}+1}} \otimes \dots \otimes \mathcal{F}_{k-1}^{y^1; x^{i_1}, \dots, x^1} \right)$$

and vanishes because μ_1 is an A_∞ -operation.

We have therefore proved the existence of $\mathcal{F}_0, \dots, \mathcal{F}_r$ from Construction 5.8 and thus obtained the desired invertible A_∞ -morphism $\mathcal{F} = \mathcal{F}_r: (\mathcal{A}, \mu) \rightarrow (\mathcal{A}, \mu_1)$.

5.3.4. We prove Proposition 5.5 (2).

In all cases except one the maps $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)$ and $\mathcal{F}(\mathcal{L}_1, \mathcal{L}_0)$ are in fact inverse to each other, so $\mathcal{F} = \mathcal{Id}_{\mathcal{A}_{\mathcal{L}_0}}$ and there is nothing to prove. The one remaining case is when $\mathcal{L}_1$ is obtained from $\mathcal{L}_0$ by a move II^{-1} . In this case the obstruction algebras $\mathcal{A}_{\mathcal{L}_0}$ and $\mathcal{A}_{\mathcal{L}_1}$ up to an A_∞ -isomorphism differ by a stabilisation and the claim follow from Lemma 5.4.

We prove Proposition 5.5 (3). We need to show that the homotopy equivalences $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1): \mathcal{A}_{\mathcal{L}_0} \rightarrow \mathcal{A}_{\mathcal{L}_1}$ constructed in §§4.1.1, 4.1.3, and 5.3.1 to 5.3.3

when $\mathcal{L}_1$ is obtained from $\mathcal{L}_0$ by a Lagrangian isotopy or a Legendrian Reidemeister move $II_{>0}^{\pm 1}$, III_a , III_b or $II^{\pm 1}$ satisfy

$$(5.13) \quad \mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)^{z^0; z^n, \dots, z^1} = \begin{cases} \mathcal{F}(\mathcal{L}_0|_J, \mathcal{L}_1|_J)^{z^0; z^n, \dots, z^1}, & \text{if } z^0 \in Z_{L_1|_J}; \\ 0, & \text{otherwise.} \end{cases}$$

for any $z^0 \in Z_{L_1}$ and $z^1, \dots, z^n \in Z_{L_0|_J}$.

For the case of Lagrangian isotopy, moves $II_{>0}^{\pm 1}$, and move III_a this is clear since $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)^n = 0$ when $n \neq 1$ and $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)^1$ is an isomorphism induced by a bijection between Z_{L_0} and Z_{L_1} .

Consider the move III_b . Then $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)$ is given by $\mathcal{F}_1: \mathcal{A}_{\mathcal{L}_0} = \mathcal{A}_0 \rightarrow \mathcal{A}_{\mathcal{L}_1} = \mathcal{A}_1$ from §5.3.2. Suppose that $\mathcal{F}_1^{z^0; z^n, \dots, z^1} \neq 0$ for some $z^0 \in Z_{L_1}$ and $z^1, \dots, z^n \in Z_{L_0|_J}$. Then there are two cases. The first is $n = 1$, $z^1 = \psi(z^0) \in Z_{L_1|_J}$, and

$$\mathcal{F}_1^{z^0; z^1} = \text{id}_{\mathcal{A}_0(z^0)} = \text{id}_{\mathcal{A}_1(\psi(z^1))} = \mathcal{F}(\mathcal{L}_1|_J, \mathcal{L}_0|_J)^{z^0; z^1}.$$

The second is $n = 2$, $b^0 = z^1 \in Z_{L_0|_J}$, $a^0 = z^2 \in Z_{L_0|_J}$. But then $c \in Z_{L_0|_J}$ and

$$\mathcal{F}_1^{c; a^0, b^0} = \mathcal{F}(\mathcal{L}_0|_J, \mathcal{L}_1|_J)^{c; a^0, b^0}$$

follows.

Consider the move II . Then $\mathcal{F}(\mathcal{L}_0, \mathcal{L}_1)$ is given by

$$\left((\mathcal{A}_0, \mu_0) \xrightarrow{F(\mathcal{L}_0, \mathcal{L}_1)} (\mathcal{A}_1, \mu_1) \right) = \left((\mathcal{A}_1, \mu) \xrightarrow{\mathcal{F}} (\mathcal{A}_1, \mu_1) \right) \circ \left((\mathcal{A}_0, \mu_0) \xrightarrow{\mathcal{I}_{c_0}} (\mathcal{A}_1, \mu) \right)$$

from §5.3.3, where $(\mathcal{A}_i, \mu_i)$ is the obstruction algebra of $\mathcal{L}_i$ for $i = 0, 1$ and $\mathcal{I}_{c_1}$ is an inclusion into stabilisation as in §5.2.2. That $\mathcal{I}_{c_1}$ and its homotopy inverse $\mathcal{P}$ respect the restriction to J is clear from their definition in §5.2.2. We prove that $\mathcal{F}$ also respects the restriction to J . Recall that $\mathcal{F} = \mathcal{I}d_{\mathcal{A}_1} + \mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_r$ where the structured morphism $\mathcal{C}_0$ is defined in (5.10) and $\mathcal{C}_k$ for $k = 1, \dots, r$ are defined in (5.11). $\mathcal{I}d_{\mathcal{A}_1}$ clearly respects the restriction to J . If $\mathcal{C}_0^{z^0; z^n, \dots, z^1}$ is non-zero for $z^1, \dots, z^n \in Z_{L_1|_J}$, then (5.10) implies that $b^1 \in Z_{L_1|_J}$ hence $a^1 \in Z_{L_1|_J}$ so the all the entries of μ_1 involved in the definition of $\mathcal{C}_0^{z^0; z^n, \dots, z^1}$ can be computed in $\mathcal{A}_{\mathcal{L}_1|_J}$ hence $\mathcal{C}_0$ respects restriction to J . Therefore, so does $\mathcal{F}_0$.

We now proceed by induction on $k = 1, \dots, r$. Suppose $\mathcal{F}_{k-1} = \mathcal{I}d_{\mathcal{A}_1} + \mathcal{C}_0 + \mathcal{C}_1 + \dots + \mathcal{C}_{k-1}$ respects restriction to J . To prove that $\mathcal{F}_k$ also respects restriction to J it suffices to show that if $\mathcal{C}_k^{b_k^0; z^n, \dots, z^1}$ is non-zero for some $z^1, \dots, z^{t-1}, z^t = b^1, z^{t+1}, \dots, z^n \in Z_{L_1|_J}$, then b_k^0 and all $y^1, \dots, y^s$ involved in the right hand side of (5.11) belong to $Z_{\mathcal{L}_1|_J}$. This is the case for $y^1, \dots, y^s$ by induction hypothesis and for b_k^0 since there exists a non-zero component $\mu_1^{b_k^0; y^s, \dots, y^1}$.

We have therefore shown that $\mathcal{F} = \mathcal{F}_r$ respects restriction to J . Since the inverse to $\mathcal{F}$ is a polynomial in $\mathcal{C}_0, \dots, \mathcal{C}_r$ and these structured morphisms respect restriction to J , so does the inverse of $\mathcal{F}$.

5.4. Homotopy equivalent objects from Legendrian isotopy

In this section we prove Theorem 2. By Lemmas 2.13 and 4.23 it remains to prove the following.

PROPOSITION 5.10. *Let $(\mathcal{L}, \delta)$ be a Lagrangian brane and $\mathcal{L}'$ a Lagrangian pre-brane that differs from $\mathcal{L}$ by a Legendrian Reidemeister move. Let $\mathcal{F}: \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\mathcal{L}'}$ be a homotopy equivalence of obstruction algebras satisfying condition (2) and (3) of Proposition 5.5. Then $(\mathcal{L}, \delta)$ and $(\mathcal{L}', \mathcal{F}_*(\delta))$ are homotopy equivalent as objects of the relative wrapped Fukaya category $\mathcal{W}$.*

In §5.4.1 we explain how to transfer the homotopy equivalence between a Lagrangian brane and its transverse perturbation along a Legendrian Reidemeister move. In §5.4.2 we finish the proof of Proposition 5.10.

5.4.1. We start the proof of Proposition 5.10. We can and will assume that all homotopy equivalences of obstruction algebras we choose satisfy (2) and (3) in Proposition 5.5.

Let $\mathbb{L}_1 = (\mathcal{L}_1 = (L_1, E), \delta_1)$ and $\mathbb{L}'_1 = (\mathcal{L}'_1 = (L'_1, E), \delta'_1)$ be two Lagrangian branes such that L'_1 is a small perturbation of a Legendrian Reidemeister move applied to L_1 , L_1 and L'_1 are transverse, and $\mathcal{F}(\mathcal{L}_1, \mathcal{L}'_1): \mathcal{A}_{\mathcal{L}_1} \rightarrow \mathcal{A}_{\mathcal{L}'_1}$ is a homotopy equivalence of obstruction algebras such that $\delta'_1 = \mathcal{F}(\mathcal{L}_1, \mathcal{L}'_1)_*(\delta_1)$. Then we can find transverse perturbations $\mathbb{L}_2 = (\mathcal{L}_2, \delta_2)$ of $\mathbb{L}_1$ and $\mathbb{L}'_2 = (\mathcal{L}'_2, \delta'_2)$ of $\mathbb{L}'_1$ that differ by the same Legendrian Reidemeister move (such perturbations can be obtained e.g. by a small shift upwards in Fig. 10). Taking $h_2 \ll 0$ we can ensure that the transverse doubles $\mathbb{L} = (\mathcal{L}, \delta = \delta_1 + \delta_2) = \mathbb{L}_{(0, h_2)}(\mathbb{L}_1, \mathbb{L}_2)$ and $\mathbb{L}' = (\mathcal{L}', \delta' = \delta'_1 + \delta'_2) = \mathbb{L}_{(0, h_2)}(\mathbb{L}'_1, \mathbb{L}'_2)$ are Legendrian isotopic, and that the composite Lagrangian branes $\mathbb{L}_{(0, 0)}(\mathbb{L}_1, \mathbb{L}'_1)$ and $\mathbb{L}_{(0, 0)}(\mathbb{L}_2, \mathbb{L}'_2)$ are Lagrangian isotopic. $\mathcal{L}'$ is obtained from $\mathcal{L}$ by a successive application of moves from Proposition 5.5 hence we get a homotopy equivalence of obstruction algebras $\mathcal{F}: \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\mathcal{L}'}$ such that $\delta' = \mathcal{F}_*\delta$.

Let $e_{12} \in \mathcal{W}(\mathbb{L}_1, \mathbb{L}_2)$ and $e'_{12} \in \mathcal{W}(\mathbb{L}'_1, \mathbb{L}'_2)$ be the homotopy equivalences as in §4.4.1. Note that $\mathbb{L}$ is adapted to e_{12} .

Lemma 5.11. *$\mathcal{F}_\delta(e_{12}) \in \mathcal{W}(\mathbb{L}'_1, \mathbb{L}'_2)$ is cohomologous to e'_{12} . In particular, $\mathcal{F}_\delta(e_{12})$ is a homotopy equivalence.*

PROOF. Note that $\mathcal{F}_\delta(e_{12})$ is closed by (5.1). Let $\mathcal{F}': \mathcal{A}_{\mathcal{L}'} \rightarrow \mathcal{A}_{\mathcal{L}}$ be a homotopy inverse to $\mathcal{F}$.

Let $\mathbb{L}'_0 = (\mathcal{L}'_0, \delta'_0)$ be a transverse perturbation of $\mathbb{L}'_1$ and $h_0 \gg 0$ be such that $\mathbb{L}_{(h_0,0,0)}(\mathbb{L}'_0, \mathbb{L}'_1, \mathbb{L}_1)$ and $\mathbb{L}_{(h_0,h_2,h_2)}(\mathbb{L}'_0, \mathbb{L}'_2, \mathbb{L}_2)$ are Lagrangian isotopic, the Lagrangian brane $\tilde{\mathbb{L}}' = (\tilde{\mathcal{L}}', \tilde{\delta}') = \mathbb{L}_{(h_0,0,h_2)}(\mathbb{L}'_0, \mathbb{L}'_1, \mathbb{L}'_2)$ contains the transverse double $\mathbb{L}'$ of $\mathbb{L}'_1$ nicely, and the Lagrangian brane $\tilde{\mathbb{L}} = (\tilde{\mathcal{L}}, \tilde{\delta}) = \mathbb{L}_{(h_0,0,h_2)}(\mathbb{L}'_0, \mathbb{L}_1, \mathbb{L}_2)$ contains the transverse double $\mathbb{L}$ of $\mathbb{L}_1$ nicely. We extend $\mathcal{F}$ and $\mathcal{F}'$ to homotopy inverse A_∞ -morphisms $\tilde{\mathcal{F}}: \mathcal{A}_{\tilde{\mathcal{L}}} \rightarrow \mathcal{A}_{\tilde{\mathcal{L}'}}$ and $\tilde{\mathcal{F}}': \mathcal{A}_{\tilde{\mathcal{L}'}} \rightarrow \mathcal{A}_{\tilde{\mathcal{L}}}$.

Let $e'_{01} \in \mathcal{W}(\mathbb{L}'_0, \mathbb{L}'_1)$ be the homotopy equivalence as in §4.4.1. Consider $a = \mathcal{F}'_{\delta'}(\tilde{\mu}(\mathcal{F}_\delta(e_{12}), e'_{01})) \in \mathcal{W}(\mathbb{L}'_0, \mathbb{L}_2)$. By (5.2) a is cohomologous to $\tilde{\mu}(\mathcal{F}'_{\delta'} \circ \mathcal{F}_\delta(e_{12}), \mathcal{F}'_{\delta'}(e'_{01}))$. Proposition 5.5 (2) and (5.4) imply that $\mathcal{F}'_{\delta'} \circ \mathcal{F}_\delta(e_{12})$ is cohomologous to e_{12} hence a is cohomologous to $\tilde{\mu}(e_{12}, \mathcal{F}'_{\delta'}(e'_{01}))$. By Lemma 4.17 a is cohomologous to $\theta_{\mathbb{L}_1, \mathbb{L}_2} \circ \mathcal{F}'_{\delta'}(e'_{01})$. Since $\mathbb{L}_{(h_0,0,0)}(\mathbb{L}'_0, \mathbb{L}'_1, \mathbb{L}_1)$ and $\mathbb{L}_{(h_0,h_2,h_2)}(\mathbb{L}'_0, \mathbb{L}'_2, \mathbb{L}_2)$ are Lagrangian isotopic and $\tilde{\mathbb{L}}'$ contains the transverse double $\mathbb{L}'$ of $\mathbb{L}'_1$ nicely,

$$\theta_{\mathbb{L}_1, \mathbb{L}_2} \circ \mathcal{F}'_{\delta'}(e'_{01}) = \mathcal{F}'_{\delta'} \circ \theta_{\mathbb{L}'_1, \mathbb{L}'_2}(e'_{01}) = \mathcal{F}'_{\delta'}(\tilde{\mu}(e'_{12}, e'_{01}))$$

We therefore conclude that $\mathcal{F}'_{\delta'}(\tilde{\mu}(\mathcal{F}_\delta(e_{12}) - e'_{12}, e'_{01}))$ is exact. Applying $\mathcal{F}_\delta$ to this and using (5.4) we obtain that $\tilde{\mu}(\mathcal{F}_\delta(e_{12}) - e'_{12}, e'_{01})$ is exact. Since e'_{01} is a homotopy equivalence, Lemma 3.3 yields that $\mathcal{F}_\delta(e_{12}) - e'_{12}$ is exact and we are done. $\square$

5.4.2. We continue the proof of Proposition 5.10. Let $\mathbb{L}_1$ and $\mathbb{L}'_1$ be as in §5.4.1. Choose transverse perturbations $\mathbb{L}_0, \mathbb{L}_2, \mathbb{L}_3$ of $\mathbb{L}_1$ and $\mathbb{L}'_3$ of $\mathbb{L}'_1$ such that the following holds

- (1) $\{\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3, \mathbb{L}'_1, \mathbb{L}'_3\}$ is a transverse set;
- (2) $\mathbb{L}_0, \mathbb{L}_1$, and $\mathbb{L}_2$ are as in Lemma 4.15, i.e. $e_{12} \circ e_{01} = e_{02}$;
- (3) $\mathbb{L}_1, \mathbb{L}_2$, and $\mathbb{L}_3$ are also as in Lemma 4.15, i.e. $e_{23} \circ e_{12} = e_{13}$;
- (4) $\mathbb{L}_1, \mathbb{L}'_1, \mathbb{L}_3, \mathbb{L}'_3$ are as in §5.4.1, i.e. $\mathbb{L}_{(0,0)}(\mathbb{L}_1, \mathbb{L}'_1)$ and $\mathbb{L}_{(0,0)}(\mathbb{L}_3, \mathbb{L}'_3)$ are Lagrangian isotopic.

Let $\mathbb{L} = (\mathcal{L}, \delta) = \mathbb{L}_{(h_0,h_1,h_2,h_3)}(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3)$ be a composite Lagrangian brane with $h_0 \gg h_1 \gg h_2 \gg h_3$ so that $\mathbb{L}' = (\mathcal{L}', \delta') = \mathbb{L}_{(h_0,h_1,h_2,h_3)}(\mathbb{L}_0, \mathbb{L}'_1, \mathbb{L}_2, \mathbb{L}'_3)$ is Legendrian isotopic to $\mathbb{L}$ and $\mathbb{L}$ is adapted to e_{01}, e_{12}, e_{13} . Let $\mathcal{F}: \mathcal{A}_{\mathcal{L}} \rightarrow \mathcal{A}_{\mathcal{L}'}$ be the homotopy equivalence obtained from successive application of Proposition 5.5.

Then (5.2) and the fact that $\mathcal{F}$ respects the restriction to sub-pre-branes implies that the following diagram is commutative (composition is given by

$\tilde{\mu})$.

$$\begin{array}{ccccc}
 & & \mathbb{L}'_1 & \xrightarrow{\mathcal{F}_\delta(e_{13})} & \mathbb{L}'_3 \\
 \mathcal{F}_\delta(e_{01}) \nearrow & & \searrow \mathcal{F}_\delta(e_{12}) & & \nearrow \mathcal{F}_\delta(e_{23}) \\
 \mathbb{L}_0 & \xrightarrow{e_{02}} & \mathbb{L}_2 & & \\
 \searrow e_{01} & & \nearrow e_{12} & & \searrow e_{23} \\
 & & \mathbb{L}_1 & \xrightarrow{e_{13}} & \mathbb{L}_3.
 \end{array}$$

e_{02} is a homotopy equivalence. By Lemma 5.11 $\mathcal{F}_\delta(e_{13})$ is also a homotopy equivalence. Hence, as $\mathcal{F}_\delta(e_{12})$ is a closed degree zero morphism, it is a homotopy equivalence by Lemma 3.1.

We have therefore proved that for any Lagrangian branes $\mathbb{L}$ and $\mathbb{L}' = (\mathcal{L}', \mathcal{F}_*(\delta))$ as in Proposition 5.10 there exist Lagrangian branes $\mathbb{L}_1$ Lagrangian isotopic to $\mathbb{L}$ and $\mathbb{L}'_1$ Lagrangian isotopic to $\mathbb{L}'$ such that $\mathbb{L}_1$ and $\mathbb{L}'_1$ are homotopy equivalent. By Lemma 4.23 this implies that $\mathbb{L}$ and $\mathbb{L}'$ are homotopy equivalent, which finishes the proof of Proposition 5.10 and hence of Theorem 2.

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